Chapter 10: Steady Heat Conduction

In thermodynamics, we considered the amount of heat transfer as a system undergoes a process from one equilibrium state to another. Thermodynamics gives no indication of how long the process takes. In heat transfer, we are more concerned about the rate of heat transfer.

The basic requirement for heat transfer is the presence of a temperature difference. The temperature difference is the driving force for heat transfer, just as voltage difference for electrical current. The total amount of heat transfer Q during a time interval can be determined from:

$$Q = \int_0^{\Delta t} Q^\bullet dt \qquad (kJ)$$

The rate of heat transfer per unit area is called heat flux, and the average heat flux on a surface is expressed as

$$q^\bullet = \frac{Q^\bullet}{A} \quad (W/m^2)$$

Steady Heat Conduction in Plane Walls

Conduction is the transfer of energy from the more energetic particles of a substance to the adjacent less energetic ones as result of interactions between the particles.

Consider steady conduction through a large plane wall of thickness $\Delta x = L$ and surface area A. The temperature difference across the wall is $\Delta T = T_2 - T_1$.

Note that heat transfer is the only energy interaction; the energy balance for the wall can be expressed:

$$Q^\bullet_{in} - Q^\bullet_{out} = \frac{dE_{wall}}{dt}$$

For steady-state operation,

$$Q^\bullet_{in} = Q^\bullet_{out} = const.$$

It has been *experimentally* observed that the rate of heat conduction through a layer is proportional to the temperature difference across the layer and the heat transfer area, but it is inversely proportional to the thickness of the layer.

$$\text{rate of heat transfer} \propto \frac{\text{(surface area)(temperature difference)}}{\text{thickness}}$$

$$Q^\bullet_{Cond} = kA\frac{\Delta T}{\Delta x} \qquad (W)$$

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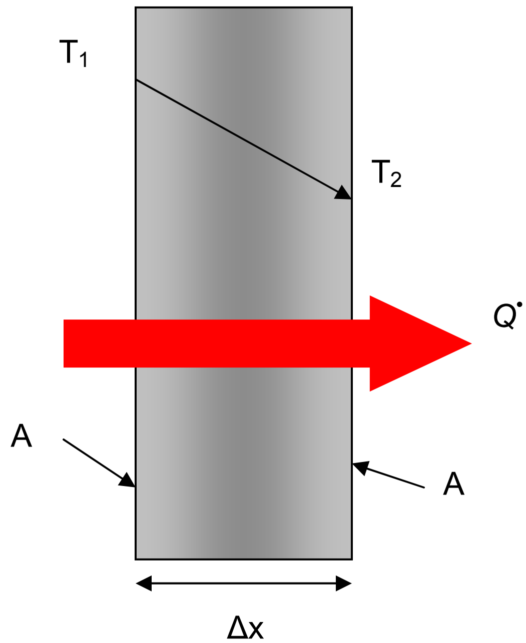

Fig. 10-1: Heat conduction through a large plane wall.

The constant proportionality *k* is the *thermal conductivity* of the material. In the limiting case where Δx→0, the equation above reduces to the differential form:

$$Q^{\bullet}_{Cond} = -kA\frac{dT}{dx} \qquad (W)$$

which is called Fourier's law of heat conduction. The term dT/dx is called the temperature gradient, which is the slope of the temperature curve (the rate of change of temperature T with length x).

Thermal Conductivity

Thermal conductivity k [W/mK] is a measure of a material's ability to conduct heat. The thermal conductivity is defined as the rate of heat transfer through a unit thickness of material per unit area per unit temperature difference.

Thermal conductivity changes with temperature and determined through experiments.

The thermal conductivity of certain materials show a dramatic change at temperatures near absolute zero, when these solids become *superconductors*.

An *isotropic* material is a material that has uniform properties in all directions.

Insulators are materials used primarily to provide resistance to heat flow. They have low thermal conductivity.

The Thermal Resistance Concept

The Fourier equation, for steady conduction through a constant area plane wall, can be written:

$$Q^{\bullet}_{Cond} = -kA\frac{dT}{dx} = kA\frac{T_1 - T_2}{L}$$

This can be re-arranged as:

$$Q^{\bullet}_{Cond} = \frac{T_2 - T_1}{R_{wall}} \qquad (W)$$

$$R_{wall} = \frac{L}{kA} \qquad ({}^{\circ}C/W)$$

R_{wall} is the *thermal resistance* of the wall against heat conduction or simply the *conduction resistance* of the wall.

The heat transfer across the fluid/solid interface is based on *Newton's law of cooling:*

$$Q^{\bullet} = hA(T_s - T_{\infty}) \qquad (W)$$

$$R_{Conv} = \frac{1}{hA} \qquad ({}^{\circ}C/W)$$

R_{conv} is the thermal resistance of the surface against heat convection or simply the *convection resistance* of the surface.

Thermal radiation between a surface of area *A* at T_s and the surroundings at T_{∞} can be expressed as:

$$Q^{\bullet}_{rad} = \varepsilon\sigma A\left(T_s^4 - T_{\infty}^4\right) = h_{rad} A(T_s - T_{\infty}) = \frac{T_s - T_{\infty}}{R_{rad}} \qquad (W)$$

$$R_{rad} = \frac{1}{h_{rad} A}$$

$$h_{rad} = \varepsilon\sigma\left(T_s^2 + T_{\infty}^2\right)(T_s + T_{\infty}) \qquad \left(\frac{W}{m^2 K}\right)$$

where σ = 5.67×10^{-8} [W/m^2K^4] is the Stefan-Boltzman constant. Also $0 < \varepsilon < 1$ is the emissivity of the surface. Note that both the temperatures must be in Kelvin.

Thermal Resistance Network

Consider steady, one-dimensional heat flow through two plane walls in series which are exposed to convection on both sides, see Fig. 10-2. Under steady state condition:

rate of heat convection into the wall = rate of heat conduction through wall 1 = rate of heat conduction through wall 2 = rate of heat convection from the wall

$$\dot{Q} = h_1 A\left(T_{\infty,1} - T_1\right) = k_1 A \frac{T_1 - T_2}{L_1} = k_2 A \frac{T_2 - T_3}{L_2} = h_2 A\left(T_2 - T_{\infty,2}\right)$$

$$\dot{Q} = \frac{T_{\infty,1} - T_1}{1/h_1 A} = \frac{T_1 - T_2}{L/k_1 A} = \frac{T_2 - T_3}{L/k_2 A} = \frac{T_2 - T_{\infty,2}}{1/h_2 A}$$

$$\dot{Q} = \frac{T_{\infty,1} - T_1}{R_{conv,1}} = \frac{T_1 - T_2}{R_{wall,1}} = \frac{T_2 - T_3}{R_{wall,2}} = \frac{T_3 - T_{\infty,2}}{R_{conv,2}}$$

$$\dot{Q} = \frac{T_{\infty,1} - T_{\infty,2}}{R_{total}}$$

$$R_{total} = R_{conv,1} + R_{wall,1} + R_{wall,2} + R_{conv,2}$$

Note that A is constant area for a plane wall. Also note that the thermal resistances are in series and equivalent resistance is determined by simply adding thermal resistances.

