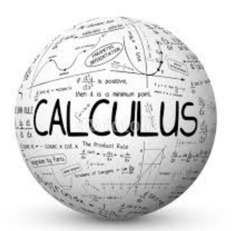
Vectors and Three Dimensional Analytic Geometry

Scalar and Vector Arithmetic



Reading

Trim 11.1 ⟶ *Rectangular Coordinates in Space*

11.4 ⟶ *Scalar and Vector Products*

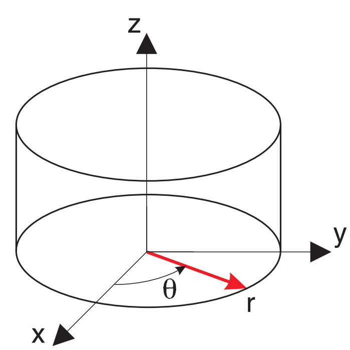
Assignment

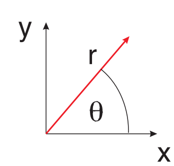
web page ⟶ *assignment #1*

Space Coordinates

1. **Cartesian Coordinates:** a system of mutually orthogonal coordinate axes in (x, y, z)
2. **Cylindrical Coordinates:**

   based on the cylindrical coordinate axes (r, θ, z). This is essentially the polar coordinates (r, θ) used instead of (x, y) coupled with the z coordinate.

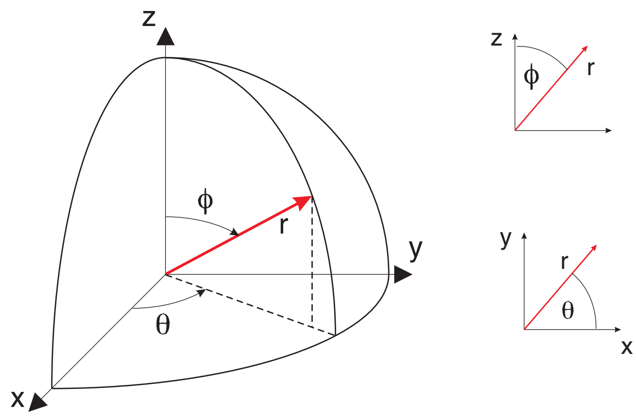




Cylindrical and Cartesian coordinates are related as follows:

Cylindrical to Cartesian	Cartesian to Cylindrical
$x = r\cos\theta$	$r = \sqrt{x^2 + y^2}$
$y = r\sin\theta$	$\theta = \tan^{-1} y/x$
$z = z$	$z = z$

3. **Spherical Coordinates:** based on the spherical coordinate system (r, θ, ϕ), where r is the distance from the origin to the surface of the sphere, ϕ is the angle from the z axis to the radial arm and θ is the angle of rotation about the z axis.

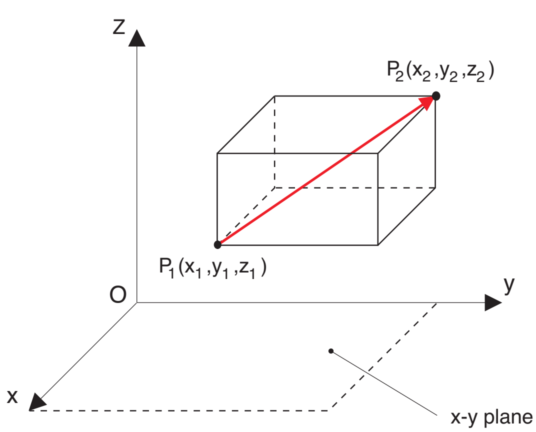


The following relations hold between spherical and Cartesian coordinates.

Spherical to Cartesian	Cartesian to Spherical
$x = r \sin\phi \cos\theta$	$r = \sqrt{x^2 + y^2 + z^2}$
$y = r \sin\phi \sin\theta$	$\theta = \tan^{-1} y/x$
$z = r \cos\phi$	$\phi = \cos^{-1} z/\sqrt{x^2 + y^2 + z^2}$

Vectors

Components of Vectors



Vector Notation

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

$$= \hat{i}(x_2 - x_1) + \hat{j}(y_2 - y_1) + \hat{k}(z_2 - z_1)$$

triple notation

$$= (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$$

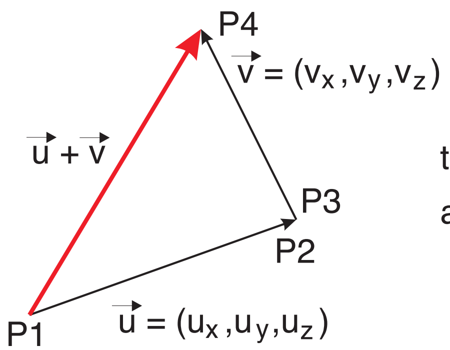
general notation

$$= (v_x,\ v_y,\ v_z)$$

Addition and Subtraction of Vectors

Two vectors $\boldsymbol{u} = (\boldsymbol{u_x}, \boldsymbol{u_y}, \boldsymbol{u_z})$ and $\boldsymbol{v} = (\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{v_z})$ can be added by:

- attaching $\boldsymbol{v} = (\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{v_z})$ to the terminal point of $\boldsymbol{u} = (\boldsymbol{u_x}, \boldsymbol{u_y}, \boldsymbol{u_z})$
- $\boldsymbol{v}$ can be arbitrarily positioned since vector location in space does not matter



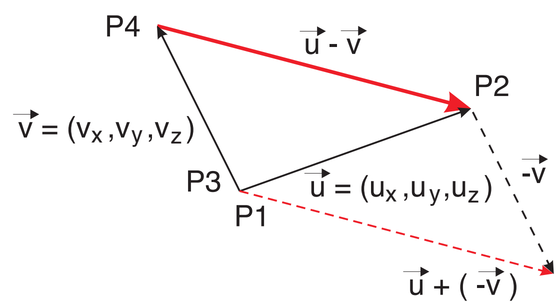
triangular addition

$$\begin{aligned} \boldsymbol{u} &= \overrightarrow{\boldsymbol{P_1P_2}} \\ \boldsymbol{v} &= \overrightarrow{\boldsymbol{P_3P_4}} \\ \boldsymbol{u}+\boldsymbol{v} &= \overrightarrow{\boldsymbol{P_1P_2}} + \overrightarrow{\boldsymbol{P_3P_4}} = \overrightarrow{\boldsymbol{P_1P_4}} \end{aligned}$$

Algebraically, vectors are added component by component, such that

$$\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{u_x} + \boldsymbol{v_x}, \boldsymbol{u_y} + \boldsymbol{v_y}, \boldsymbol{u_z} + \boldsymbol{v_z})$$

