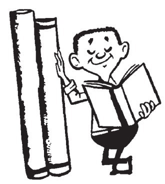
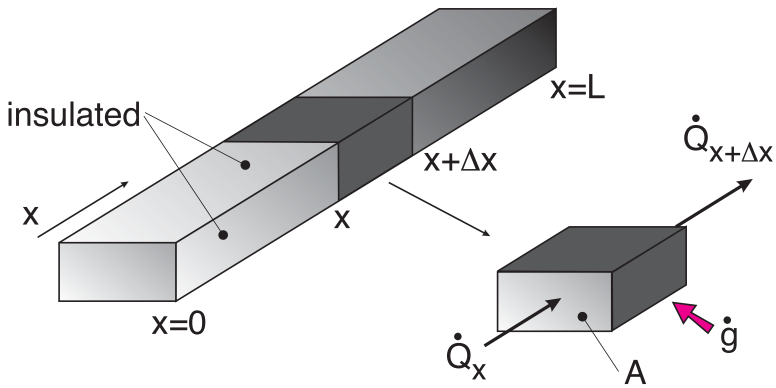
Conduction Heat Transfer



Reading	Problems
17-1 → 17-6	17-35, 17-57, 17-68, 17-81, 17-88, 17-110
18-1 → 18-2	18-14, 18-20, 18-34, 18-52, 18-80, 18-104

Fourier Law of Heat Conduction



The general 1-D conduction equation is given as

$$\underbrace{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)}_{\substack{\text{longitudinal}\\ \text{conduction}}} + \underbrace{\dot{g}}_{\substack{\text{internal}\\ \text{heat}\\ \text{generation}}} = \underbrace{\rho C\frac{\partial T}{\partial t}}_{\substack{\text{thermal}\\ \text{inertia}}}$$

where the heat flow rate, $\dot{Q}_x$, in the axial direction is given by *Fourier's law of heat conduction*.

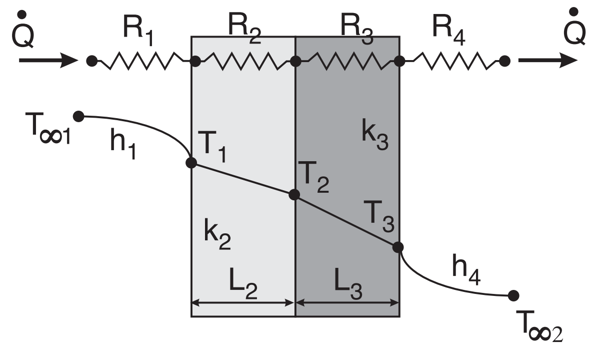
$$\dot{Q}_x = -kA\frac{\partial T}{\partial x}$$

Thermal Resistance Networks

Resistances in Series

The heat flow through a solid material of conductivity, k is

$$\dot{Q} = \frac{kA}{L}(T_{in} - T_{out}) = \frac{T_{in} - T_{out}}{R_{cond}} \qquad \text{where} \quad R_{cond} = \frac{L}{kA}$$

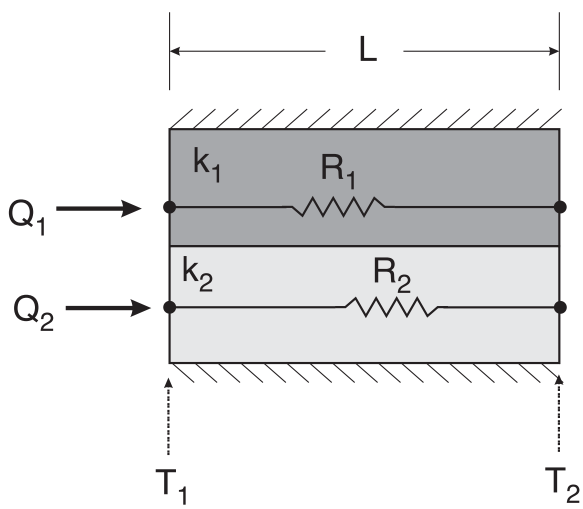


The total heat flow across the system can be written as

$$\dot{Q} = \frac{T_{\infty_1} - T_{\infty_2}}{R_{total}} \qquad \text{where} \quad R_{total} = \sum_{i=1}^{4} R_i$$

