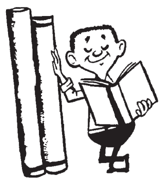
Conduction Heat Transfer

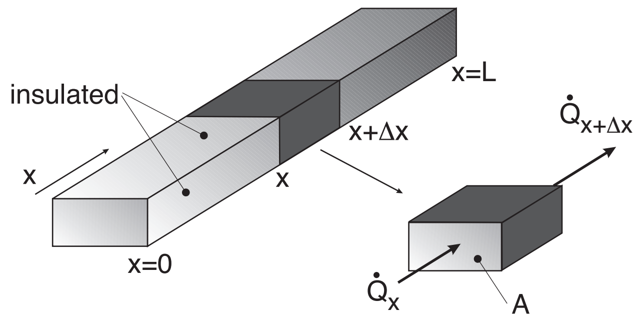


Reading	Problems
10-1 → 10-6	10-20, 10-35, 10-49, 10-54, 10-59, 10-69, 10-71, 10-92, 10-126, 10-143, 10-157, 10-162
11-1 → 11-2	11-14, 11-17, 11-36, 11-41, 11-46, 11-97, 11-104

General Heat Conduction

From a 1^{st} law energy balance:

$$\frac{\partial E}{\partial t} = \dot{Q}_x - \dot{Q}_{x+\Delta x}$$



If the volume to the element is given as $V = A \cdot \Delta x$, then the mass of the element is

$$m = \rho \cdot A \cdot \Delta x$$

The energy term ($KE = PE = 0$) is

$$E = m \cdot u = (\rho \cdot A \cdot \Delta x) \cdot u$$

For an incompressible substance the internal energy is $du = C\ dT$ and we can write

$$\frac{\partial E}{\partial t} = \rho C A \Delta x \frac{\partial T}{\partial t}$$

Heat flow along the $x-$direction is a product of the temperature difference.

$$\dot{Q}_x = \frac{kA}{\Delta x}(T_x - T_{x+\Delta x})$$

where k is the thermal conductivity of the material. In the limit as $\Delta x \rightarrow 0$

$$\dot{Q}_x = -kA\frac{\partial T}{\partial x}$$

This is *Fourier's law of heat conduction.* The $-ve$ in front of k guarantees that we adhere to the 2^{nd} law and that heat always flows in the direction of lower temperature.

We can write the heat flow rate across the differential length, Δx as a truncated Taylor series expansion as follows

$$\dot{Q}_{x+\Delta x} = \dot{Q}_x + \frac{\partial \dot{Q}_x}{\partial x}\Delta x$$

when combined with Fourier's equation gives

$$\dot{Q}_{x+\Delta x} = \underbrace{-kA\frac{\partial T}{\partial x}}_{\dot{Q}_x} - \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right)\Delta x$$

Noting that

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} = \frac{\partial E}{\partial t} = \rho C A \Delta x \frac{\partial T}{\partial t}$$

By removing the common factor of $A\Delta x$ we can then write the general 1-D conduction equation as

$$\underbrace{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)}_{\substack{\textit{longitudinal}\\ \textit{conduction}}} = \underbrace{\rho C\frac{\partial T}{\partial t}}_{\substack{\textit{thermal}\\ \textit{inertia}}}$$

$$\swarrow \qquad \searrow$$

Steady Conduction

- $\dfrac{\partial T}{\partial t} \rightarrow 0$
- properties are constant
- temperature varies in a linear manner
- heat flow rate defined by Fourier's equation
- resistance to heat flow: $R = \dfrac{\Delta T}{\dot{Q}}$

Transient Conduction

- properties are constant
- therefore $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{\rho C}{k}\dfrac{\partial T}{\partial t} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$

  where thermal diffusivity is defined as $\alpha = \dfrac{k}{\rho C}$
- exact solution is complicated
- partial differential equation can be solved using approximate or graphical methods

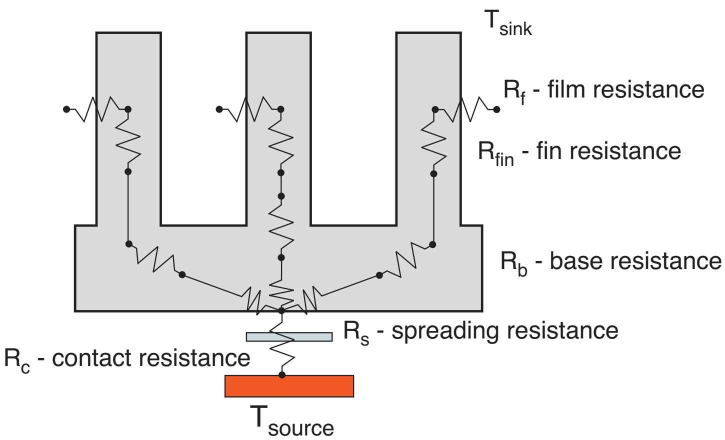
Steady Heat Conduction

Thermal Resistance Networks

Thermal circuits based on heat flow rate, $\dot{Q}$, temperature difference, ΔT and thermal resistance, R, enable analysis of complex systems.

Thermal Resistance

The thermal resistance to heat flow ($°C/W$) can be constructed for all heat transfer mechanisms, including conduction, convection, and radiation as well as contact resistance and spreading resistance.