Two vectors $\boldsymbol{u} = (\boldsymbol{u_x}, \boldsymbol{u_y}, \boldsymbol{u_z})$ and $\boldsymbol{v} = (\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{v_z})$ are subtracted by attaching $\boldsymbol{u}$ and $\boldsymbol{v}$ to the same starting point. The difference between $\boldsymbol{u}$ and $\boldsymbol{v}$ is given as the vector from the tip of $\boldsymbol{v}$ to the tip of $\boldsymbol{u}$. (or by adding one vector to the negative of the second vector as shown below)



parallelogram addition (subtraction)

$$\boldsymbol{u} - \boldsymbol{v} = (\boldsymbol{u_x} - \boldsymbol{v_x}, \boldsymbol{u_y} - \boldsymbol{v_y}, \boldsymbol{u_z} - \boldsymbol{v_z})$$

Scalar Multiplication

The multiplication of a vector, $\boldsymbol{v} = (\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{v_z})$ by a positive scalar value, $\boldsymbol{\lambda}$ is obtained by multiplying each component of the vector by the scalar value, such that

$$\boldsymbol{\lambda v} = \boldsymbol{\lambda}(\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{v_z}) = (\boldsymbol{\lambda} \cdot \boldsymbol{v_x}, \boldsymbol{\lambda} \cdot \boldsymbol{v_y}, \boldsymbol{\lambda} \cdot \boldsymbol{v_z})$$

The Scalar Product

The scalar (also referred to as the dot product or the inner product) of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}|\,\cos\boldsymbol{\theta}$$

where $\boldsymbol{\theta}$ is an angle between $0 \leq \boldsymbol{\theta} \leq \pi$ defined by the vector pair when the initial points coincide.

Note: We can also find the angle between two vectors as: $\cos\boldsymbol{\theta} = \dfrac{\boldsymbol{A} \cdot \boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}$

Useful Properties

	$\boldsymbol{A}(a_x, a_y, a_z) \qquad \boldsymbol{B}(b_x, b_y, b_z)$
Scalar Product	$\boldsymbol{A} \cdot \boldsymbol{B} = a_x b_x + a_y b_y + a_z b_z$
Commutative law	$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$
Distributive law	$\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$
	$(\boldsymbol{A} + \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{C}$
	$(\boldsymbol{A} + \boldsymbol{B}) \cdot (\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{A} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{D}$
Orthogonal vectors	$\boldsymbol{A} \cdot \boldsymbol{B} = 0$ if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular (since $\cos\,90^o = 0$)
Coincident vectors	$\boldsymbol{B} = \boldsymbol{A}$, then $\boldsymbol{\theta} = 0$ and $\cos\boldsymbol{\theta} = 1 \;\rightarrow\; \boldsymbol{A} \cdot \boldsymbol{A} = |\boldsymbol{A}|^2$

Note:

$\boldsymbol{\theta} = 0$ vectors are parallel

$\boldsymbol{\theta} = \pi/2$ $(\cos\boldsymbol{\theta} = 0)$ vectors are perpendicular

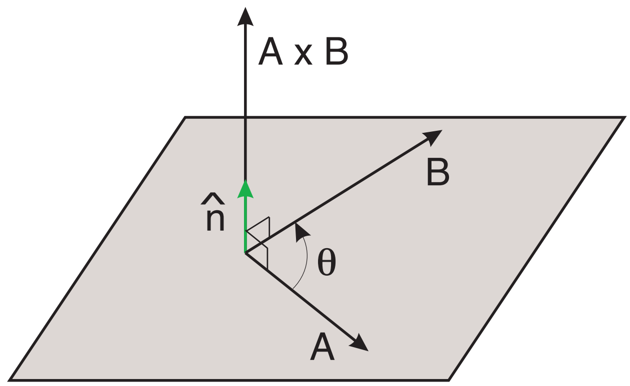
The Vector Product

Also referred to as the cross product or the outer product.

Two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ sharing the same origin and separated by a angle $\boldsymbol{\theta}$ form a plane. If we let $\vec{\boldsymbol{n}}$ be a unit vector perpendicular to this plane, pointing in a direction dictated by the right hand rule, the vector or cross product can be defined as

$$\boldsymbol{A} \times \boldsymbol{B} = \hat{n}|\boldsymbol{A}|\ |\boldsymbol{B}|\ \sin\boldsymbol{\theta}$$

↪ since it is a vector we must give it direction



Useful Properties

$$\boldsymbol{A}(a_x, a_y, a_z) \qquad \boldsymbol{B}(b_z, b_y, b_z)$$

$$\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$$

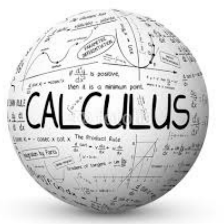
$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C}$$

$$(\boldsymbol{B} + \boldsymbol{C}) \times \boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{A} + \boldsymbol{C} \times \boldsymbol{A}$$

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\hat{i}(a_y b_z - a_z b_y) - \hat{j}(a_x b_z - a_z b_x) + \hat{k}(a_x b_y - a_y b_x)$$

Vectors and Points, Curves, Lines in 3-D



Reading

Trim 11.2 ⟶ *Curves and Surfaces*

11.3 ⟶ *Vectors*

11.5 ⟶ *Planes and Lines*

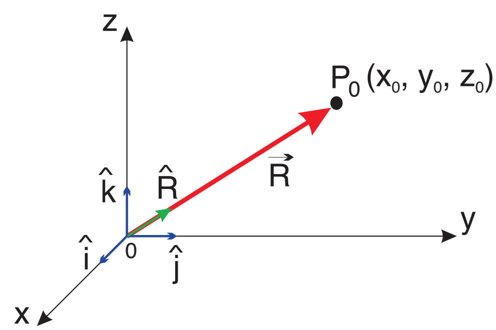
Assignment

web page ⟶ *assignment #1*

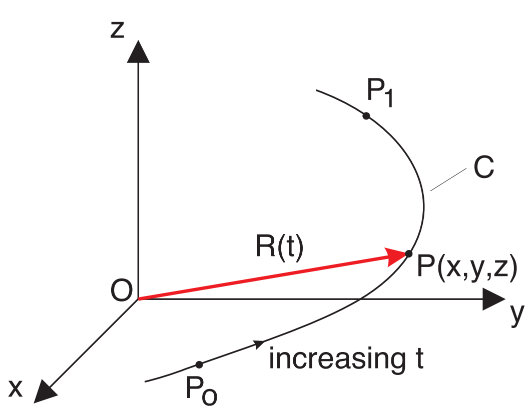
Point in 3D

The position vector for this point is given by the directed line segment $\overrightarrow{OP_0}$.

$$\begin{aligned}\vec{R} &= \hat{i}x_0 + \hat{j}y_0 + \hat{k}z_0 \\ &= \left|\vec{R}\right| \hat{R} = \sqrt{x_0^2 + y_0^2 + z_0^2}\ \hat{R} \\ &= \text{magnitude of } \vec{R} \times \text{ unit} \\ &\qquad \text{vector in } \vec{R} \text{ direction}\end{aligned}$$



Curve in 3D



Imagine that point P moves through 3D space in time. These equations are denoted as the parametric equations of motion for the particle and we refer to C as the trajectory of the particle in 3D space.