This is analogous to current flow through electrical circuits where, $I = \Delta V / R$

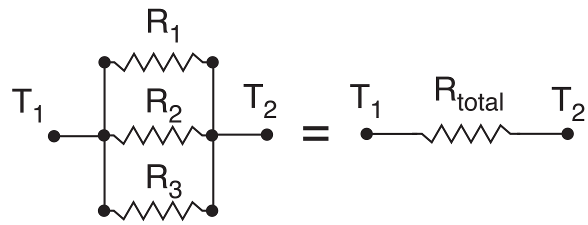
Resistances in Parallel



In general, for parallel networks we can use a parallel resistor network as follows:

where

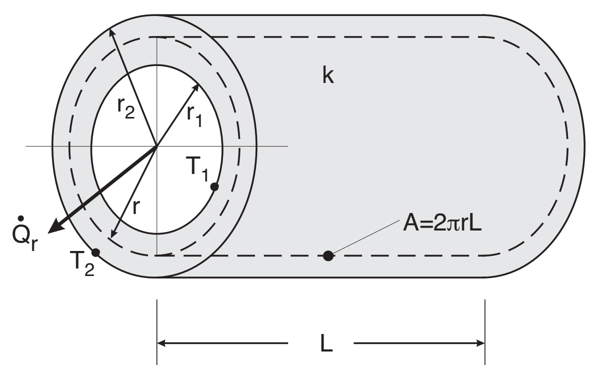
$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$



and

$$\dot{Q} = \frac{T_1 - T_2}{R_{total}}$$

Cylindrical Systems

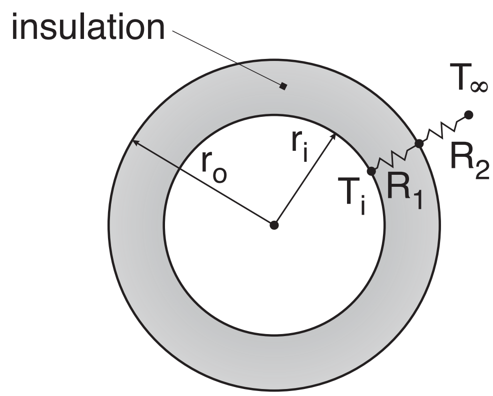


Steady, 1D heat flow from T_1 to T_2 in a cylindrical systems occurs in a radial direction where the lines of constant temperature (isotherms) are concentric circles, as shown by the dotted line in the figure above and $T = T(r)$.

$$T_2 - T_1 = -\frac{\dot{Q}_r}{2\pi k \mathcal{L}}(\ln r_2 - \ln r_1) = -\frac{\dot{Q}_r}{2\pi k \mathcal{L}} \ln \frac{r_2}{r_1}$$

Therefore we can write

$$\dot{Q}_r = \frac{T_2 - T_1}{\left(\frac{\ln(r_2/r_1)}{2\pi k \mathcal{L}}\right)} \qquad \text{where} \quad R = \left(\frac{\ln(r_2/r_1)}{2\pi k \mathcal{L}}\right)$$



Critical Thickness of Insulation

Consider a steady, 1-D problem where an insulation cladding is added to the outside of a tube with constant surface temperature T_i. What happens to the heat transfer as insulation is added, i.e. we increase the thickness of the insulation?

The resistor network can be written as a series combination of the resistance of the insulation, R_1 and the convective resistance, R_2

$$R_{total} = R_1 + R_2 = \frac{\ln(r_o/r_i)}{2\pi k\mathcal{L}} + \frac{1}{h2\pi r_o\mathcal{L}}$$

Note: as the thickness of the insulation is increased the outer radius, r_o increases.