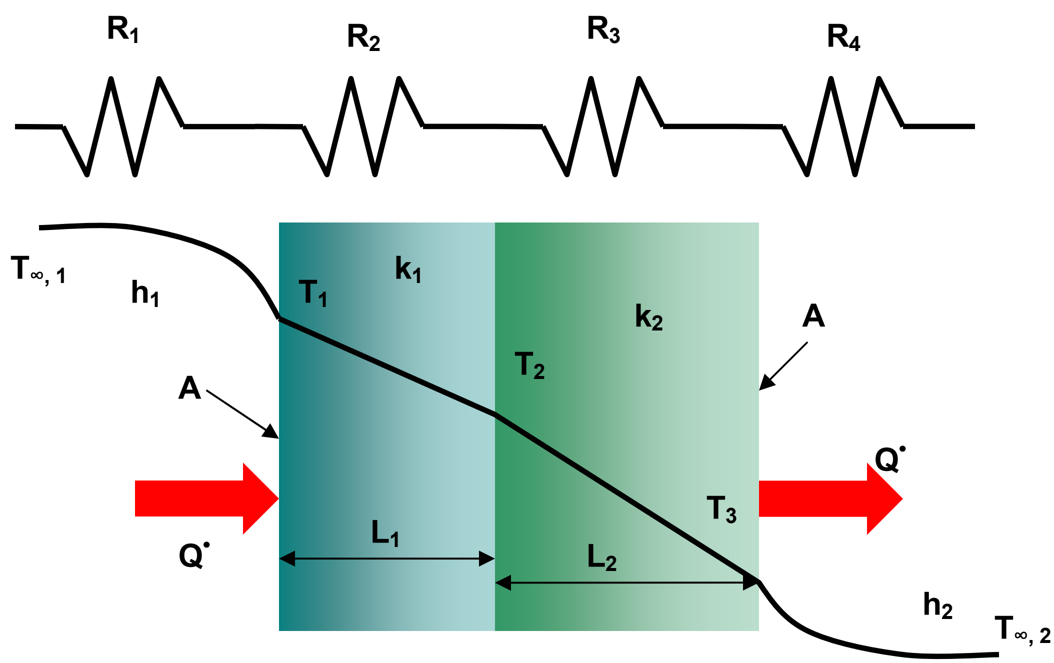

Fig. 10-2: Thermal resistance network.

The rate of heat transfer between two surfaces is equal to the temperature difference divided by the total thermal resistance between two surfaces.

It can be written:

$$\Delta T = \dot{Q} R$$

The thermal resistance concept is widely used in practice; however, its use is limited to systems through which the rate of heat transfer remains constant. It other words, to systems involving *steady* heat transfer with no *heat generation.*

Thermal Resistances in Parallel

The thermal resistance concept can be used to solve steady state heat transfer problem in parallel layers or combined series-parallel arrangements.

It should be noted that these problems are often two- or three dimensional, but approximate solutions can be obtained by assuming one dimensional heat transfer (using thermal resistance network).

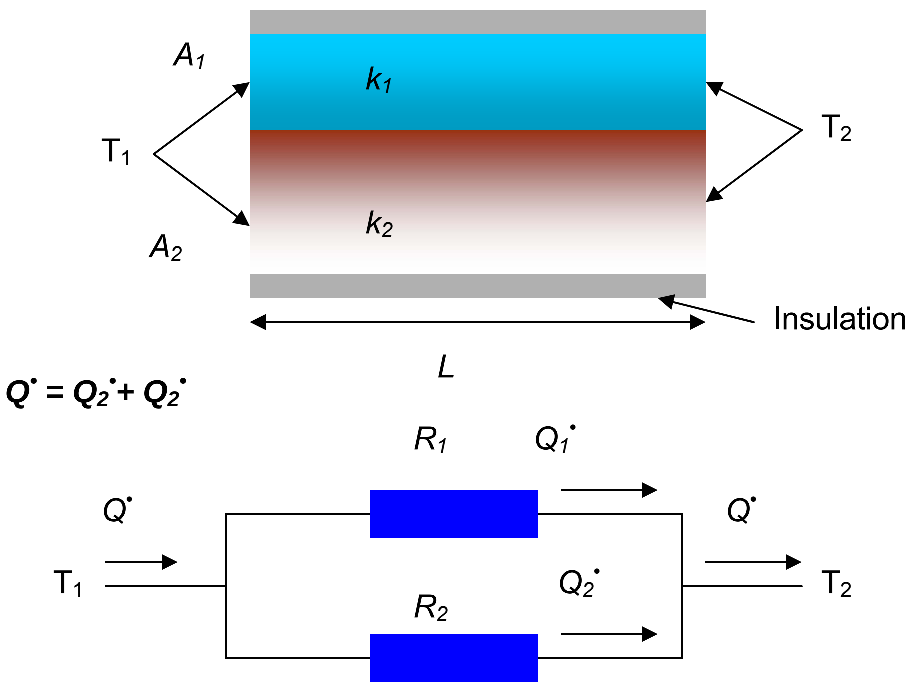

Fig. 10-3 Parallel resistances.

$$Q^{\bullet} = Q_1^{\bullet} + Q_2^{\bullet} = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2} = (T_1 - T_2)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$Q^{\bullet} = \frac{T_1 - T_2}{R_{total}}$$

$$\frac{1}{R_{total}} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \rightarrow \frac{1}{R_{total}} = \frac{R_1 R_2}{R_1 + R_2}$$

Example 10-1: Thermal Resistance Network

Consider the combined series-parallel arrangement shown in figure below. Assuming one –dimensional heat transfer, determine the rate of heat transfer.