The position vector is now a function of t as well.

$$\vec{R}(t) = \hat{i}\ x(t) + \hat{j}\ y(t) + \hat{k}\ z(t)$$

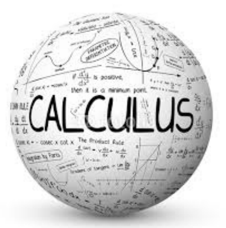
Example 1.1

Suppose P moves from $(-1, 0, 1)$ to $(2, 2, -1)$ in 4 seconds in a straight line and the x coordinate increases linearly with time. i.e. $x(t) = at + b$

a) find the position vector vs. time, for points on the line

b) find the equation of the line in 3D space

Vectors and Planes in 3-D



`Reading`

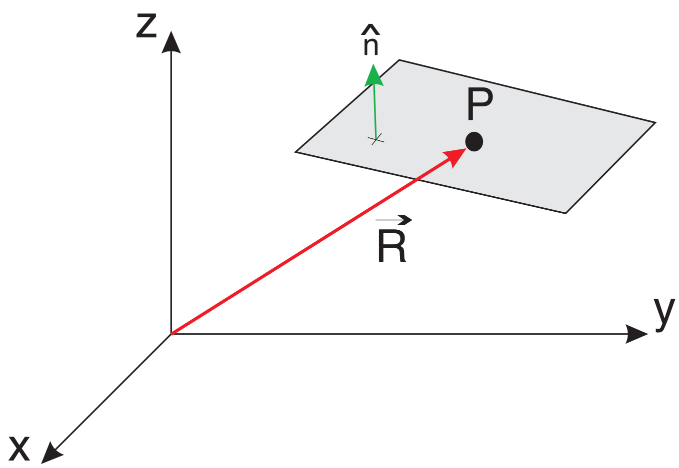
Trim 11.5 $\longrightarrow$ *Planes and Lines*

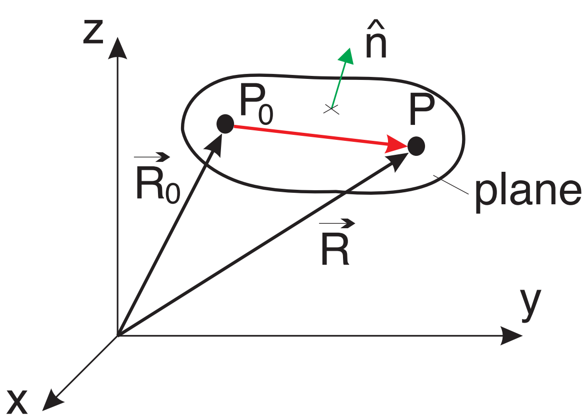
11.6 $\longrightarrow$ *Geometric Applications of Scalar and Vector Products*

`Assignment`

web page $\longrightarrow$ *assignment #2*

Suppose point $\boldsymbol{P}$, with position vector $\vec{\boldsymbol{R}}$, lies in a plane in 3D space. One way to define the plane is through a unit normal vector, $\hat{\boldsymbol{n}}$.





$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{i}}\boldsymbol{n_x} + \hat{\boldsymbol{j}}\boldsymbol{n_y} + \hat{\boldsymbol{k}}\boldsymbol{n_z}$$

If we assume there is a fixed point, $\boldsymbol{P_0(x_0, y_o, z_0)}$, lying **in the plane** with position vector

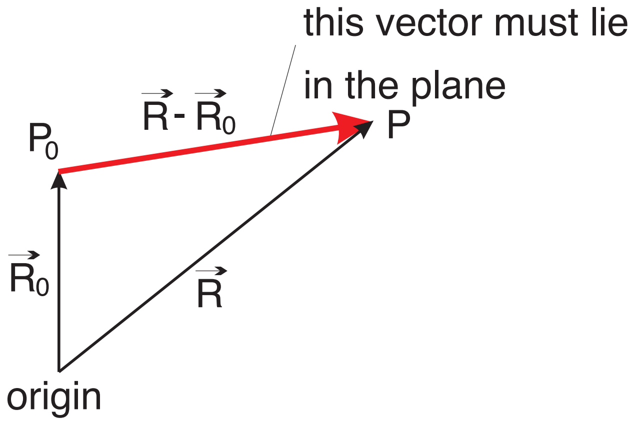
$$\vec{\boldsymbol{R}}_0 = \hat{\boldsymbol{i}}\boldsymbol{x_0} + \hat{\boldsymbol{j}}\boldsymbol{y_0} + \hat{\boldsymbol{k}}\boldsymbol{z_0}$$

We can find the equation for the arbitrary point $\boldsymbol{P(x, y, z)}$ **in the plane**, by first defining the position vector to $\boldsymbol{P}$ as

$$\vec{\boldsymbol{R}} = \hat{\boldsymbol{i}}\boldsymbol{x} + \hat{\boldsymbol{j}}\boldsymbol{y} + \hat{\boldsymbol{k}}\boldsymbol{z}$$

Consider

$$\vec{R} - \vec{R}_0 = \hat{i}(x - x_0) + \hat{j}(y - y_0) + \hat{k}(z - z_0)$$



For the points to be in the plane, $\vec{R} - \vec{R}_0$ must be perpendicular to $\hat{n}$. Therefore

$$(\vec{R} - \vec{R}_0) \cdot \hat{n} = 0$$

The dot product gives

$$\left[\underbrace{\hat{i}(x - x_0) + \hat{j}(y - y_0) + \hat{k}(z - z_0)}_{\vec{R} - \vec{R}_0}\right] \cdot \left[\underbrace{\hat{i}n_x + \hat{j}n_y + \hat{k}n_z}_{\hat{n}}\right] = 0$$

which leads to

$$\boxed{n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0}$$

This is the general equation for a plane through a point (x_0, y_0, z_0) that is perpendicular to (n_x, n_y, n_z).

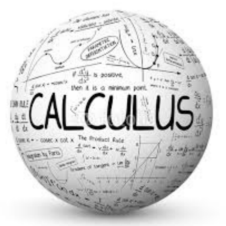
Example 1.2

Given that points $(1, 0, 0)$ $(0, 3, 0)$ $(0, 0, 2)$ lie on a plane in 3D, find

a) the unit normal vector $\hat{n}$ to the plane

b) the equation describing points (x, y, z) lying in the plane

Calculation of Distance in 3-D



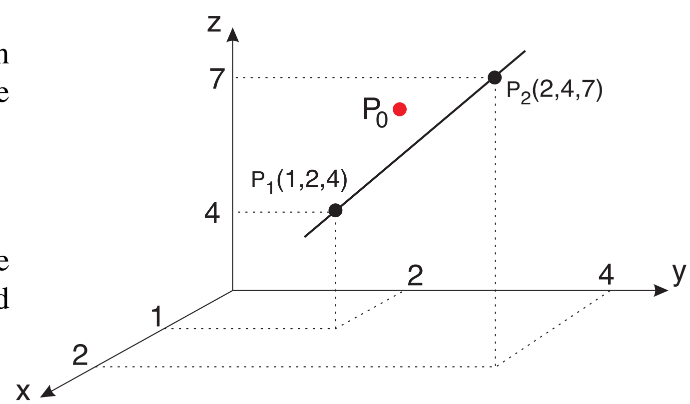
Reading
Trim 11.6 ⟶ *Geometric Applications of Scalar and Vector Products*

Assignment
web page ⟶ *assignment #2*

i) shortest distance between a point and a line

3 steps are required to find the shortest distance between a point and a line:

1. determine the position vector between an arbitrary point on the line and the point of interest itself
2. find the unit vector along the line
3. the shortest distance is the length of the cross product of the position vector and the unit vector along the line



We will use the case of robot arm moving along a straight line to illustrate the technique. The equation of the straight line is given in symmetric form as

$$x - 1 = \frac{y-2}{2} = \frac{z-4}{3}$$

How close does it pass to the point $P_0(1, 3, 6)$?