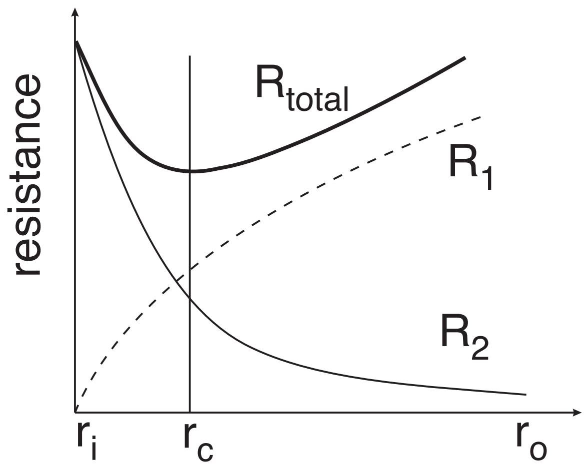
Could there be a situation in which adding insulation <u>*increases*</u> *the overall heat transfer?*

To find the critical radius, r_c, where adding more insulation begins to decrease heat transfer, set

$$\frac{dR_{total}}{dr_o} = 0$$

$$\frac{dR_{total}}{dr_o} = \frac{1}{2\pi k r_o\mathcal{L}} - \frac{1}{h2\pi r_o^2\mathcal{L}} = 0$$

$$r_c = \frac{k}{h}$$



Heat Generation in a Solid

Heat can be generated within a solid as a result of resistance heating in wires, chemical reactions, nuclear reactions, etc.

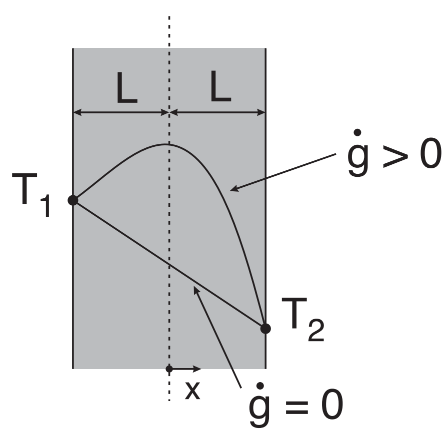
A volumetric heat generation terms will be defined as follows:

$$\dot{g} = \frac{\dot{E}_g}{V} \quad (W/m^3)$$

for heat generation in wires, we will define $\dot{g}$ as

$$\dot{g} = \frac{I^2 R_e}{\pi r_o^2 \mathcal{L}}$$

Slab System



$$T = \frac{T_1 + T_2}{2} - \left(\frac{T_1 - T_2}{2}\right)\frac{x}{L} + \frac{\dot{q}L^2}{2k}\left(1 - \left(\frac{x}{L}\right)^2\right)$$

Cylindrical System

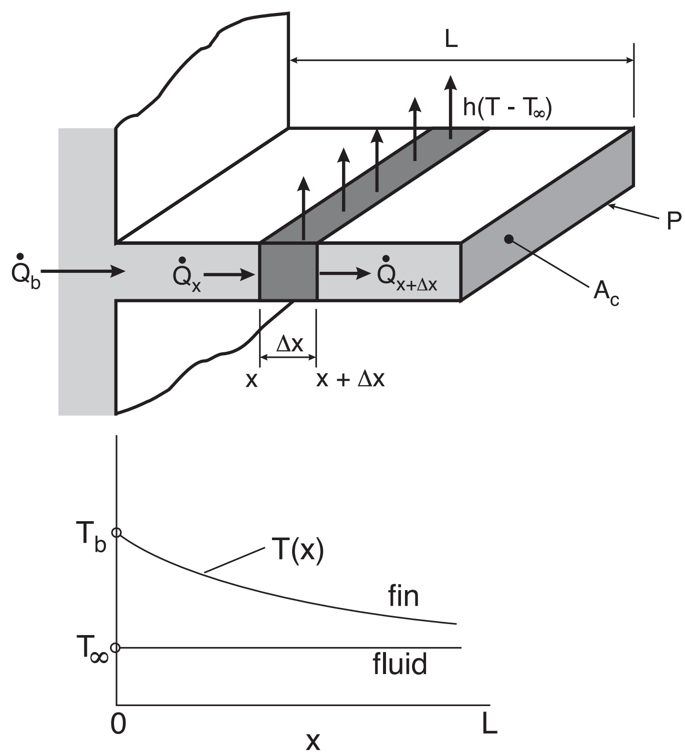
$$T = T_s + \frac{\dot{g}r_0^2}{4k}\left(1 - \left(\frac{r}{r_0}\right)^2\right)$$