Conduction: $$R_{cond} = \frac{L}{kA}$$

Convection: $$R_{conv} = \frac{1}{hA}$$

Radiation: $$R_{rad} = \frac{1}{h_{rad}A} \longrightarrow h_{rad} = \epsilon\sigma(T_s^2 + T_{surr}^2)(T_s + T_{surr})$$

Contact: $$R_c = \frac{1}{h_c A} \longrightarrow h_c \text{ see Table 10-2}$$

Cartesian Systems

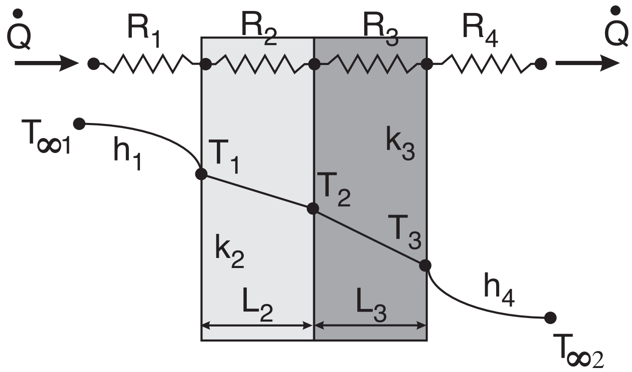
Resistances in Series

The heat transfer across the fluid/solid interface is based on *Newton's law of cooling*

$$\dot{Q} = hA(T_{in} - T_{out}) = \frac{T_{in} - T_{out}}{R_{conv}} \qquad \text{where} \quad R_{conv} = \frac{1}{hA}$$

The heat flow through a solid material of conductivity, $\boldsymbol{k}$ is

$$\dot{Q} = \frac{kA}{L}(T_{in} - T_{out}) = \frac{T_{in} - T_{out}}{R_{cond}} \qquad \text{where} \quad R_{cond} = \frac{L}{kA}$$



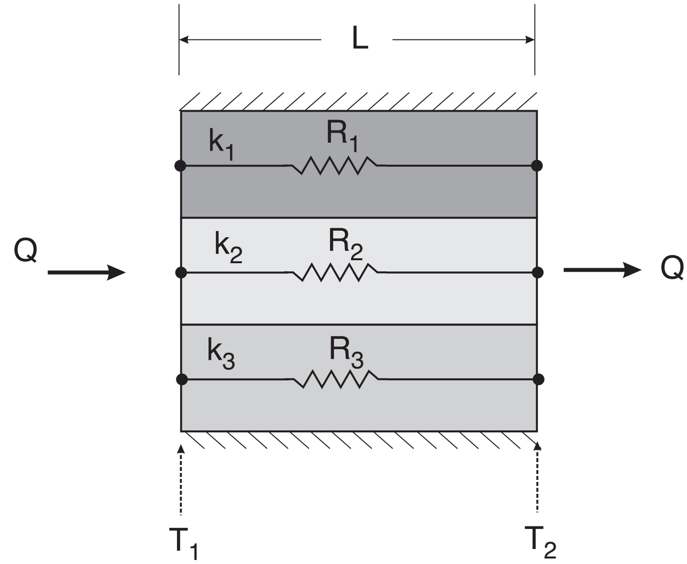
By summing the temperature drop across each section, we can write:

$$\begin{aligned}
\dot{Q}\,R_1 &= (T_{\infty_1} - T_1)\\
\dot{Q}\,R_2 &= (T_1 - T_2)\\
\dot{Q}\,R_3 &= (T_2 - T_3)\\
\dot{Q}\,R_4 &= (T_3 - T_{\infty_2})\\
\hline
\dot{Q}\left(\sum_{i=1}^{4} R_i\right) &= (T_{\infty_1} - T_{\infty_2})
\end{aligned}$$

The total heat flow across the system can be written as

$$\dot{Q} = \frac{T_{\infty_1} - T_{\infty_2}}{R_{total}} \qquad \text{where} \quad R_{total} = \sum_{i=1}^{4} R_i$$

Resistances in Parallel



For systems of parallel flow paths as shown above, we can use the 1^{st} law to preserve the total energy

$$\dot{Q} = \dot{Q}_1 + \dot{Q}_2$$

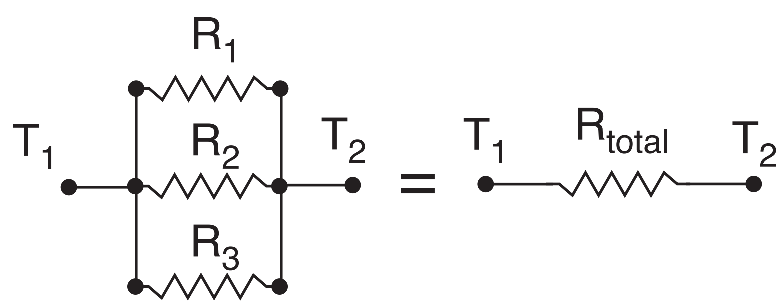
where we can write

$$\dot{Q}_1 = \frac{T_1 - T_2}{R_1} \qquad R_1 = \frac{L}{k_1 A_1}$$

$$\dot{Q}_2 = \frac{T_1 - T_2}{R_2} \qquad R_2 = \frac{L}{k_2 A_2}$$

$$\dot{Q} = \sum \dot{Q}_i = (T_1 - T_2)\left(\sum \frac{1}{R_i}\right) \qquad \text{where} \qquad \frac{1}{R_{total}} = \sum \frac{1}{R_i} = UA$$