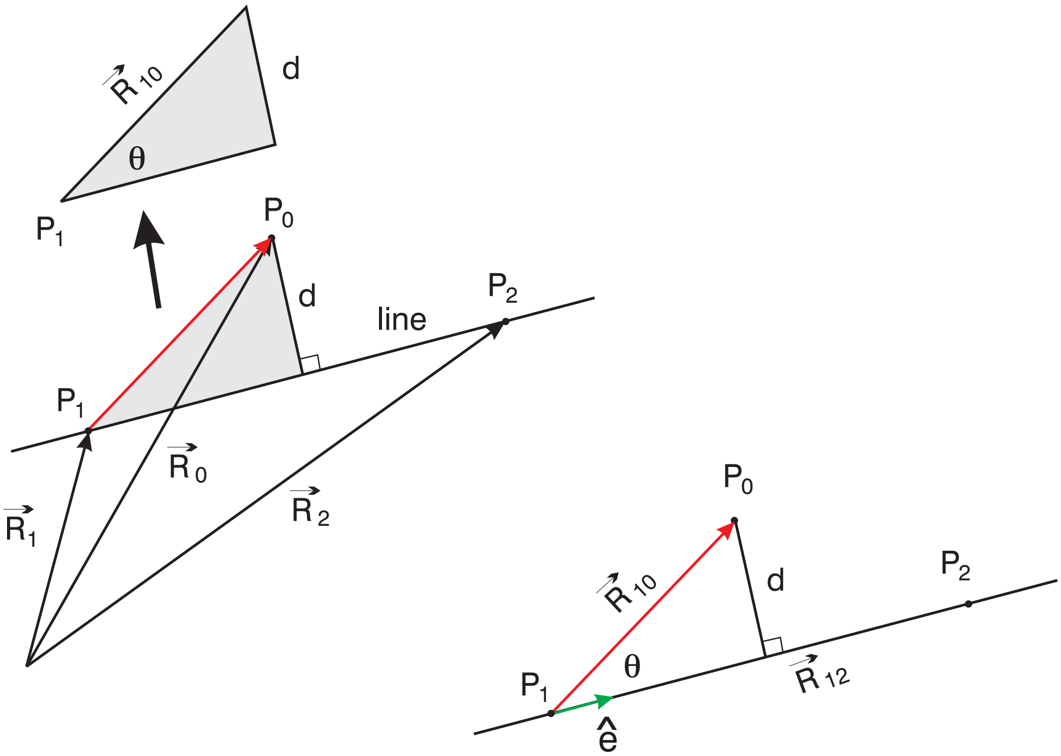
Step 1: Find a position vector between the line and the point

set $x = 1$ to find point P_1 and its position vector

$$P_1 : \quad \vec{R}_1 = \hat{i} + 2\hat{j} + 4\hat{k} \qquad \Rightarrow \qquad P_1(1, 2, 4)$$

find the position vector to our point of interest

$$P_0 : \quad \vec{R}_0 = \hat{i} + 3\hat{j} + 6\hat{k} \qquad \Rightarrow \qquad P_0(1, 3, 6)$$



The position from the line to the point is

$$\vec{R}_{10} = \vec{R}_0 - \vec{R}_1 = (0)\hat{\imath} + \hat{\jmath} + (2)\hat{k}$$

Step 2: Find the unit vector along the line

The position vector to a second point along the line can be determined by letting $x = 2$

$$P_2 : \quad \vec{R}_2 = 2\hat{\imath} + 4\hat{\jmath} + 7\hat{k} \qquad \Rightarrow \qquad P_2(2, 4, 7)$$

By definition, the unit vector along the line, $\hat{e}$, is given as the vector divided by its magnitude

$$\hat{e} = \frac{\vec{R}_2 - \vec{R}_1}{\left|\vec{R}_2 - \vec{R}_1\right|} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|}$$

where

$$\vec{R}_{12} = \vec{R}_2 - \vec{R}_1 = \hat{\imath} + (2)\hat{\jmath} + (3)\hat{k}$$

Step 3: Find the shortest distance between the point and the line

Method 1: Cross Product Approach

$$d = \frac{|\vec{R}_{10} \times \vec{R}_{12}|}{|\vec{R}_{12}|}$$

Method 2: Dot Product Approach

$$d = \left|\vec{R}_{10}\right| \, \sin\theta$$

Using the cross product approach:

$$|\vec{R}_{10} \times \vec{R}_{12}| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}$$

$$|\vec{R}_{12}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

The magnitude of d is

$$\text{mag } d \;=\; \frac{\sqrt{6}}{\sqrt{14}} = 0.655$$

ii) shortest distance between a point and a plane

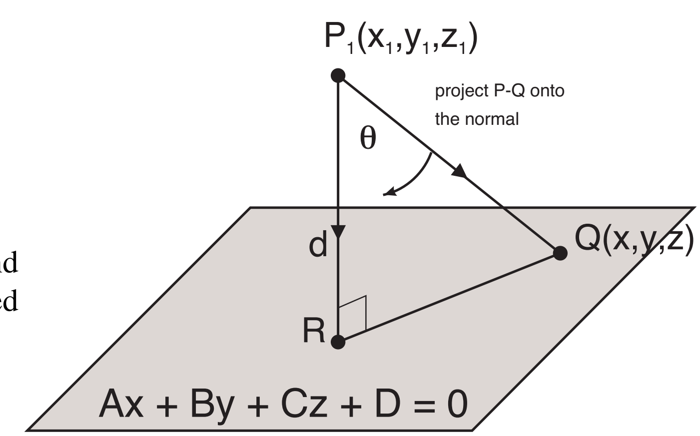
3 steps are required to find the shortest distance between a point and a plane:

1. determine a vector between an arbitrary point in space and a point on the plane
2. find the unit normal from the point to the plane
3. use the dot product of the vector and the unit normal to find the shortest distance

The equation of a plane is given as

$$Ax + By + Cz + D = 0$$

Let $P_1(x_1, y_1, z_1)$ be **any point in space** and $Q(x, y, z)$ be a **point on the plane** specified above as shown below