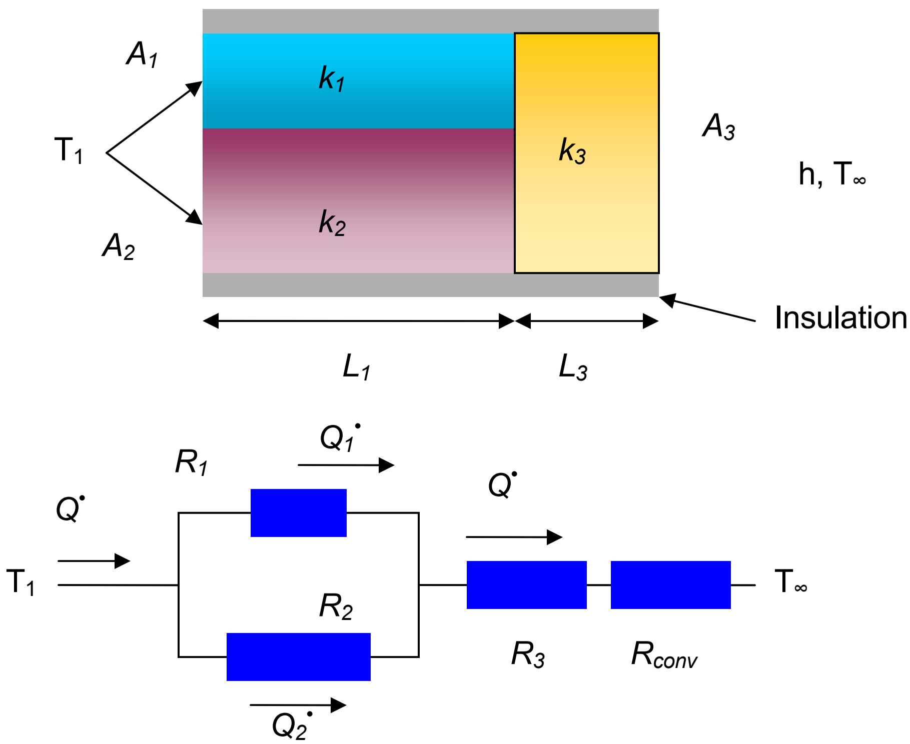

Fig. 10-4 Schematic for example 10-1.

Solution:

The rate of heat transfer through this composite system can be expressed as:

$$Q^{\bullet} = \frac{T_1 - T_{\infty}}{R_{total}}$$

$$R_{total} = R_{12} + R_3 + R_{conv} = \frac{R_1 R_2}{R_1 + R_2} + R_3 + R_{conv}$$

Two approximations commonly used in solving complex multi-dimensional heat transfer problems by transfer problems by treating them as one dimensional, using the thermal resistance network:

1- Assume any plane wall normal to the x-axis to be isothermal, i.e. temperature to vary in one direction only T = T(x)

2- Assume any plane parallel to the x-axis to be adiabatic, i.e. heat transfer occurs in the x-direction only.

These two assumptions result in different networks (different results). The actual result lies between these two results.

Heat Conduction in Cylinders and Spheres

Steady state heat transfer through pipes is in the normal direction to the wall surface (no significant heat transfer occurs in other directions). Therefore, the heat transfer can be modeled as steady-state and one-dimensional, and the temperature of the pipe will depend only on the radial direction, T = T (r).

Since, there is no heat generation in the layer and thermal conductivity is constant, the Fourier law becomes:

$$\dot{Q}_{cond,cyl} = -kA\frac{dT}{dr} \qquad (W)$$

$$A = 2\pi r L$$

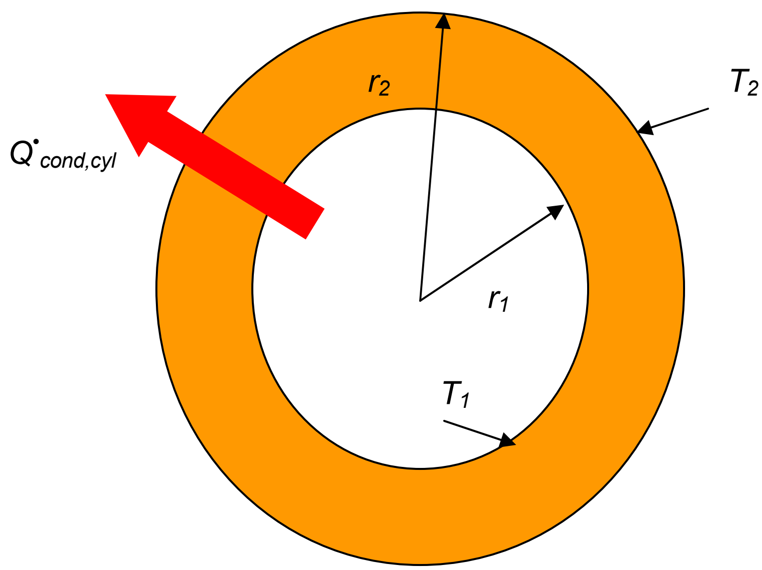

Fig. 10-5: Steady, one-dimensional heat conduction in a cylindrical layer.

After integration:

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$$\int_{r1}^{r2} \frac{\dot{Q}_{cond,cyl}}{A} dr = -\int_{T1}^{T2} k dT \qquad A = 2\pi r L$$

$$\dot{Q}_{cond,cyl} = 2\pi k L \frac{T_1 - T_2}{\ln(r_2 / r_1)}$$

$$\dot{Q}_{cond,cyl} = \frac{T_1 - T_2}{R_{cyl}}$$

$$R_{cyl} = \frac{\ln(r_2 / r_1)}{2\pi k L}$$

where R_{cyl} is the conduction resistance of the cylinder layer.

Following the analysis above, the conduction resistance for the spherical layer can be found:

$$\dot{Q}_{cond,sph} = \frac{T_1 - T_2}{R_{sph}}$$

$$R_{sph} = \frac{r_2 - r_1}{4\pi r_1 r_2 k}$$

The convection resistance remains the same in both cylindrical and spherical coordinates, $R_{conv} = 1/hA$. However, note that the surface area $A = 2\pi rL$ (cylindrical) and $A = 4\pi r^2$ (spherical) are functions of radius.

Example 10-2: Multilayer cylindrical thermal resistance network

Steam at $T_{\infty,1}$ = 320 °C flows in a cast iron pipe [k = 80 W/ m.°C] whose inner and outer diameter are D_1 = 5 cm and D_2 = 5.5 cm, respectively. The pipe is covered with a 3-cm-thick glass wool insulation [k = 0.05 W/ m.°C]. Heat is lost to the surroundings at $T_{\infty,2}$ = 5°C by natural convection and radiation, with a combined heat transfer coefficient of h_2 = 18 W/m^2. °C. Taking the heat transfer coefficient inside the pipe to be h_1 = 60 W/m^{2}K, determine the rate of heat loss from the steam per unit length of the pipe. Also determine the temperature drop across the pipe shell and the insulation.