In general, for parallel networks we can use a parallel resistor network as follows:

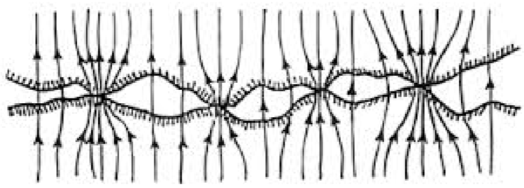


$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$

and

$$\dot{Q} = \frac{T_1 - T_2}{R_{total}}$$

Thermal Contact Resistance



- <u>real surfaces</u> have microscopic roughness, leading to non-perfect contacts where
  - 1 - 4% of the surface area is in solid-solid contact, the remainder consists of air gaps

- the total heat flow rate can be written as

$$\dot{Q}_{total} = h_c A \Delta T_{interface}$$

where:

h_c = thermal contact conductance
A = apparent or projected area of the contact
$\Delta T_{interface}$ = average temperature drop across the interface

The conductance, h_c and the contact resistance, R_c can be written as

$$h_c A = \frac{\dot{Q}_{total}}{\Delta T_{interface}} = \frac{1}{R_c}$$

Table 10-2 can be used to obtain some representative values for contact conductance

Table 10-2: Contact Conductances

Chapter 10 | 415

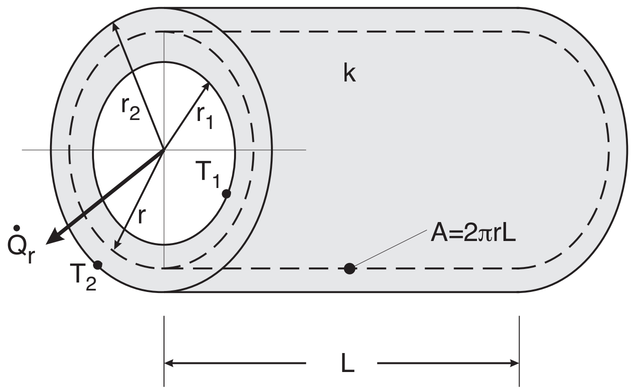
TABLE 10–2

Thermal contact conductance of some metal surfaces in air (from various sources)

Material	Surface condition	Roughness, μm	Temperature, °C	Pressure, MPa	h_c,* W/m² · °C
Identical Metal Pairs					
416 Stainless steel	Ground	2.54	90–200	0.17–2.5	3800
304 Stainless steel	Ground	1.14	20	4–7	1900
Aluminum	Ground	2.54	150	1.2–2.5	11,400
Copper	Ground	1.27	20	1.2–20	143,000
Copper	Milled	3.81	20	1–5	55,500
Copper (vacuum)	Milled	0.25	30	0.17–7	11,400
Dissimilar Metal Pairs					
Stainless steel–				10	2900
Aluminum		20–30	20	20	3600
Stainless steel–				10	16,400
Aluminum		1.0–2.0	20	20	20,800
Steel Ct-30–				10	50,000
Aluminum	Ground	1.4–2.0	20	15–35	59,000
Steel Ct-30–				10	4800
Aluminum	Milled	4.5–7.2	20	30	8300
				5	42,000
Aluminum-Copper	Ground	1.17–1.4	20	15	56,000
				10	12,000
Aluminum-Copper	Milled	4.4–4.5	20	20–35	22,000

*Divide the given values by 5.678 to convert to Btu/h · ft² · °F.

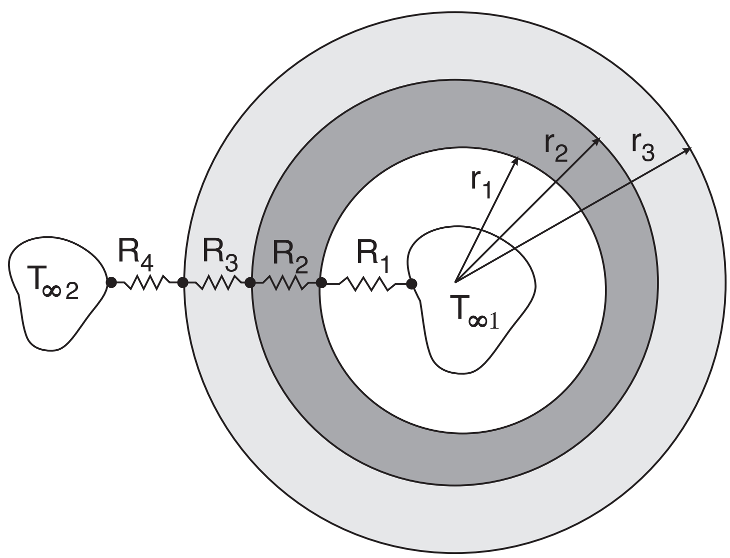
Cylindrical Systems



Steady, 1D heat flow from T_1 to T_2 in a cylindrical system occurs in a radial direction where the lines of constant temperature (isotherms) are concentric circles, as shown by the dotted line and $T = T(r)$.
Performing a 1^{st} law energy balance on a *control mass* from the annular ring of the cylindrical cylinder gives:

$$\dot{Q}_r = \frac{T_1 - T_2}{\left(\dfrac{\ln(r_2/r_1)}{2\pi k \mathcal{L}}\right)} \qquad \text{where} \quad R = \left(\frac{\ln(r_2/r_1)}{2\pi k \mathcal{L}}\right)$$