Step 1: Determine the vector between the point in space and the point on the plane

$$\vec{PQ} = (x - x_1), (y - y_1), (z - z_1)$$

Step 2: Determine the unit normal vector between the point in space and the plane

$$\text{normal to plane} = (A, B, C)$$

$$\hat{PR} = \frac{\vec{PR}}{|\vec{PR}|} = \frac{\pm(A, B, C)}{\sqrt{A^2 + B^2 + C^2}}$$

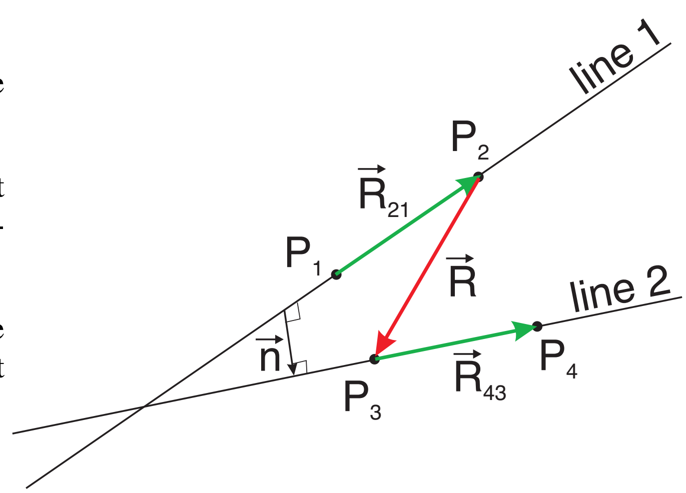
Step 3: Use the dot product approach to find the shortest distance

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

iii) shortest distance between two skewed lines

3 steps are required to find the shortest distance between two non-instersecting lines:

1. find the unit normal vector between the two lines, $\hat{n}$
2. determine a position vector staring at any point on the first line and terminating at any point on the second line
3. the shortest distance is the length of the position vector projected onto the unit normal vector



We will use the case of the following two non-intersecting, skewed lines to illustrate.

line 1 $\quad x = y = z$

line 2 $\quad y = 2 \quad z = 0$

Step 1: Find the unit normal vector (either dot product <u>or</u> cross product approach

<u>Dot Product Approach</u> $\Rightarrow$ $\cos\theta = \dfrac{A \cdot B}{|A|\,|B|} = 0 \quad (\text{for } \perp)$

We know that the dot product of a line and its normal equals zero.

$$n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0$$

If $(0, 0, 0)$ and $(1, 1, 1)$ are two points on line 1, then

$$n_x + n_y + n_z = 0 \qquad (1)$$

If $(1, 2, 0)$ and $(0, 2, 0)$ are two points on line 2, then

$$n_x = 0 \qquad (2)$$

Therefore $n_x = 0, n_y = 1$ and $n_z = -1$ is a solution and

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{n_x\,\hat{i} + n_y\,\hat{j} - n_z\,\hat{k}}{\sqrt{n_y^2 + n_z^2}} = \frac{1}{\sqrt{2}}\,(\hat{j} - \hat{k})$$

<u>Cross Product Approach</u> $\Rightarrow$ $\hat{n} = \dfrac{\vec{n}}{|\vec{n}|} = \dfrac{A \times B}{|A|\,|B| \sin\theta} = \dfrac{A \times B}{|\vec{n}|}$

2 points on line 1 $\quad P_1(0, 0, 0)$ and $P_2(1, 1, 1)$

vector on line 1 $\quad \vec{R}_{21} = \vec{R}_2 - \vec{R}_1 = \hat{i} + \hat{j} + \hat{k}$

2 points on line 2 $\quad (0, 2, 0)$ and $(1, 2, 0)$

vector on line 2 $\quad \vec{R}_{43} = \vec{R}_4 - \vec{R}_3 = +\hat{i}$

$$\vec{n} = \underbrace{(\hat{i} + \hat{j} + \hat{k})}_{vector\ 1} \times \underbrace{\hat{i}}_{vector\ 2} = \hat{i}(0) - \hat{j}(0 - 1) + \hat{k}(0 - 1) = \hat{j} - \hat{k}$$

and we know

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\hat{j} - \hat{k}}{\sqrt{2}}$$

Step 2: Find a position vector between points on each line

The position vector between any 2 points on the 2 lines, say $(0, 0, 0)$ and $(1, 2, 0)$ is

$$\vec{R} = \hat{i} + 2\hat{j}$$

Step 3: Find the shortest distance, d

Dot Product

$\vec{n} \cdot$ line 1 $= 0$

$\vec{n} \cdot$ line 2 $= 0 \Rightarrow$ solve for $n_x,\ n_y,\ n_z$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|}$$

$d = |\text{line } 3 \cdot \hat{n}|$

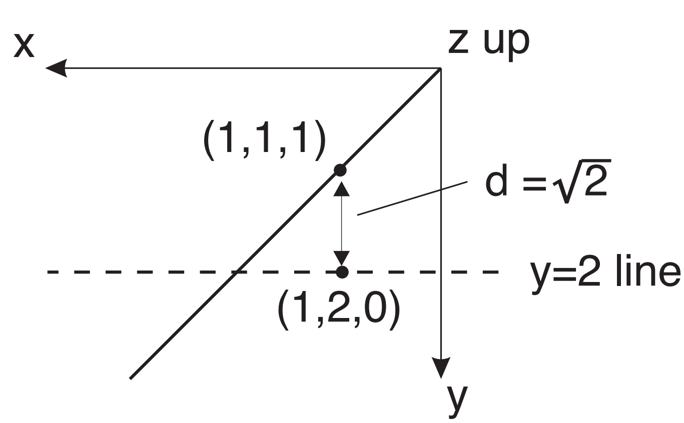
Cross Product

$$\hat{n} = \frac{\text{line } 1 \times \text{line } 2}{|\text{line } 1 \times \text{line } 2|}$$

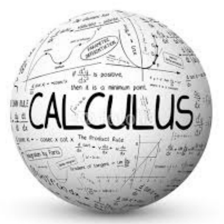
$d = |\text{line } 3 \cdot \hat{n}|$

<u>Dot Product Approach</u>

$$d = \left| (\hat{i} + 2\hat{j}) \cdot \frac{\hat{j} - \hat{k}}{\sqrt{2}} \right| = \frac{2}{\sqrt{2}} = \sqrt{2}$$



Differentiation of Vectors



Reading

Trim 11.9 $\longrightarrow$ *Differentiation and Integration of Vectors*

Assignment

web page $\longrightarrow$ *assignment #2*

Think of a dynamics problem of a particle moving along some curve in 3D space where a twist may be possible, i.e. not planar. The particles position is defined by a position vector

$$\vec{R}(t) = \hat{\imath}\, x(t) + \hat{\jmath}\, y(t) + \hat{k}\, z(t) \qquad \text{for } a \le t \le b$$