Assumptions:

Steady-state and one-dimensional heat transfer.

Solution:

Taking L = 1 m, the areas of the surfaces exposed to convection are:

$A_1 = 2\pi r_1 L = 0.157\ m^2$

$A_2 = 2\pi r_2 L = 0.361\ m^2$

$$R_{conv,1} = \frac{1}{h_1 A_1} = \frac{1}{(60\ W/m^2.^\circ C)(0.157 m^2)} = 0.106\ ^\circ C/W$$

$$R_1 = R_{pipe} = \frac{\ln(r_2/r_1)}{2\pi k_1 L} = 0.0002^\circ C/W$$

$$R_2 = R_{insulation} = \frac{\ln(r_3/r_2)}{2\pi k_2 L} = 2.35^\circ C/W$$

$$R_{conv,2} = \frac{1}{h_2 A_2} = 0.154\ ^\circ C/W$$

$$R_{total} = R_{conv,1} + R_1 + R_2 + R_{conv,2} = 2.61^\circ C/W$$

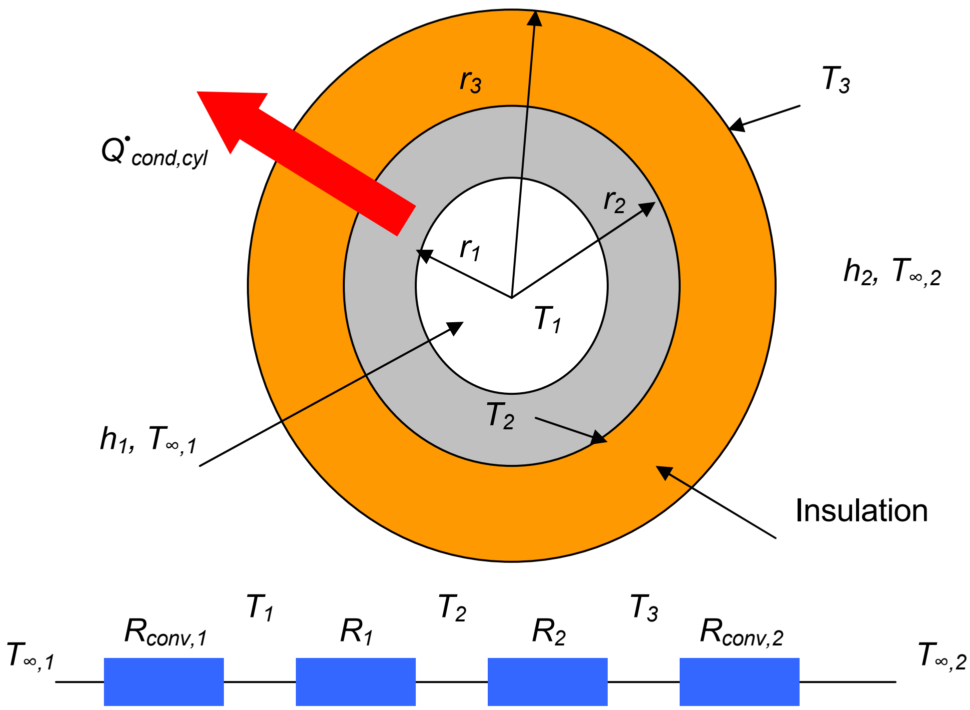

Fig. 10-6: Schematic for example 10-1.

The steady-state rate of heat loss from the steam becomes

$$Q^\bullet = \frac{T_{\infty,1} - T_{\infty,2}}{R_{total}} = 120.7\ W \qquad \text{(per m pipe length)}$$

The total heat loss for a given length can be determined by multiplying the above quantity by the pipe length.

The temperature drop across the pipe and the insulation are:

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$$\Delta T_{pipe} = Q^{\bullet} R_{pipe} = (120.7\ W)(0.0002\ ^{\circ}C/W) = 0.02^{\circ}C$$

$$\Delta T_{insulation} = Q^{\bullet} R_{insulation} = (120.7\ W)(2.35\ ^{\circ}C/W) = 284^{\circ}C$$

Note that the temperature difference (thermal resistance) across the pipe is too small relative to other resistances and can be ignored.

Critical Radius of Insulation

To insulate a plane wall, the thicker the insulator, the lower the heat transfer rate (since the area is constant). However, for cylindrical pipes or spherical shells, adding insulation results in increasing the surface area which in turns results in increasing the convection heat transfer. As a result of these two competing trends the heat transfer may increase or decrease.

$$Q^{\bullet} = \frac{T_1 - T_{\infty}}{R_{ins} + R_{conv}} = \frac{T_1 - T_{\infty}}{\dfrac{\ln(r_2/r_1)}{2\pi kL} + \dfrac{1}{(2\pi r_2 L)h}}$$

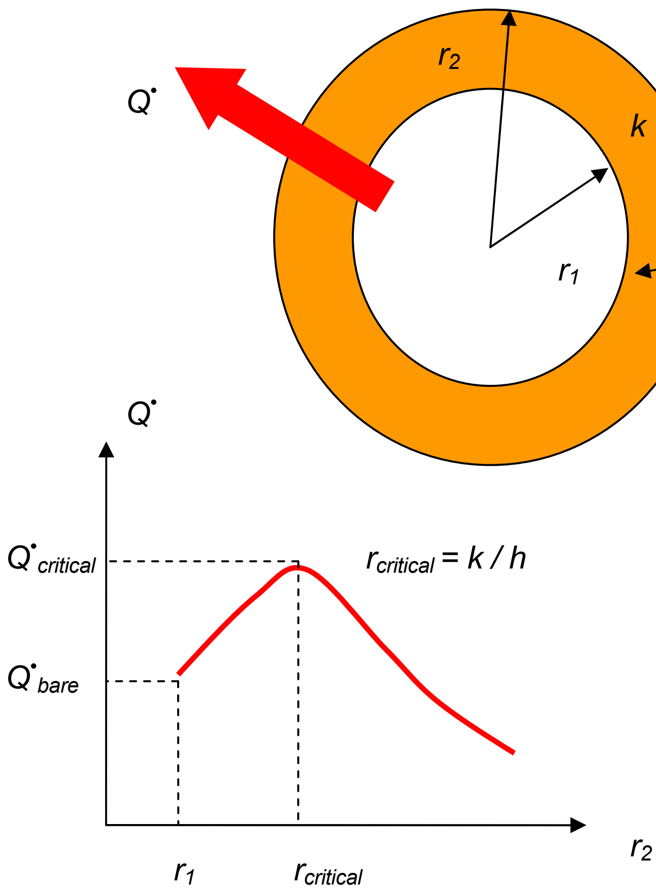

Fig. 10-7: Critical radius of insulation.