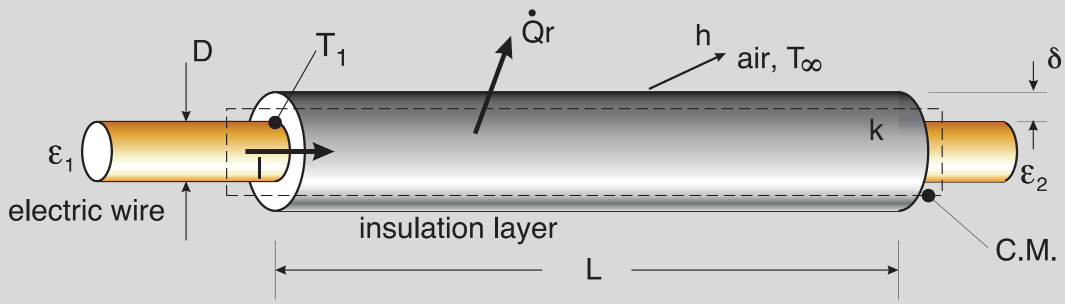
Composite Cylinders



Then the total resistance can be written as

$$\begin{aligned} R_{total} &= R_1 + R_2 + R_3 + R_4 \\ &= \frac{1}{h_1 A_1} + \frac{\ln(r_2/r_1)}{2\pi k_2 \mathcal{L}} + \frac{\ln(r_3/r_2)}{2\pi k_3 \mathcal{L}} + \frac{1}{h_4 A_4} \end{aligned}$$

Example 5-1: Determine the temperature (T_1) of an electric wire surrounded by a layer of plastic insulation with a thermal conductivity if $0.15\ W/mK$ when the thickness of the insulation is a) $2\ mm$ and b) $4\ mm$, subject to the following conditions:



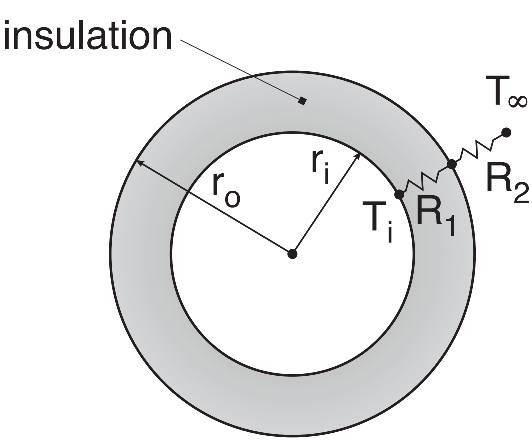
Given:

$$I = 10\ A$$
$$\Delta\epsilon = \epsilon_1 - \epsilon_2 = 8\ V$$
$$D = 3\ mm$$
$$\mathcal{L} = 5\ m$$
$$k = 0.15\ W/mK$$
$$T_\infty = 30\ ^\circ C$$
$$h = 12\ W/m^2 \cdot K$$

Find:

$$T_1 = ???$$
when:
$$\delta = 2\ mm$$
$$\delta = 4\ mm$$

Critical Radius of Insulation



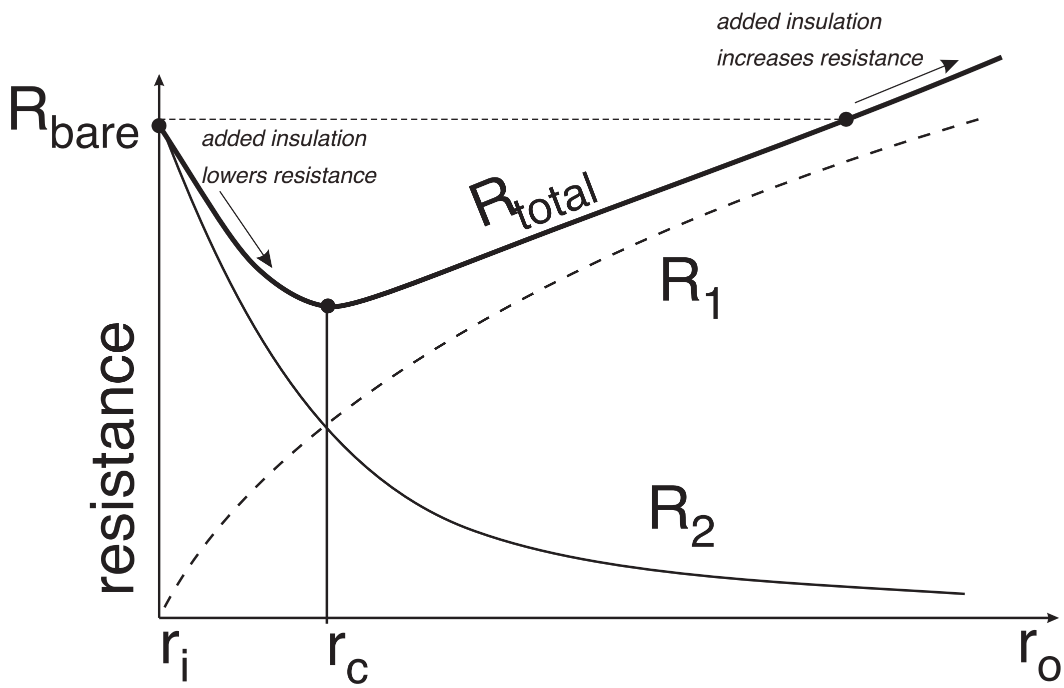
Consider a steady, 1-D problem where an insulation cladding is added to the outside of a tube with constant surface temperature T_i. What happens to the heat transfer as insulation is added, i.e. we increase the thickness of the insulation?

