ME201 Advanced Calculus

Project #2: Volume Integration

Assigned date: March 10, 2017 Due date: March 24, 2017

Project Requirements

Integration of complex geometric shapes, in particular the volume integration of multivariable functions, is a difficult and sometimes impossible task. Software tools such as Mathcad, MAPLE and Mathematica can be used to perform integrations either numerically or in symbolic form but that is not the purpose of this project. In the following project you will perform the preliminary setup and develop of the triple integrals necessary to calculate the volumes of two complex solid bodies.

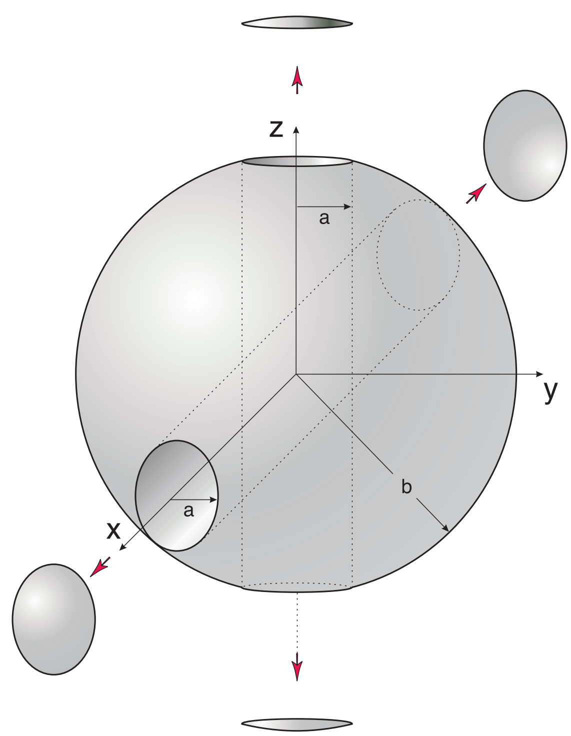
Reports will be submitted on an individual basis. **Due date: Friday, March 24.**

Present your analysis and results in a clear, concise report, typed or neatly handwritten. The report should explain the techniques used, show the problem geometry and the range of integration using graphical images, show all coordinate transformations, summarize the analyses, describe any problems encountered, and present the results.

Detailed Description

Part 1

A casting is in the form of a sphere of radius $\boldsymbol{b}$ with two cylindrical holes of radius $\boldsymbol{a}$ passing through the center of the sphere at right angles, as shown in the adjoining figure. You can assume that $\boldsymbol{a} < \boldsymbol{b}$. In this exercise you are required to calculate the total volume of metal required for the casting as a function of the sphere and hole radii.



The total volume is calculated starting with the volume of the sphere *minus* the volume of the two cylinders. These volumes can be calculated using the formulas given in the text or Schaums.

You will also need to subtract the 4 caps formed between the ends of the cylinders and the spheres and *add* the intersection of the two cylinders back into the problem (since you will double account for this volume when you include the volume of both cylinders).

Set up the volume integrals in circular cylinder coordinates. The total volume is determined by:

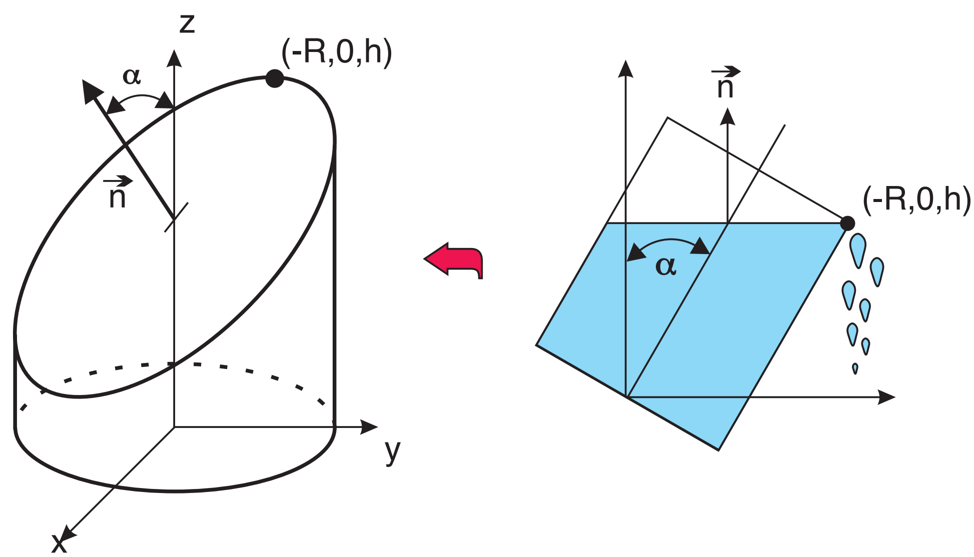
$$V_{total} = V_{sphere} - 2 \times V_{cylinder} - 4 \times V_{cap} + V_{intersection,cylinder}$$

i) Present the volume formulations for the sphere and the two cylinders. Clearly show using properly labeled graphical representations of the geometries for which these calculations apply.

i) Derive the triple integral expressions for V_{cap} and $V_{intersection,cylinder}$ in terms of a and b. Show your complete development of these equations including the transformation to cylindrical coordinates along with properly labeled graphical representations of the geometries for which these calculations apply.

Part 2

A cup in the form of a circular cylinder of radius R and height h is full of water. As the axis of the cup is tilted from vertical, water pours out over the side. In this problem, you are required to set up a triple integral expression to find the volume of water remaining in the cup as a function of the tilt angle.



Using the coordinate scheme shown in the figure, the water is "poured" out of the cup by increasing the tilt angle. The point at which the water is poured out of the cup, $(-R, 0, h)$, is always in the plane formed at the water surface.

i) Using the unit normal vector, $\hat{n}$ and the point $(-R, 0, h)$, derive an equation for the elevation of the surface of the water, z, as a function of the tilt angle, α, the cup radius, R and the cup height, h. Note: you will need to examine two cases separately:

Case 1: the early part of the pouring sequence when the upper surface of the water is only in contact with the side walls of the container

Case 2: the latter part of the pouring sequence when the upper surface of the water is in contact with the side walls and the bottom of the container.

ii) Using the equations for the surface elevation, z from part i), derive triple integral expressions for the volume of water in the cup as a function of the tilt angle, α using circular cylindrical coordinates. Again you will need to perform these calculations for the two separate cases described above.

Note: for Case 2, when the surface of the water touches the bottom of the cup, the θ boundary becomes discontinuous. You will need to be especially careful how you set up the integrals in this region.

Provide an integral analysis for all possible scenarios and thoroughly discuss how and why the integral formulations change for each scenario.