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The variation of $\dot{Q}$ with the outer radius of the insulation reaches a maximum that can be determined from $d\dot{Q} / dr_2 = 0$. The value of the critical radius for the cylindrical pipes and spherical shells are:

$$r_{cr,cylinder} = \frac{k}{h} \qquad (m)$$

$$r_{cr,spherer} = \frac{2k}{h} \qquad (m)$$

Note that for most applications, the critical radius is so small. Thus, we can insulate hot water or steam pipes without worrying about the possibility of increasing the heat transfer by insulating the pipe.

Heat Generation in Solids

Conversion of some form of energy into heat energy in a medium is called *heat generation*. Heat generation leads to a temperature rise throughout the medium.

Some examples of heat generation are resistance heating in wires, exothermic chemical reactions in solids, and nuclear reaction. Heat generation is usually expressed per unit volume (W/m^3).

In most applications, we are interested in maximum temperature T_{max} and surface temperature T_s of solids which are involved with heat generation.

The maximum temperature T_{max} in a solid that involves uniform heat generation will occur at a location furthest away from the outer surface when the outer surface is maintained at a constant temperature, T_s.

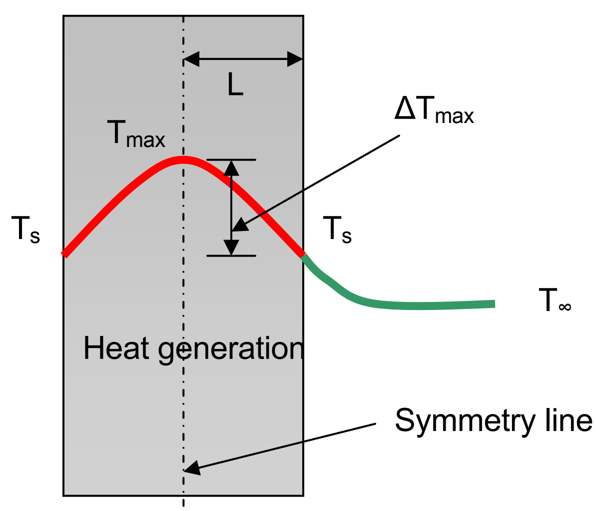


Fig. 10-8: Maximum temperature with heat generation.

Consider a solid medium of surface area *A*, volume *V*, and *constant thermal conductivity k*, where heat is *generated at a constant rate of* $\dot{g}$ per unit volume.

Heat is transferred from the solid to the surroundings medium at T_∞. Under steady conditions, the energy balance for the solid can be expressed as:

rate of heat transfer from the solid	=	rate of energy generation within the solid

$$\dot{Q} = \dot{g}\, V \quad (W)$$

From the Newton's law of cooling, $\dot{Q} = hA\,(T_s - T_\infty)$. Combining these equations, a relationship for the surface temperature can be found:

$$T_s = T_\infty + \frac{\dot{g} V}{hA}$$

Using the above relationship, the surface temperature can be calculated for a plane wall of thickness *2L*, a long cylinder of radius r_0, and a sphere of radius r_0, as follows:

$$T_{\text{s,plane wall}} = T_\infty + \frac{\dot{g} L}{h}$$

$$T_{\text{s,cylinder}} = T_\infty + \frac{\dot{g} r_0}{2h}$$

$$T_{\text{s,sphere}} = T_\infty + \frac{\dot{g} r_0}{3h}$$

Note that the rise in temperature is due to heat generation.

Using the Fourier's law, we can derive a relationship for the center (maximum) temperature of long cylinder of radius r_0.

$$-kA_r \frac{dT}{dr} = \dot{g} V_r \qquad A_r = 2\pi r L \qquad V_r = \pi r^2 L$$

After integrating,

$$\Delta \mathrm{T}_{\max} = T_0 - T_s = \frac{\dot{g} r_0^2}{4k}$$

where T0 is the centerline temperature of the cylinder (T_{max}). Using the approach, the maximum temperature can be found for plane walls and spheres.

$$\Delta \mathrm{T}_{\text{max,cylinder}} = \frac{\dot{g} r_0^2}{4k}$$

$$\Delta \mathrm{T}_{\text{max,plane wall}} = \frac{\dot{g} L^2}{2k}$$

$$\Delta \mathrm{T}_{\text{max,sphere}} = \frac{\dot{g} r_0^2}{6k}$$

Heat Transfer from Finned Surfaces

From the Newton's law of cooling, $\dot{Q}_{conv} = h\,A\,(T_s - T_\infty)$, the rate of convective heat transfer from a surface at a temperature T_s can be increased by two methods:

1) Increasing the convective heat transfer coefficient, h

2) Increasing the surface area A.

Increasing the convective heat transfer coefficient may not be practical and/or adequate. An increase in surface area by attaching *extended surfaces* called *fins* to the surface is more convenient.

Finned surfaces are commonly used in practice to enhance heat transfer. In the analysis of the fins, we consider *steady* operation with *no heat generation* in the fin. We also assume that the convection heat transfer coefficient h to be constant and uniform over the entire surface of the fin.