The resistor network can be written as a series combination of the resistance of the insulation, R_1 and the convective resistance, R_2

$$R_{total} = R_1 + R_2 = \frac{\ln(r_o/r_i)}{2\pi k\mathcal{L}} + \frac{1}{h2\pi r_o\mathcal{L}}$$

Could there be a situation in which adding insulation <u>*increases*</u> *the overall heat transfer?*

$$\frac{dR_{total}}{dr_o} = \frac{1}{2\pi k r_o \mathcal{L}} - \frac{1}{h 2\pi r_o^2 \mathcal{L}} = 0 \qquad \Rightarrow \qquad r_{cr,cyl} = \frac{k}{h} \quad [m]$$



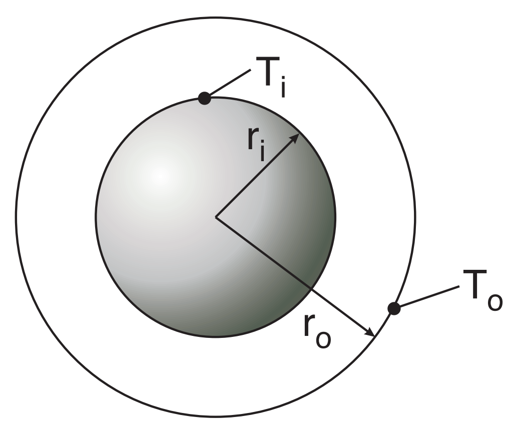
There is always a value of $r_{cr,cal}$, but there is a minimum in heat transfer only if $r_{cr,cal} > r_i$

Spherical Systems

For steady, 1D heat flow in spherical geometries we can write the heat transfer in the radial direction as

$$\dot{Q} = \frac{4\pi k r_i r_o}{(r_0 - r_i)} (T_i - T_o) = \frac{(T_i - T_o)}{R}$$

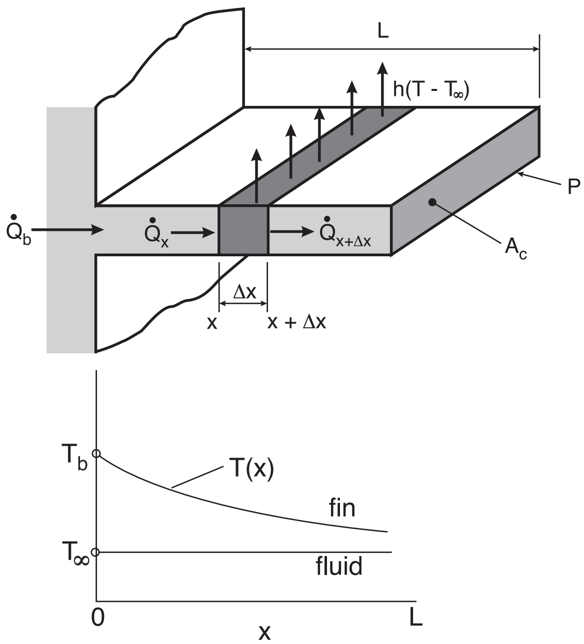
where: $R = \frac{r_o - r_i}{4\pi k r_i r_o}$



The critical radius of insulation for a spherical shell is given as

$$r_{cr,sphere} = \frac{2k}{h} \quad [m]$$

Heat Transfer from Finned Surfaces



We can establish a $\mathbf{1^{st}}$ law balance over the thin slice of the fin between x and $x + \Delta x$ such that

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} - \underbrace{P\Delta x}_{A_{surface}} h(T - T_\infty) = 0$$

From Fourier's law we know

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} = kA_c \frac{d^2T}{dx^2}\Delta x$$

Therefore the conduction equation for a fin with constant cross section is

$$\underbrace{kA_c \frac{d^2T}{\partial x^2}}_{\textit{longitudinal conduction}} - \underbrace{hP(T - T_\infty)}_{\textit{lateral convection}} = 0$$

Let the temperature difference between the fin and the surroundings (temperature excess) be $\theta = T(x) - T_\infty$ which allows the 1-D fin equation to be written as

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad \text{where} \qquad m = \left(\frac{hP}{kA_c}\right)^{1/2}$$

The solution to the differential equation for θ is

$$\theta(x) = C_1 \sinh(mx) + C_2 \cosh(mx) \qquad [\equiv \theta(x) = C_1 e^{mx} + C_2 e^{-mx}]$$

Potential boundary conditions include:

Base: $\rightarrow$ $@x = 0$ $\theta = \theta_b$
Tip: $\rightarrow$ $@x = L$ $\theta = \theta_L$ [T-specified tip]

$\theta = \left.\frac{d\theta}{dx}\right|_{x=L} = 0$ [adiabatic (insulated) tip]

$\theta \rightarrow 0$ [infinitely long fin]