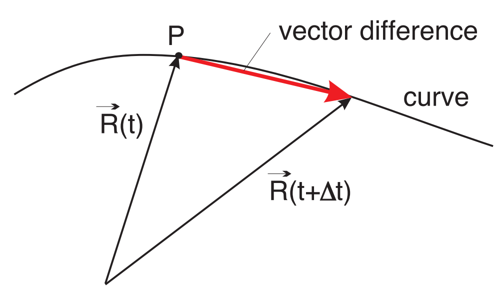
The formal definition of the derivative can be written as

$$\left.\frac{d\vec{R}}{dt}\right|_{at\ P} = \lim_{\Delta t \to 0} \frac{\vec{R}(t+\Delta t) - \vec{R}(t)}{\Delta t}$$

How do we calculate this from the definition?

$$\begin{aligned}
\left.\frac{d\vec{R}}{dt}\right|_{at\ P} &= \lim_{\Delta t \to 0} \frac{\hat{\imath}x(t+\Delta t) + \hat{\jmath}y(t+\Delta t) + \hat{k}z(t+\Delta t) - \hat{\imath}x(t) - \hat{\jmath}y(t) - \hat{k}z(t)}{\Delta t} \\
&= \hat{\imath} \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} + \hat{\jmath} \lim \Big(\Big) + \hat{k} \lim \Big(\Big) \\
&= \hat{\imath}\frac{dx(t)}{dt} + \hat{\jmath}\frac{dy(t)}{dt} + \hat{k}\frac{dz(t)}{dt}
\end{aligned}$$

We can just differentiate each component in the position vector $\vec{R}(t)$. The differentiation process produces another vector. The physical meaning of differentiation is that we have a slope but not just a slope – it is tied back to the particle motion where $\vec{R}(t+\Delta t) - \vec{R}(t)$ is a vector difference that in the limit will give us the tangent vector at point P.



Therefore we must have the rate of change position vector along the curve in the $\hat{\boldsymbol{T}}$ direction. (Let $\hat{\boldsymbol{T}}$ be the unit vector in the tangent direction)

$$\frac{d\vec{R}}{dt} = \left|\frac{d\vec{R}}{dt}\right| \hat{T}$$

Given the particle trajectory for curve $\vec{R}(t)$

$$\left.\frac{d\vec{R}}{dt}\right|_{at\ P} = \hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt} = \left|\vec{V}\right| \hat{T}$$

where the velocity of the particle, $\vec{V}$, is the rate of change of position.

Similar ideas apply for higher derivatives.

2nd derivative of $\vec{R}(t)$

It is easy to see

$$\left.\frac{d^2\vec{R}}{dt^2}\right|_{at\ P} = \hat{i}\frac{d^2x}{dt^2} + \hat{j}\frac{d^2y}{dt^2} + \hat{k}\frac{d^2z}{dt^2}$$

Since we have just shown that

$$\frac{d\vec{R}}{dt} = \left|\frac{d\vec{R}}{dt}\right| \hat{T} = \left|\vec{V}\right| \hat{T}$$

Then the second derivative is

$$\begin{aligned} \frac{d^2\vec{R}}{dt^2} &= \frac{d}{dt}\left(\frac{d\vec{R}}{dt}\right) = \frac{d}{dt}\left(\left|\vec{V}\right| \hat{T}\right) \\ &= \underbrace{\hat{T}\frac{d\left|\vec{V}\right|}{dt}}_{acceleration} + \left|\vec{V}\right| \frac{d\hat{T}}{dt} \end{aligned}$$

The first term, $\hat{T}\frac{d\left|\vec{V}\right|}{dt}$, is the **acceleration component along the curve** in the $\hat{\boldsymbol{T}}$ direction, i.e. the speed change along the curve.

The second term is the **centripetal acceleration** due to a changing trajectory path

$$\left|\vec{V}\right| \left|\frac{d\hat{T}}{dt}\right| \hat{N}$$

where $\hat{N}$ is called the principal normal unit vector.

Therefore for a given curve

$$\frac{d^2\vec{R}}{dt^2} = \hat{i}\frac{d^2x}{dt^2} + \hat{j}\frac{d^2y}{dt^2} + \hat{k}\frac{d^2z}{dt^2} = \hat{T}\frac{d\left|\vec{V}\right|}{dt} + \left|\vec{V}\right| \left|\frac{d\hat{T}}{dt}\right| \hat{N}$$

where

1st term	velocity acceleration at P
2nd term	Cartesian components of acceleration
3rd term	components of acceleration in coordinates relative to the trajectory - speed along the trajectory plus the centripetal acceleration due to trajectory shape

Note: we can turn this around and use to calculate the principal normal to any given curve at $\vec{R}(t)$

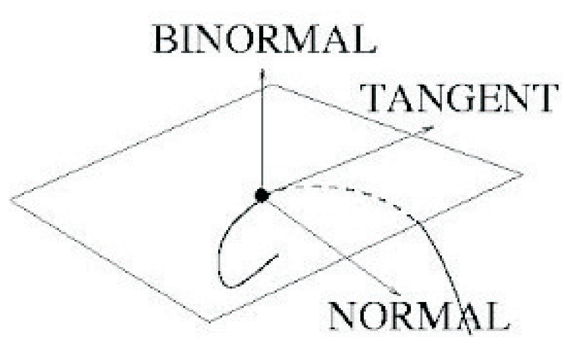
$$\hat{N} = \frac{\dfrac{d\hat{T}}{dt}}{\left|\dfrac{d\hat{T}}{dt}\right|}$$

where

$$\hat{T} = \frac{\dfrac{d\hat{R}}{dt}}{\left|\dfrac{d\hat{R}}{dt}\right|}$$

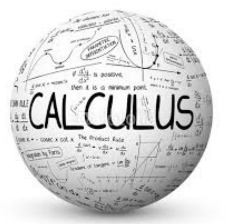
and

$$\hat{B} = \hat{T} \times \hat{N}$$



gives the third local coordinate.

Integration of Vectors



Reading
Trim 11.9 $\longrightarrow$ *Differentiation and Integration of Vectors*

Assignment
web page $\longrightarrow$ *assignment #2*

Some of the ideas learned in first year calculus carry over to vectors in 3-D space.

For a moving particle, the position vector $\vec{R}(t)$ and its derivative $d\vec{R}(t)/dt = \vec{V}(t) = \left|\vec{V}(t)\right|\hat{T}(t)$ can be used.

We see that

$$\int \frac{d\vec{R}(t)}{dt} \cdot dt = \int \vec{V}(t)dt = \vec{R}(t) + \vec{C}$$

where $\vec{C}$ is the vector constant.

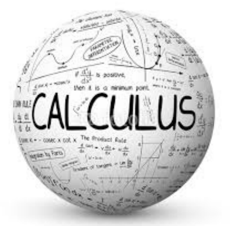
Example 1.3

Given a particle at $t = 0$ and position $(2, 0, 2)$ that moves with a velocity

$$\vec{V}(t) = \hat{\imath}(-2\sin t) + \hat{\jmath}(2\cos t) + \hat{k}(2\sin t - 2\cos t)$$

for $0 \leq t \leq 2\pi$, find the curve that $\vec{R}(t)$ traces in 3D space.