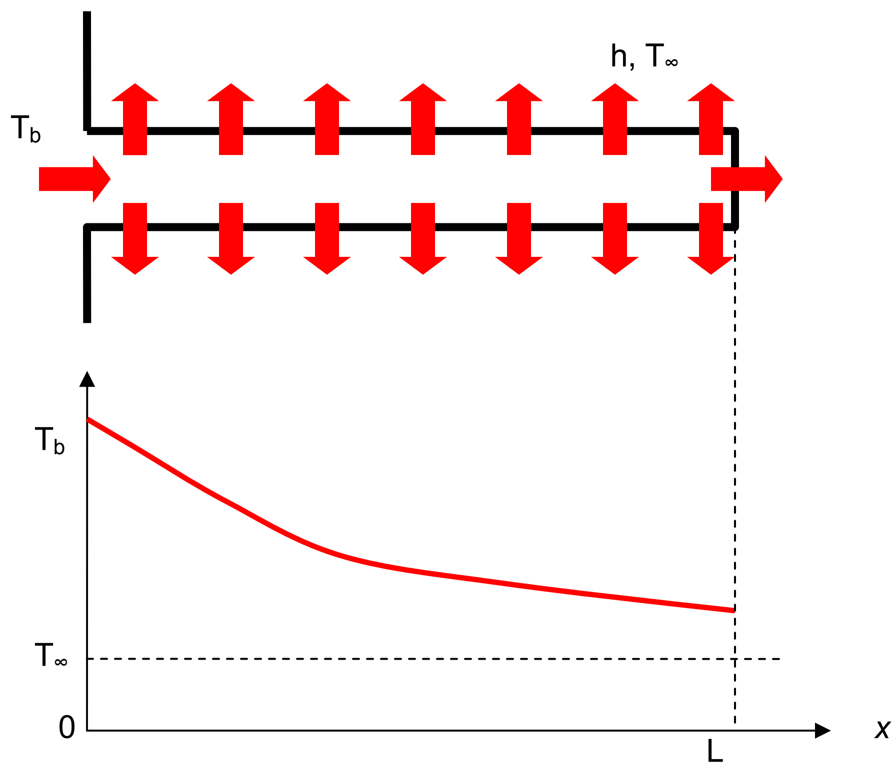


Fig. 10-9: Temperature of a fin drops gradually along the fin.

In the limiting case of zero thermal resistance ($k\rightarrow\infty$), the temperature of the fin will be uniform at the base value of T_b. The heat transfer from the fin will be maximized in this case:

$$\dot{Q}_{fin,\max} = hA_{fin}\left(T_b - T_\infty\right)$$

.

Fin efficiency can be defined as:

$$\eta_{fin} = \frac{\overset{\bullet}{Q}_{fin}}{\overset{\bullet}{Q}_{fin,\max}} = \frac{\text{actual heat transfer rate from the fin}}{\text{ideal heat transfer rate from the fin (if the entire fin were at base temperature)}}$$

where A_{fin} is total surface area of the fin. This enables us to determine the heat transfer from a fin when its efficiency is known:

$$\overset{\bullet}{Q}_{fin} = \eta_{fin} \overset{\bullet}{Q}_{fin,\max} = \eta_{fin} h A_{fin} \left(T_b - T_\infty\right)$$

Fin efficiency for various profiles can be read from Fig. 10-59, 10-60 in Cengel's book.

The following must be noted for a proper fin selection:

- the longer the fin, the larger the heat transfer area and thus the higher the rate of heat transfer from the fin
- the larger the fin, the bigger the mass, the higher the price, and larger the fluid friction
- also, the fin efficiency decreases with increasing fin length because of the decrease in fin temperature with length.

Fin Effectiveness

The performance of fins is judged on the basis of the enhancement in heat transfer relative to the no-fin case, and expressed in terms of the fin effectiveness:

$$\varepsilon_{fin} = \frac{\overset{\bullet}{Q}_{fin}}{\overset{\bullet}{Q}_{no\ fin}} = \frac{\overset{\bullet}{Q}_{fin}}{hA_b\left(T_b - T_\infty\right)} = \frac{\text{heat transfer rate from the fin}}{\text{heat transfer rate from the surface area of } \mathrm{A_b}}$$

$$\varepsilon_{fin} = \begin{cases} <1 & \text{fin acts as insulation} \\ =1 & \text{fin does not affect heat transfer} \\ >1 & \text{fin enhances heat transfer} \end{cases}$$

For a sufficiently long fin of uniform cross-section A_c, the temperature at the tip of the fin will approach the environment temperature, T_∞. By writing energy balance and solving the differential equation, one finds:

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \exp\left(-x\sqrt{\frac{hp}{kA_c}}\right)$$

$$\overset{\bullet}{Q}_{long\ fin} = \sqrt{hpkA_c}\left(T_b - T_\infty\right)$$

where A_c is the cross-sectional area, x is the distance from the base, and p is perimeter. The effectiveness becomes:

$$\varepsilon_{long\ fin} = \sqrt{\frac{kp}{hA_c}}$$

To increase fin effectiveness, one can conclude:

- the thermal conductivity of the fin material must be as high as possible
- the ratio of perimeter to the cross-sectional area p/A_c should be as high as possible
- the use of fin is most effective in applications that involve low convection heat transfer coefficient, i.e. natural convection.

Chapter 11: Transient Heat Conduction

In general, temperature of a body varies with time as well as position.

Lumped System Analysis

Interior temperatures of some bodies remain essentially uniform at all times during a heat transfer process. The temperature of such bodies are only a function of time, T = T(t). The heat transfer analysis based on this idealization is called *lumped system analysis*.

Consider a body of arbitrary shape of mass m, volume V, surface area A, density ρ and specific heat C_p initially at a uniform temperature T_i.

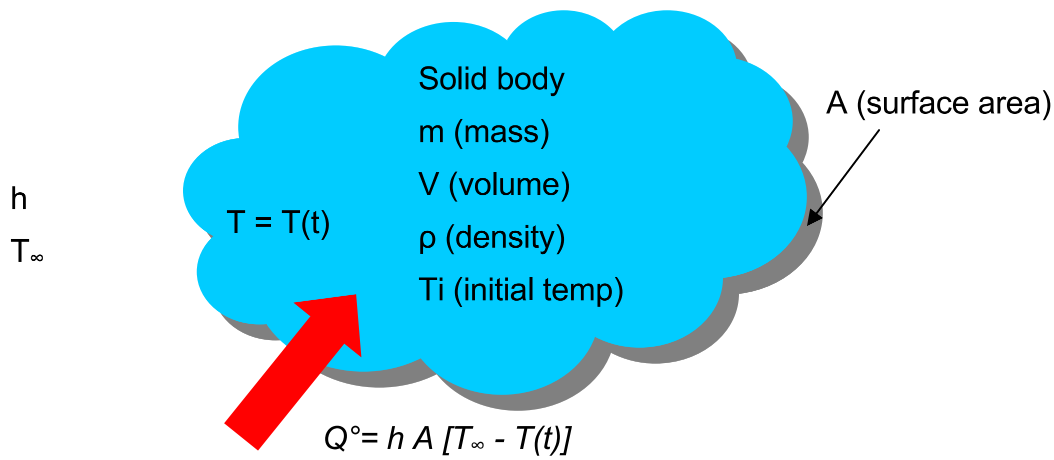


Fig. 11-1: Lumped system analysis.