Substituting the boundary conditions to find the constants of integration, the temperature distribution and fin heat transfer rate can be determined as follows:

Case 1: Prescribed temperature ($\theta_{@\ x+L} = \theta_L$)

$$\frac{\theta(x)}{\theta_b} = \frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$$

$$\dot{Q}_b = M \frac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$$

Case 2: Adiabatic tip $\left(\left. \frac{d\theta}{dx} \right|_{x=L} = 0 \right)$

$$\frac{\theta(x)}{\theta_b} = \frac{\cosh m(L-x)}{\cosh mL} \qquad \dot{Q}_b = M \tanh mL$$

Case 3: Infinitely long fin ($\theta \to 0$)

$$\frac{\theta(x)}{\theta_b} = e^{-mx} \qquad \dot{Q}_b = M$$

where

$$m = \sqrt{hP/(kA_c)}$$

$$M = \sqrt{hPkA_c}\ \theta_b$$

$$\theta_b = T_b - T_\infty$$

Fin Efficiency and Effectiveness

The dimensionless parameter that compares the actual heat transfer from the fin to the ideal heat transfer from the fin is the *fin efficiency*

$$\eta = \frac{\text{actual heat transfer rate}}{\text{maximum heat transfer rate when the entire fin is at } T_b} = \frac{\dot{Q}_b}{hPL\theta_b}$$

If the fin has a constant cross section then

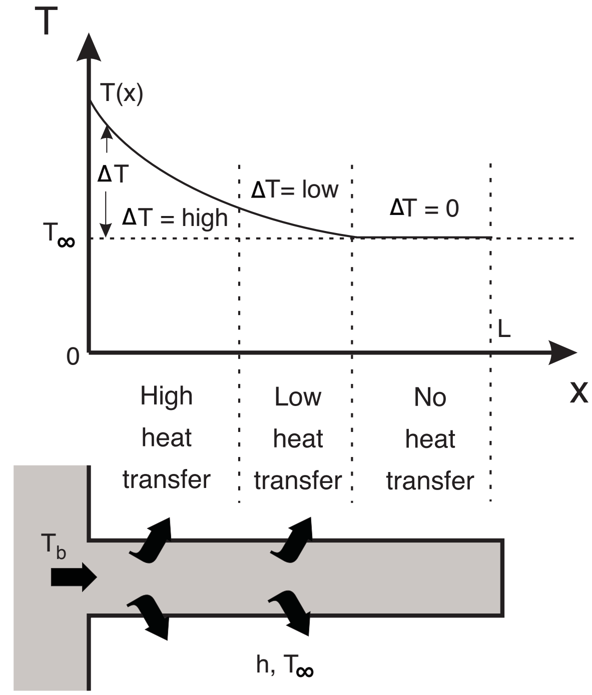
$$\eta = \frac{\tanh(mL)}{mL}$$

An alternative figure of merit is the *fin effectiveness* given as

$$\epsilon_{fin} = \frac{\text{total fin heat transfer}}{\text{the heat transfer that would have occurred through the base area in the absence of the fin}} = \frac{\dot{Q}_b}{hA_c\theta_b}$$

How to Determine the Appropriate Fin Length

- theoretically an infinitely long fin will dissipate the most heat
- but practically, an extra long fin is inefficient given the exponential temperature decay over the length of the fin
- so what is a realistic fin length in order to optimize performance and cost



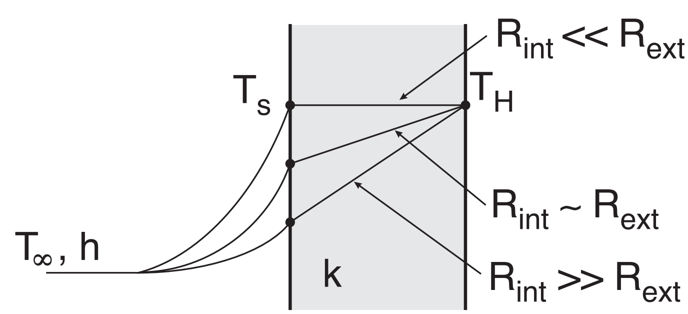
If we determine the ratio of heat flow for a fin with an insulated tip (Case 2) versus an infinitely long fin (Case 3) we can assess the relative performance of a conventional fin

$$\frac{\dot{Q}_{Case\ 2}}{\dot{Q}_{Case\ 3}} = \frac{M \tanh mL}{M} = \tanh mL$$

Transient Heat Conduction

Performing a 1^{st} law energy balance on a plane wall gives

$$\begin{aligned} \dot{Q}_{cond} &= \frac{T_H - T_s}{L/(k \cdot A)} \\ &= \dot{Q}_{conv} = \frac{T_s - T_\infty}{1/(h \cdot A)} \end{aligned}$$



where the Biot number can be obtained as follows:

$$\frac{T_H - T_s}{T_s - T_\infty} = \frac{L/(k \cdot A)}{1/(h \cdot A)} = \frac{\text{internal resistance to H.T.}}{\text{external resistance to H.T.}} = \frac{hL}{k} = Bi$$