where

$$BC1: \quad \frac{dT}{dr} = 0 \quad @r = 0$$

$$BC2: \quad T = T_s \quad @r = r_0$$

Heat Transfer from Finned Surfaces



The temperature difference between the fin and the surroundings (temperature excess) is usually expressed as

$$\theta = T(x) - T_\infty$$

which allows the 1-D fin equation to be written as

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

where the fin parameter m is

$$m = \left(\frac{hP}{kA_c}\right)^{1/2}$$

and the boundary conditions are

$$\theta = \theta_b \quad @ \quad x = 0$$

$$\theta \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty$$

The solution to the differential equation for θ is

$$\theta(x) = C_1 \sinh(mx) + C_2 \cosh(mx)$$

substituting the boundary conditions to find the constants of integration

$$\theta = \theta_b \, \frac{\cosh[m(L-x)]}{\cosh(mL)}$$

The heat transfer flowing through the base of the fin can be determined as

$$\begin{aligned} \dot{Q}_b &= A_c \left(-k\frac{dT}{dx}\right)_{@x=0} \\ &= \theta_b (kA_c hP)^{1/2} \, \tanh(mL) \end{aligned}$$

Fin Efficiency and Effectiveness

The dimensionless parameter that compares the actual heat transfer from the fin to the ideal heat transfer from the fin is the *fin efficiency*

$$\eta = \frac{\text{actual heat transfer rate}}{\begin{array}{c}\text{maximum heat transfer rate when}\\ \text{the entire fin is at } T_b\end{array}} = \frac{\dot{Q}_b}{hPL\theta_b}$$

If the fin has a constant cross section then

$$\eta = \frac{\tanh(mL)}{mL}$$

An alternative figure of merit is the *fin effectiveness* given as

$$\epsilon_{fin} = \frac{\text{total fin heat transfer}}{\begin{array}{c}\text{the heat transfer that would have}\\ \text{occurred through the base area}\\ \text{in the absence of the fin}\end{array}} = \frac{\dot{Q}_b}{hA_c\theta_b}$$

Transient Heat Conduction

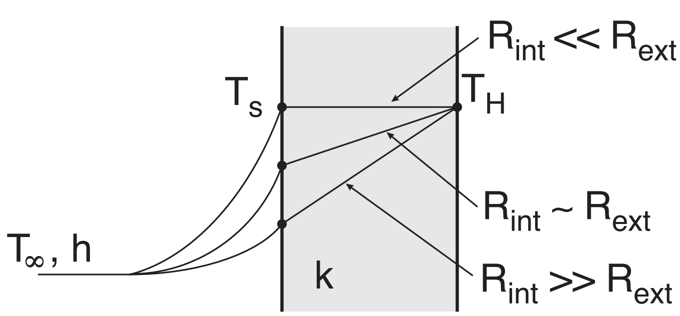
Performing a 1^{st} law energy balance on a plane wall gives

$$\dot{E}_{in} - \dot{E}_{out} \quad \Rightarrow \dot{Q}_{cond} = \frac{T_H - T_s}{L/(k \cdot A)} = \dot{Q}_{conv} = \frac{T_s - T_\infty}{1/(h \cdot A)}$$

where

$$\frac{T_H - T_s}{T_s - T_\infty} = \frac{L/(k \cdot A)}{1/(h \cdot A)} = \frac{\text{internal resistance to H.T.}}{\text{external resistance to H.T.}}$$

$$= \frac{hL}{k} = Bi \equiv \text{Biot number}$$



$R_{int} << R_{ext}$: the Biot number is small and we can conclude

$$T_H - T_s << T_s - T_\infty \qquad \text{and in the limit} \quad T_H \approx T_s$$