At time t = 0, the body is placed into a medium at temperature T_∞ ($T_\infty > T_i$) with a heat transfer coefficient h. An energy balance of the solid for a time interval *dt* can be expressed as:

heat transfer into the body during *dt*	=	the increase in the energy of the body during *dt*

$$h\,A\,(T_\infty - T)\,dt = m\,C_p\,dT$$

With m = ρV and change of variable $dT = d(T - T_\infty)$, we find:

$$\frac{d(T - T_\infty)}{T - T_\infty} = -\frac{hA}{\rho V C_p} dt$$

Integrating from t = 0 to $T = T_i$

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt}$$

$$b = \frac{hA}{\rho V C_p} \quad (1/s)$$

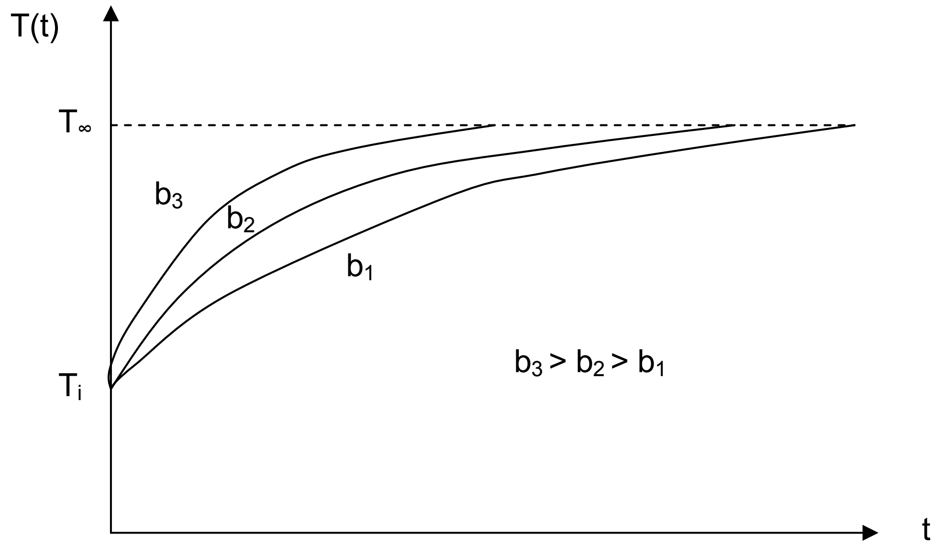


Fig. 11-2: Temperature of a lump system.

Using above equation, we can determine the temperature T(t) of a body at time t, or alternatively, the time t required for the temperature to reach a specified value T(t).

Note that the temperature of a body approaches the ambient temperature T_∞ exponentially.

A large value of b indicates that the body will approach the environment temperature in a short time.

b is proportional to the surface area, but inversely proportional to the mass and the specific heat of the body.

The total amount of heat transfer between a body and its surroundings over a time interval is:

$$Q = m\, C_p\, [T(t) - T_i]$$

Electrical Analogy

The behavior of lumped systems, shown in Fig. 9-2 can be interpreted as a thermal time constant

$$\tau_t = \left(\frac{1}{hA}\right)\rho V C_p = R_t C_t$$

$$\tau_t = \frac{1}{b}$$

where R_t is the resistance to convection heat transfer and C_t is the lumped thermal capacitance of the solid. Any increase in R_t or C_t will cause a solid to respond more slowly to changes in its thermal environment and will increase the time respond required to reach thermal equilibrium.

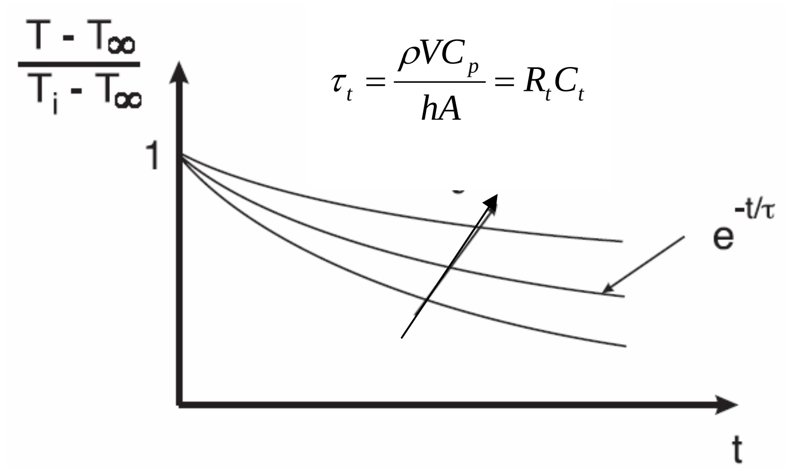


Fig. 11-3: Thermal time constant.

Criterion for Lumped System Analysis

Lumped system approximation provides a great convenience in heat transfer analysis. We want to establish a criterion for the applicability of the lumped system analysis.

A characteristic length scale is defined as:

$$L_c = \frac{V}{A}$$

A Biot number is defined:

$$Bi = \frac{hL_c}{k}$$

$$Bi = \frac{h\Delta T}{\frac{k}{L_c}\Delta T} = \frac{\text{convection at the surface of the body}}{\text{conduction within the body}}$$

$$Bi = \frac{L_c / k}{1/h} = \frac{\text{conduction resistance within the body}}{\text{convection resistance at the surface of the body}}$$

The Biot number is the ratio of the internal resistance (conduction) to the external resistance to heat convection.

Lumped system analysis assumes a uniform temperature distribution throughout the body, which implies that the conduction heat resistance is zero. Thus, the lumped system analysis is exact when Bi = 0.

It is generally accepted that the lumped system analysis is applicable if

$$Bi \leq 0.1$$

Therefore, small bodies with high thermal conductivity are good candidates for lumped system analysis.

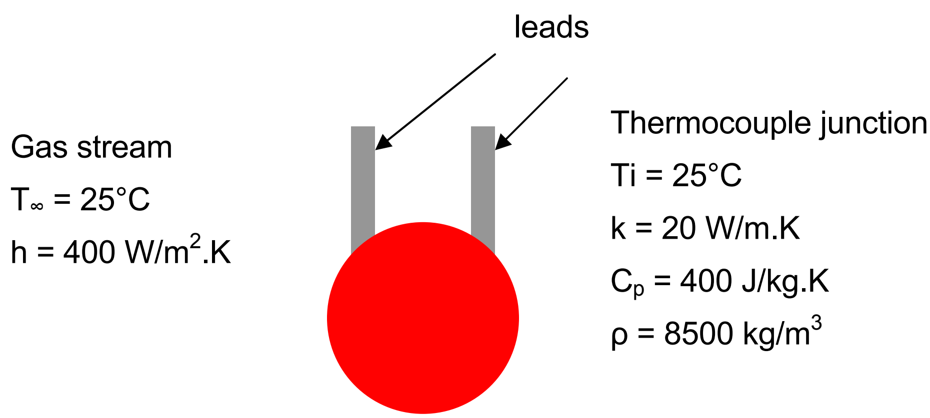
Note that assuming h to be constant and uniform is an approximation.