$R_{int} << R_{ext}$**:** the Biot number is small and we can conclude

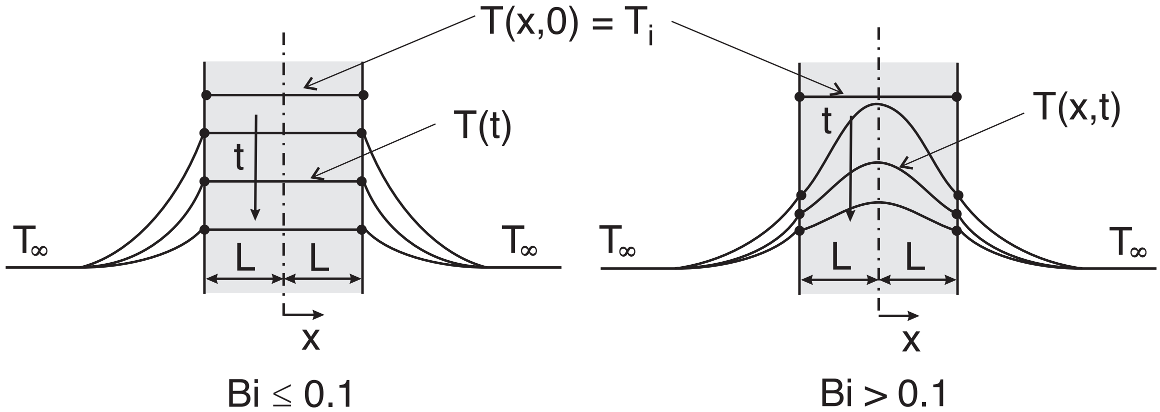
$$T_H - T_s << T_s - T_\infty \qquad \text{and in the limit} \quad T_H \approx T_s$$

$R_{ext} << R_{int}$**:** the Biot number is large and we can conclude

$$T_s - T_\infty << T_H - T_s \qquad \text{and in the limit} \quad T_s \approx T_\infty$$

Transient Conduction Analysis

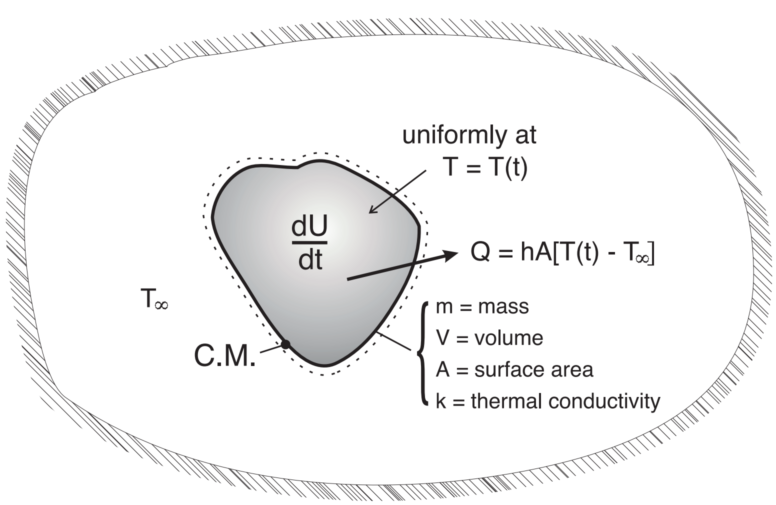
- if the internal temperature of a body remains relatively constant with respect to time
  - can be treated as a lumped system analysis
  - heat transfer is a function of time only, $T = T(t)$



$Bi \leq 0.1$: temperature profile is not a function of position
temperature profile only changes with respect to time $\rightarrow T = T(t)$
use lumped system analysis

$Bi > 0.1$: temperature profile changes with respect to time and position $\rightarrow T = T(x, t)$
use approximate analytical or graphical solutions (Heisler charts)

Lumped System Analysis



At $t > 0$, $T = T(x, y, z, t)$, however, when $Bi \leq 0.1$ then we can assume $T \approx T(t)$.

Performing a 1^{st} law energy balance on the control volume shown below

$$\frac{dE_{C.M.}}{dt} = \dot{E}_{in} - \dot{E}_{out} + \cancel{\dot{E}_g}^{0}$$

If we assume PE and KE to be negligible then

$$\frac{dU}{dt} = -\dot{Q} \qquad \Leftarrow \frac{dU}{dt} < 0 \text{ implies } U \text{ is decreasing}$$

For an incompressible substance specific heat is constant and we can write

$$\underbrace{mC}_{\equiv C_{th}} \frac{dT}{dt} = -\underbrace{Ah}_{1/R_{th}}(T - T_\infty)$$

where C_{th} = lumped capacitance

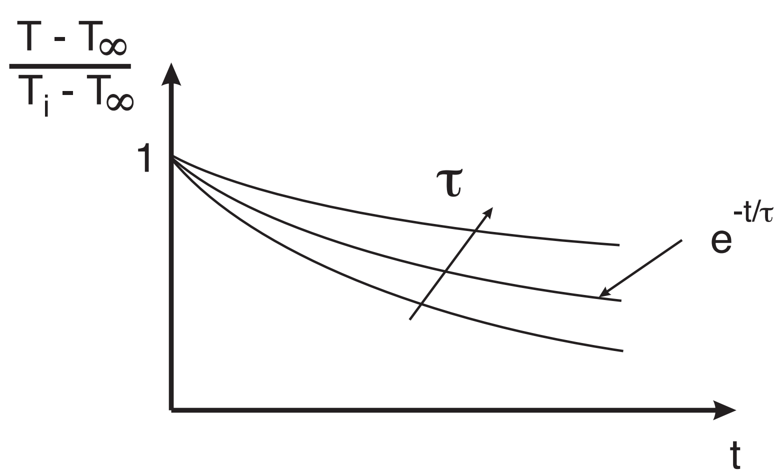
$$C_{th}\frac{dT}{dt} = -\frac{1}{R_{th}}(T - T_\infty)$$