Tangent Vectors and Arc Length of Curves in 3-D



Reading

Trim 11.10 $\longrightarrow$ *Parametric and Vector Representation of Curves*

11.11 $\longrightarrow$ *Tangent Vectors and Lengths of Curves*

Assignment

webpage $\longrightarrow$ *assignment #2 & #3*

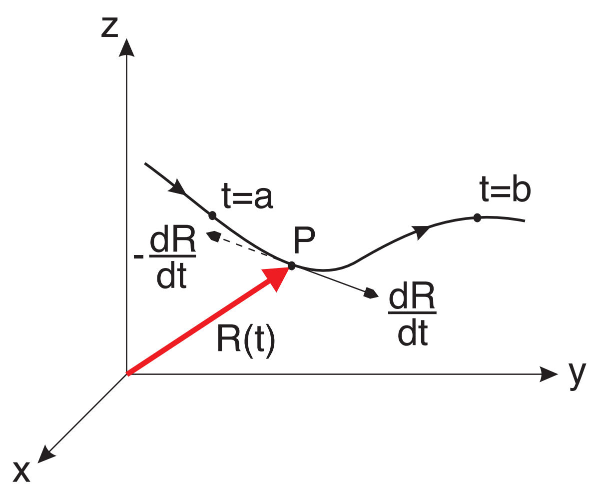
The tangent vector to the curve at point P is given by

$$\vec{T} = \frac{d\vec{R}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

There are two possible tangent directions at each point, however, the tangent vector is always defined in the direction of increasing t along the curve as shown below.

To find the unit tangent vector to a curve, $\hat{T}$ at any point along the curve, the tangent vector is normalized with respect to its length as follows

$$\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{d\vec{R}/dt}{|d\vec{R}/dt|}$$



We recall that for the 2D case that the length is given by

$$\Delta L^2 = \Delta x(t)^2 + \Delta y(t)^2$$

$$\Delta L = \sqrt{\Delta x(t)^2 + \Delta y(t)^2} = \sqrt{\frac{\Delta x(t)^2}{\Delta t^2} + \frac{\Delta y(t)^2}{\Delta t^2}}\,\Delta t$$

In differential form this becomes

$$dL = \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2}\,dt$$

Summing all the differential lengths, starting at $t = a$

$$
\begin{aligned}
L &= \int_{t=a} \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2} dt \qquad \text{or} \\
&= \int_{x=x_0}^{x_1} \sqrt{\frac{dx^2 + dy^2}{dx^2}}\, dx \\
&= \int_{x=x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx
\end{aligned}
$$

We can extend this to 3D space, where the position vector

$$\vec{R}(t) = \hat{\imath}x(t) + \hat{\jmath}y(t) + \hat{k}z(t)$$

for $a \leq t \leq b$.

If we consider a small section of the curve for a small Δt

$$\Delta S \approx \left|\vec{R}(t + \Delta t) - \vec{R}(t)\right|$$

If we divide through by Δt

$$\frac{\Delta S}{\Delta t} = \left|\frac{\vec{R}(t + \Delta t) - \vec{R}(t)}{\Delta t}\right|$$

In the limit as $\Delta t \to 0$ the equality is exact, therefore

$$\frac{dS}{dt} = \left|\frac{d\vec{R}(t)}{dt}\right|$$

Note the signs, +'ve ΔS in the direction of +'ve Δt.

We have

$$\frac{dS}{dt} = \left|\frac{d\vec{R}}{dt}\right| = \left|\hat{\imath}\frac{dx(t)}{dt} + \hat{\jmath}\frac{dy(t)}{dt} + \hat{k}\frac{dz(t)}{dt}\right|$$

Therefore

$$\frac{dS}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

or

$$dS = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

where dS is the arc length along the curve.

The total length of the curve is given by

$$L = \int_a^b dS = \int_{t=a}^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

If we use the distance traveled along the curve, $S(t)$ to denote position, where

$$S(t) = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

then the unit tangent vector can be specified independent of the length of the tangent vector, $|T|$.

$$\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{d\vec{R}/dt}{|d\vec{R}/dt|} = \frac{d\vec{R}/dt}{dS/dt} = \frac{d\vec{R}}{dS}$$

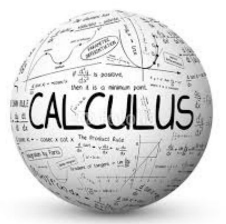
$$\hat{T} = \frac{d\vec{R}}{dS} = \frac{dx}{dS}\hat{i} + \frac{dy}{dS}\hat{j} + \frac{dz}{dS}\hat{k}$$

Example 1.4

Find the length of the curve of intersection between

$$\left.\begin{array}{lll} \text{surface 1} & y = 2x & \text{(a plane)} \\ \text{surface 2} & z = \dfrac{x^2 + y^2}{5} & \end{array}\right\} \quad \text{for } z \leq 25$$

Curvature and Centripetal Acceleration



Reading

Trim 11.12 $\longrightarrow$ *Normal Vectors, Curves, and Radius of Curvature*

11.13 $\longrightarrow$ *Displacement, Velocity, and Acceleration*

Assignment

web page $\longrightarrow$ *assignment #3*

Normal Vector

The normal to a point is the line which is perpendicular to the tangent vector, $\boldsymbol{T}$, given in the previous section. If the unit tangent vector in 2D space is given in terms of the length along the curve, $\boldsymbol{S}$

$$\hat{\boldsymbol{T}} = \frac{d\vec{\boldsymbol{R}}}{d\boldsymbol{S}} = \frac{d\boldsymbol{x}}{d\boldsymbol{S}}\hat{\boldsymbol{i}} + \frac{d\boldsymbol{y}}{d\boldsymbol{S}}\hat{\boldsymbol{j}}$$

then the unit normal vector, $\hat{\boldsymbol{N}}$, is

$$\hat{\boldsymbol{N}} = -\frac{d\boldsymbol{y}}{d\boldsymbol{S}}\hat{\boldsymbol{i}} + \frac{d\boldsymbol{x}}{d\boldsymbol{S}}\hat{\boldsymbol{j}}$$

Because of orthogonality $\hat{\boldsymbol{T}} \cdot \hat{\boldsymbol{N}} = \boldsymbol{0}$. In 3D space there is not a single normal vector but an entire plane of normal vectors.