Example 11-1

A thermocouple junction, which may be approximated by a sphere, is to be used for temperature measurement in a gas stream. The convection heat transfer coefficient between the junction surface and the gas is known to be h = 400 W/m2.K, and the junction thermophysical properties are k = 20 W/m.K, C_p = 400 J/kg.K, and ρ = 8500 kg/m^3. Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?

Assumptions:

1. Temperature of the junction is uniform at any instant.
2. Radiation is negligible.
3. Losses through the leads, by conduction, are negligible.
4. Constant properties.



Solution:

To find the diameter of the junction, we can use the time constant:

$$\tau_t = \frac{1}{hA}\rho V C_p = \frac{1}{h\pi D^2} \times \frac{\rho \pi D^3}{6} C_p$$

Rearranging and substituting numerical values, one finds, D = 0.706 mm.

Now, we can check the validity of the lumped system analysis. With $L_c = r_0 / 3$

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$$Bi = \frac{hL_c}{k} = 2.35 \times 10^{-4} \leq 0.1 \rightarrow \text{Lumped analysis is OK.}$$

Bi << 0.1, therefore, the lumped approximation is an excellent approximation.

The time required for the junction to reach T = 199°C is

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt}$$

$$b = \frac{hA}{\rho V C_p}$$

$$t = \frac{1}{b} \ln \frac{T_i - T_\infty}{T(t) - T_\infty}$$

$$t = 5.2\, s$$

Transient Conduction in Large Plane Walls, Long Cylinders, and Spheres

The lumped system approximation can be used for small bodies of highly conductive materials. But, in general, temperature is a function of position as well as time.

Consider a plane wall of thickness 2L, a long cylinder of radius r_0, and a sphere of radius r_0 initially at a uniform temperature T_i.

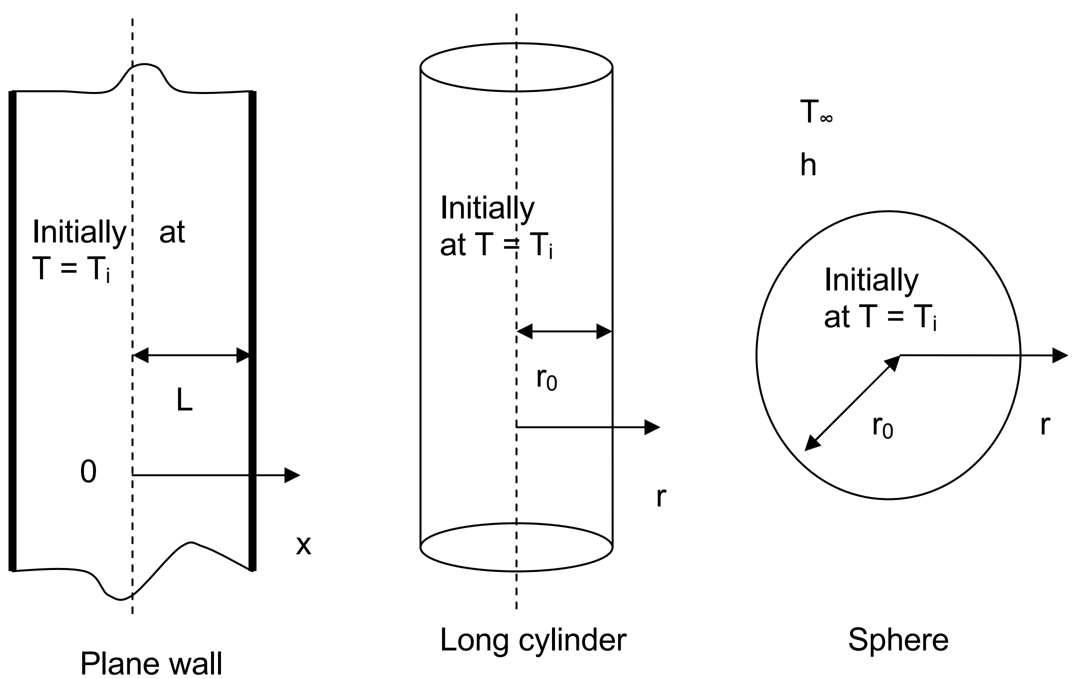

Fig. 11-4: Schematic for simple geometries in which heat transfer is one-dimensional.

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We also assume a constant heat transfer coefficient h and neglect radiation. The formulation of the one-dimensional transient temperature distribution T(x,t) results in a partial differential equation, which can be solved using advanced mathematical methods. For plane wall, the solution involves several parameters:

$$T = T\,(x, L, k, \alpha, h, T_i, T_\infty)$$

By using dimensional groups, we can reduce the number of parameters. So, one can write:

$$\theta = \theta(x, Bi, \tau)$$

where,

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} \qquad \text{dimensionless temperature}$$

$$x = \frac{x}{L} \qquad \text{dimensionless distance}$$

$$Bi = \frac{hL}{k} \qquad \text{Biot number}$$

$$\tau = \frac{\alpha t}{L^2} \qquad \text{Fourier number}$$

There are two approaches:

1. Use the first term of the infinite series solution. This method is only valid for Fourier number > 0.2

2. Use the *Heisler charts* for each geometry as shown in Figs. 11-13, 11-14 and 11-15.

Using the First Term Solution

The maximum error associated with method is less than 2%. For different geometries we have:

$$\theta(x,t)_{wall} = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = A_1 \exp\left(-\lambda_1^2 \tau\right)\cos(\lambda_1 x / L)$$

$$\theta(x,t)_{cylinder} = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 \exp\left(-\lambda_1^2 \tau\right) J_0(\lambda_1 r / r_0)$$

$$\theta(x,t)_{sphere} = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 \exp\left(-\lambda_1^2 \tau\right) \frac{\sin(\lambda_1 r / r_0)}{(\lambda_1 r / r_0)}$$

$$\text{where} \qquad \tau > 0.2$$

where A_1 and λ_1 can be found from Table 11-1 Cengel book.

Using Heisler Charts

There are three charts associated with each geometry:

1. The first chart is to determine the temperature at the center T_0 at a given time.
2. The second chart is to determine the temperature at other locations at the same time in terms of T_0.
3. The third chart is to determine the total amount of heat transfer up to the time t.