We can integrate and apply the initial condition, $T = T_i$ @$t = 0$ to obtain

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-t/(R_{th}\cdot C_{th})} = e^{-t/\tau} = e^{-bt}$$

where

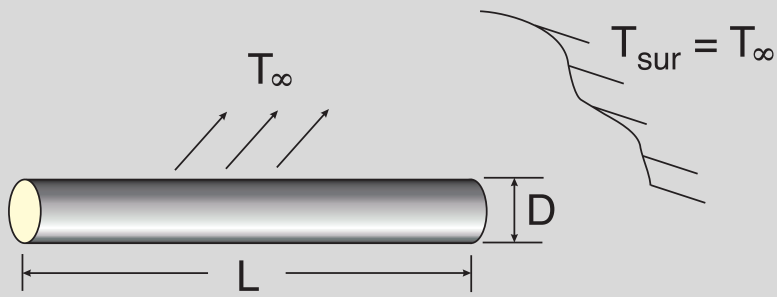
$$\begin{aligned} \frac{1}{b} &= \tau \\ &= R_{th} \cdot C_{th} \\ &= \text{thermal time constant} \\ &= \frac{mC}{Ah} = \frac{\rho V C}{Ah} \end{aligned}$$



The total heat transferred over the time period $0 \rightarrow t^*$ is

$$Q_{total} = mC(T_i - T_\infty)[1 - e^{-t^*/\tau}]$$

Example 5-2: Determine the time it takes a fuse to melt if a current of $3\ A$ suddenly flows through the fuse subject to the following conditions:



Given:

$$D = 0.1\ mm \qquad T_{melt} = 900\ ^\circ C \qquad k = 20\ W/mK$$

$$L = 10\ mm \qquad T_\infty = 30\ ^\circ C \qquad \alpha = 5 \times 10^{-5}\ m^2/s \equiv k/\rho C_p$$

Assume:

- constant resistance $\mathcal{R} = 0.2\ ohms$
- the overall heat transfer coefficient is $h = h_{conv} + h_{rad} = 10\ W/m^2K$
- neglect any conduction losses to the fuse support

Approximate Analytical and Graphical Solutions (Heisler Charts)

If $Bi > 0.1$

- need to solve the partial differential equation for temperature
- leads to an infinite series solution $\Rightarrow$ difficult to obtain a solution
  (see pp. 481 - 483 for exact solution by separation of variables)

We must find a solution to the PDE

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \Rightarrow \qquad \frac{T(x,t) - T_\infty}{T_i - T_\infty} = \sum_{n=1,3,5...}^{\infty} A_n e^{\left(-\frac{\lambda_n}{L}\right)^2 \alpha t} \cos\left(\frac{\lambda_n x}{L}\right)$$

By using dimensionless groups, we can reduce the temperature dependence to 3 dimensionless parameters

Dimensionless Group	Formulation
temperature	$\theta(x,t) = \dfrac{T(x,t) - T_\infty}{T_i - T_\infty}$
position	$X = x/L$
heat transfer	$Bi = hL/k$ Biot number
time	$Fo = \alpha t/L^2$ Fourier number

note: Cengel uses τ instead of Fo.

Now we can write

$$\theta(x,t) = f(X, Bi, Fo)$$

The characteristic length for the Biot number is

slab	$\mathcal{L} = L$
cylinder	$\mathcal{L} = r_o$
sphere	$\mathcal{L} = r_o$

contrast this versus the characteristic length for the lumped system analysis.

With this, two approaches are possible

1. use the first term of the infinite series solution. This method is only valid for $Fo > 0.2$
2. use the Heisler charts for each geometry as shown in Figs. 11-15, 11-16 and 11-17

First term solution: $Fo > 0.2 \rightarrow$ error about 2% max.

Plane Wall:
$$\theta_{wall}(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \cos(\lambda_1 x/L)$$

Cylinder:
$$\theta_{cyl}(r,t) = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \mathrm{J}_0(\lambda_1 r/r_o)$$

Sphere:
$$\theta_{sph}(r,t) = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}$$

λ_1, A_1 can be determined from Table 11-2 based on the calculated value of the Biot number (will likely require some interpolation). The Bessel function, J_0 can be calculated using Table 11-3.

Using Heisler Charts

- find T_0 at the center for a given time (Table 11-15 a, Table 11-16 a or Table 11-17 a)
- find T at other locations at the same time (Table 11-15 b, Table 11-16 b or Table 11-17 b)
- find Q_{tot} up to time t (Table 11-15 c, Table 11-16 c or Table 11-17 c)

Example 5-3: An aluminum plate made of Al 2024-T6 with a thickness of $0.15\ m$ is initially at a temperature of $300\ K$. It is then placed in a furnace at $800\ K$ with a convection coefficient of $500\ W/m^2K$.

Find:
i) the time (s) for the plate midplane to reach $700\ K$
ii) the surface temperature at this condition. Use both the Heisler charts and the approximate analytical, first term solution.