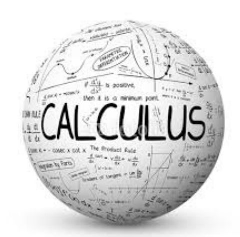
Multiple Integrals

Review of Single Integrals



Reading

Trim 7.1 $\longrightarrow$ *Review Application of Integrals: Area*

7.2 $\longrightarrow$ *Review Application of Integrals: Volumes*

7.3 $\longrightarrow$ *Review Application of Integrals: Lengths of Curves*

Assignment

web page $\longrightarrow$

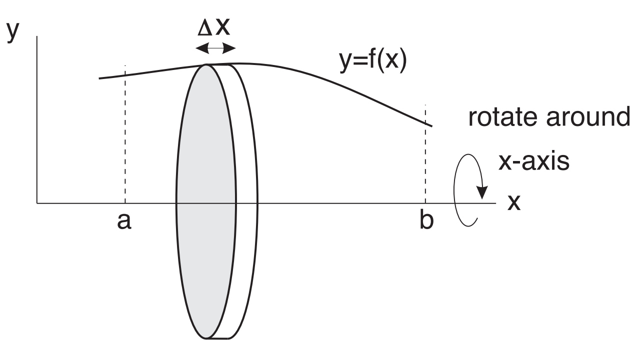
Planar Area

In the limit as $\Delta x \to dx$ the total number of panels $\to \infty$

$$A = \int_a^b y \cdot dx = \int_a^b f(x)dx$$

Volume of Solid of Revolution

a) **Disk Method** : rotate $y = f(x)$ about the $x-$ axis to form a solid.



The disk has a volume of $\mathcal{V} = \pi y^2 \Delta x$.

The total volume between a and b can be determined as:

$$\mathcal{V} = \int_a^b \pi y^2 dx$$

Note: The value of $y = f(x)$ is substituted into the formulation for area and the resulting equation is integrated between a and b.

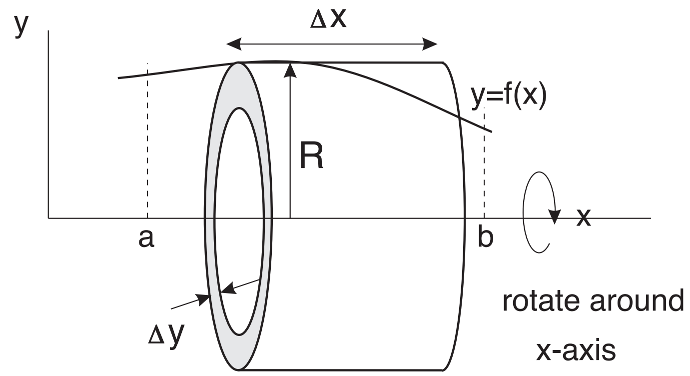
b) **Shell Method:** Find a ring defined with ring area: $2\pi y \cdot \Delta y$.

The volume of the ring is given by

$$\Delta \mathcal{V} = (2\pi y \cdot \Delta y)\Delta x$$

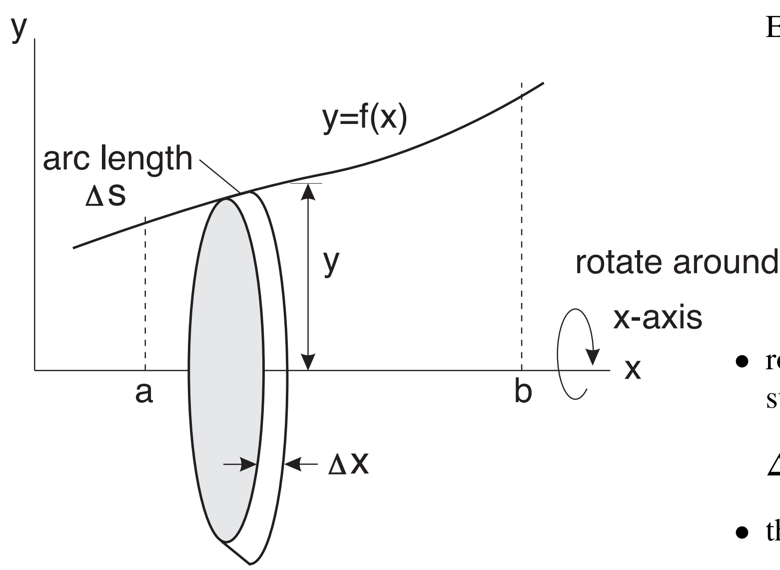
The volume of the solid is determined by solving the integral

$$\mathcal{V} = \int_0^R 2\pi\, x\, y\, dy$$



Either method can be used, which ever is most convenient.