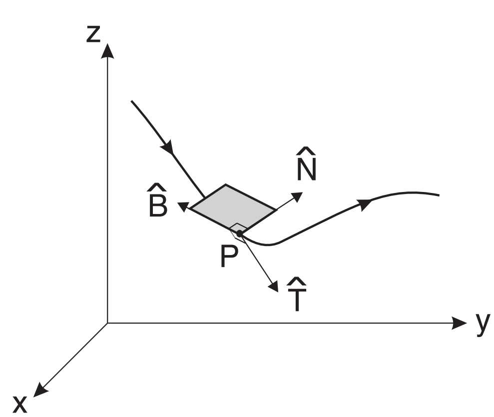
Two unique vectors can be identified in this plane of vectors, they are the *principal normal*, given as

$$\hat{\boldsymbol{N}} = \frac{\vec{\boldsymbol{N}}}{|\vec{\boldsymbol{N}}|} = \frac{d\hat{\boldsymbol{T}}/d\boldsymbol{t} \cdot (d\boldsymbol{t}/d\boldsymbol{S})}{\left|d\hat{\boldsymbol{T}}/d\boldsymbol{t} \cdot (d\boldsymbol{t}/d\boldsymbol{S})\right|} = \frac{d\hat{\boldsymbol{T}}/d\boldsymbol{S}}{\left|d\hat{\boldsymbol{T}}/d\boldsymbol{S}\right|}$$

and the *binormal* given as

$$\hat{\boldsymbol{B}} = \hat{\boldsymbol{T}} \times \hat{\boldsymbol{N}}$$

These vectors are shown in the following figure.



Curvature and Radius of Curvature

The rate of change of the unit tangent vector with respect to the distance travelled along the curve, i.e. $d\hat{T}/dS$ can be thought of as a measure of the rate of change of direction of $\hat{T}$ or a measure of the curvature of the curve.

Since the vector $\hat{T}$ has both direction and magnitude, the curvature of a curve is more aptly defined in terms of a curvature parameter, $\kappa(S)$, where

$$\kappa(S) = \text{rate of change of the unit tangent with respect to distance travelled}$$

$$\kappa(S) = \left|\frac{d\hat{T}}{dS}\right| = \left|\frac{d\hat{T}/dt}{dS/dt}\right| \qquad (1)$$

but since

$$\frac{dS}{dt} = \left|\frac{d\vec{R}(t)}{dt}\right| = \left|\vec{V}\right| \qquad (2)$$

and [from page 32 of the notes]

$$|\vec{a}_{\hat{N}}| = \left|\vec{V}\right|\left|\frac{d\hat{T}}{dt}\right| \qquad \Rightarrow \qquad \left|\frac{d\hat{T}}{dt}\right| = \frac{|\vec{a}_{\hat{N}}|}{\left|\vec{V}\right|} \qquad (3)$$

If we substitute Eqs. 2 and 3 into 1, we get

$$\kappa(S) = \frac{|\vec{a}_{\hat{N}}|}{\left|\vec{V}\right|} \cdot \frac{1}{\left|\vec{V}\right|}$$

Noting that

$$|\vec{a}_{\hat{N}}| = \frac{|\vec{V} \times \vec{a}|}{|\vec{V}|} \qquad \text{where} \qquad \vec{a}_N = |\vec{a}|\sin\theta = \frac{|\vec{V}|}{|\vec{V}|}|\vec{a}|\sin\theta = \frac{\vec{V} \times \vec{a}}{|\vec{V}|}$$

$$\kappa(S) = \frac{|\vec{V} \times \vec{a}|}{|\vec{V}|^3} \qquad \Rightarrow \qquad \boxed{\kappa(S) = \frac{|\vec{R}' \times \vec{R}''|}{|\vec{R}'|^3}} \qquad \text{and} \qquad \boxed{\rho(S) = \frac{1}{\kappa(S)}}$$

Centripetal Acceleration

Recall that the 2nd derivative (acceleration) can be written as

$$\begin{aligned}\frac{d^2\vec{R}}{dt^2} &= \hat{i}\frac{d^2x}{dt^2} + \hat{j}\frac{d^2y}{dt^2} + \hat{k}\frac{d^2z}{dt^2} \\ &= \underbrace{\frac{d\left|\vec{V}\right|}{dt}\hat{T}}_{tangent\ to\ curve} + \underbrace{\left|\vec{V}\right|\left|\frac{d\hat{T}}{dt}\right|\hat{N}}_{normal\ to\ curve}\end{aligned}$$

where the acceleration normal to the path is given as

$$|\vec{a}_{\hat{N}}| = \left|\vec{V}\right|^2 \kappa = |\vec{V}|^2/\rho$$

Therefore

$$\frac{d^2\vec{R}}{dt^2} = \frac{d\left|\vec{V}\right|}{dt}\hat{T} + \frac{\left|\vec{V}\right|^2}{\rho}\hat{N}$$

This clearly relates the centripetal acceleration locally to the curvature of path followed by the particle.

The limiting case can be shown by looking at the motion along a straight line path

$$\rho \rightarrow \infty \quad \text{(everywhere)}$$

$$\frac{d^2\vec{R}}{dt^2} = \frac{d\left|\vec{V}\right|}{dt}\hat{T}$$

since the only acceleration is along the path.

For a circular path of radius R, with a particle moving at a constant speed $\vec{V}_0$

$$\frac{d^2\vec{R}}{dt^2} = \frac{\left|\vec{V}\right|^2}{\rho}\hat{N} = \frac{\left|\vec{V}\right|^2}{R}\hat{N}$$

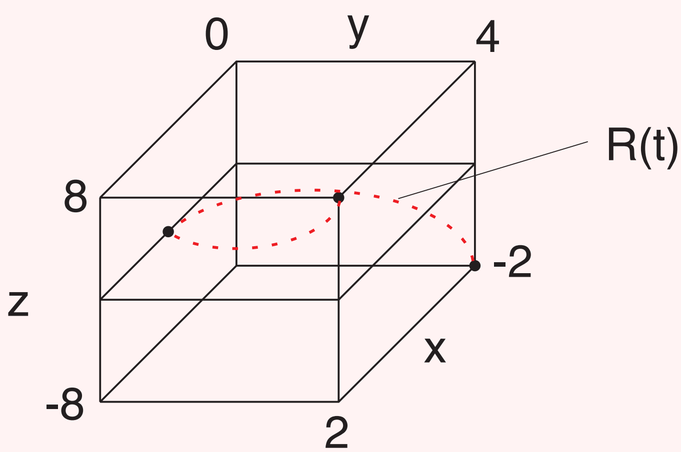
The only acceleration is perpendicular to the path.

Example 1.5

A particle moves through $3-$D space so that its position vector at time t is

$$\vec{R}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

The path is referred to as a twisted cube as shown below



Find the tangential and normal components of acceleration at time t.

Example 1.6: This example gives a good summary of the types of vector/3D curve calculations you should be able to do after working through chapter 11 of Trim.

A particle follows a trajectory in space given by

$$\begin{aligned} x(t) &= 2\cos t \\ y(t) &= 2\sin t \\ z(t) &= 2\pi - t \end{aligned}$$

$$\boxed{\begin{array}{l} x^2 = 4\cos^2 t \\ y^2 = 4\sin^2 t \\ x^2 + y^2 = 4(\cos^2 t + \sin^2 t) = 4 \end{array}}$$

with x, y, z in meters and $0 \le t \le 2\pi$ in seconds.

a) sketch the trajectory curve
b) calculate local coordinates $\hat{T}, \hat{N}, \hat{B}$ to the curve at any time t
c) calculate the length of the curve
d) find the curvature at any time t
e) express the particle velocity and acceleration at any time t in $\hat{T}, \hat{N}$ coordinates