$R_{ext} << R_{int}$:

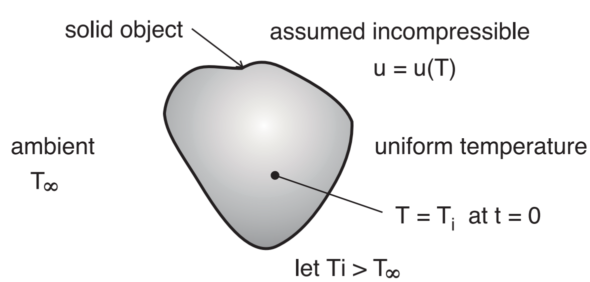
$R_{int} << R_{ext}$: the Biot number is large and we can conclude

$$T_s - T_\infty << T_H - T_s \qquad \text{and in the limit} \quad T_s \approx T_\infty$$

Lumped System Analysis

- if the internal temperature of a body remains relatively constant with respect to position
  - can be treated as a lumped system analysis
  - heat transfer is a function of time only, $T = T(t)$
- internal temperature is relatively constant at low Biot number
- typical criteria for lumped system analysis → $Bi \leq 0.1$

Transient Conduction Analysis

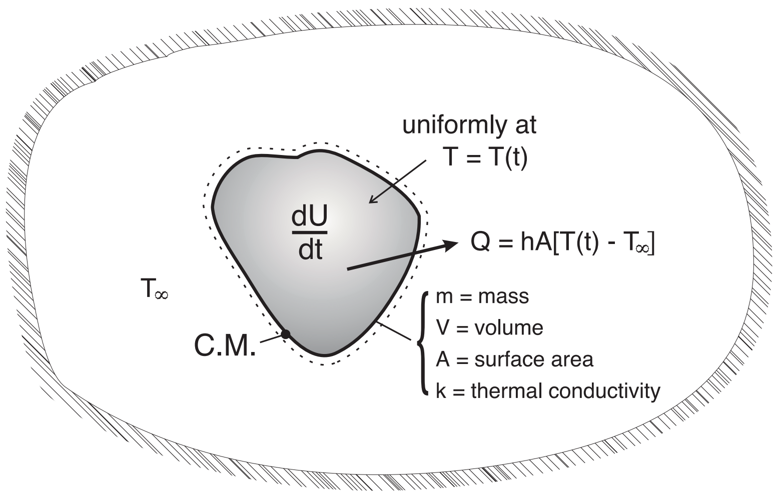


For the 3-D body of volume V and surface area A, we can use a lumped system analysis if

$$Bi = \frac{hV}{kA} < 0.1 \qquad \Leftarrow \text{results in an error of less that } 5\%$$

The characteristic length for the 3-D object is given as $\mathcal{L} = V/A$. Other characteristic lengths for conventional bodies include:

Slab	$\dfrac{V}{A_s} = \dfrac{WH2L}{2WH} = L$
Rod	$\dfrac{V}{A_s} = \dfrac{\pi r_o^2 L}{2\pi r_0 L} = \dfrac{r_0}{2}$
Sphere	$\dfrac{V}{A_s} = \dfrac{4/3\pi r_o^3}{4\pi r_0^2} = \dfrac{r_0}{3}$



For an incompressible substance we can write

$$\underbrace{mC}_{\equiv C_{th}} \frac{dT}{dt} = - \underbrace{Ah}_{1/R_{th}} (T - T_\infty)$$

where

$C_{th} =$ lumped thermal capacitance

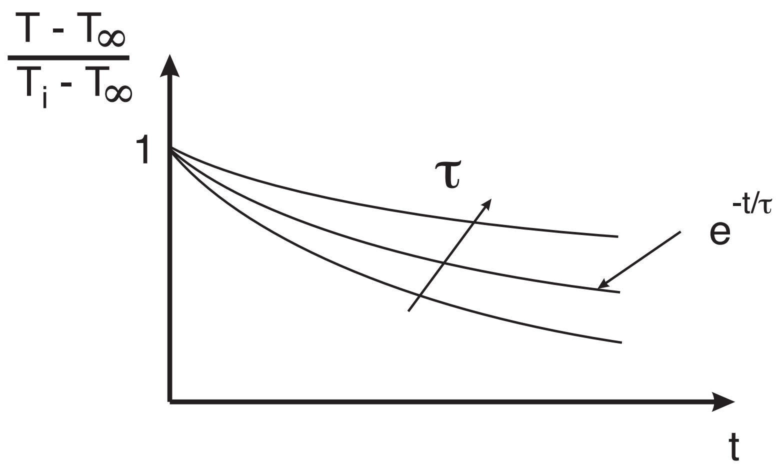
It should be clearly noted that we have neglected the spatial dependence of the temperature within the object. This type of an approach is only valid for $Bi = \dfrac{hV}{kA} < 0.1$

We can integrate and apply the initial condition, $T = T_i \;\; @t = 0$ to obtain

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-t/(R_{th} \cdot C_{th})} = e^{-t/\tau}$$

where

$$\tau = R_{th} \cdot C_{th} = \text{thermal time constant} = \frac{mC}{Ah}$$



The total heat transfer rate can be determined by integrating $\dot{Q}$ with respect to time.

$$\dot{Q}_{total} \;\; = \;\; hA(T_i - T_\infty)(\tau)[1 - e^{-t^*/\tau}]$$

Therefore

$$\dot{Q}_{total} = \dot{m}C(T_i - T_\infty)[1 - e^{-t^*/\tau}]$$

Heisler Charts

The lumped system analysis can be used if $Bi = hL/k < 0.1$ but what if $Bi > 0.1$

- need to solve the partial differential equation for temperature
- leads to an infinite series solution $\Rightarrow$ difficult to obtain a solution

The solution procedure for temperature is a function of several parameters

$$T(x,t) = f(x, L, t, k, \alpha, h, T_i, T_\infty)$$

By using dimensionless groups, we can reduce the temperature dependence to 3 dimensionless parameters

Dimensionless Group	**Formulation**
temperature	$\theta(x,t) = \dfrac{T(x,t) - T_\infty}{T_i - T_\infty}$
position	$x = x/L$
heat transfer	$Bi = hL/k$ Biot number
time	$Fo = \alpha t/L^2$ Fourier number

note: Cengel uses τ instead of Fo.

Now we can write

$$\theta(x,t) = f(x, Bi, Fo)$$

The characteristic length for the Biot number is

slab	$\mathcal{L} = L$
cylinder	$\mathcal{L} = r_o$
sphere	$\mathcal{L} = r_o$

contrast this versus the characteristic length for the lumped system analysis.

With this, two approaches are possible

1. use the first term of the infinite series solution. This method is only valid for $Fo > 0.2$
2. use the Heisler charts for each geometry as shown in Figs. 18-13, 18-14 and 18-15

First term solution: $Fo > 0.2 \rightarrow$ error about 2% max.

Plane Wall:
$$\theta_{wall}(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \cos(\lambda_1 x/L)$$

Cylinder:
$$\theta_{cyl}(r,t) = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \, J_0(\lambda_1 r/r_o)$$

Sphere:
$$\theta_{sph}(r,t) = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}$$

where λ_1, A_1 can be determined from Table 18-1 based on the calculated value of the Biot number (will likely require some interpolation).

Heisler Charts

- find T_0 at the center for a given time
- find T at other locations at the same time
- find Q_{tot} up to time t