Surface Area of Solid of Revolution



- the arc length can be defined using Eq. 7.15:

$$\Delta s = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\Delta x$$

$$= \frac{\sqrt{dx^2 + dy^2}}{dx}\Delta x$$

- rotate about the $x-$ axis, where the surface area is defined as

$$\Delta A_{surf} = 2\pi y \Delta s$$

- the total surface area is given as

$$A_{surface} = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$$

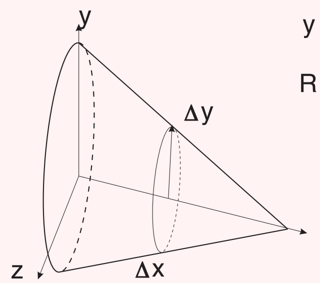
Example: 3.1

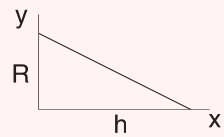
Find the area in the positive quadrant bounded by

$$y = \frac{1}{4}x \qquad \text{and} \qquad y = x^3$$

Example: 3.2

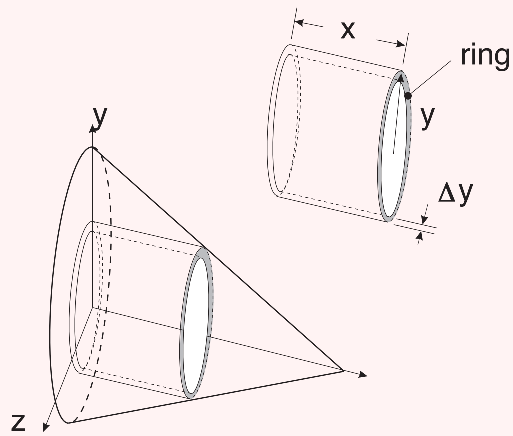
Find the volume of a cone with base radius $\boldsymbol{R}$ and height $\boldsymbol{h}$, rotated about the $\boldsymbol{x}$ axis using the disk method.





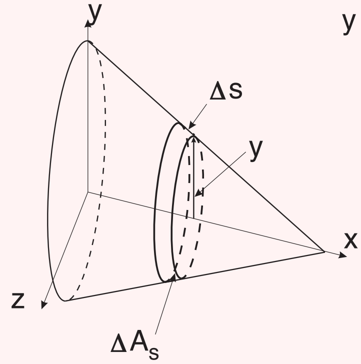
Example: 3.3

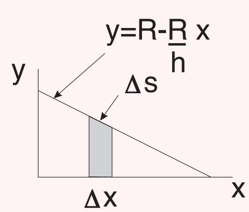
Find the volume of a cone with base radius $\boldsymbol{R}$ and height $\boldsymbol{h}$, rotated about the $\boldsymbol{x}$ axis using the shell method.



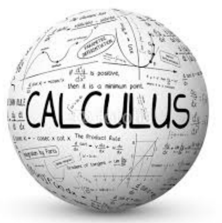
Example: 3.4

Find the surface area of a cone with base radius $\boldsymbol{R}$ and height $\boldsymbol{h}$, rotated about the $\boldsymbol{x}$ axis.





Numerical Integration



Reading
Trim 8.7 ⟶ *Numerical Integration*

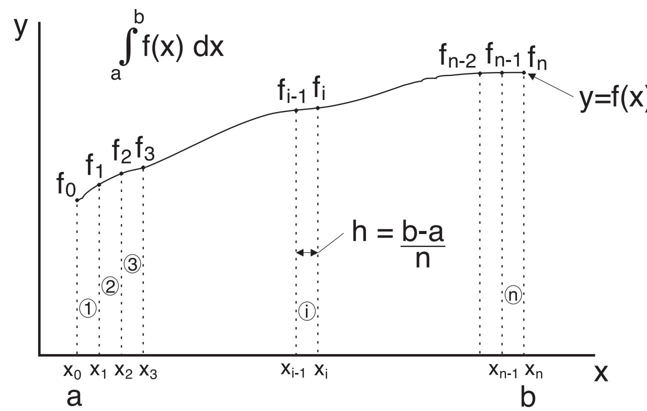
Assignment
web page ⟶

To find:

$$I = \int_a^b f(x)\, dx$$

First, subdivide the interval $a - b$ into n panels or strips of finite width, where

$$\Delta x \quad \text{or} \quad h = (b-a)/n$$



Trapezoidal Rule

The objective of the trapezoidal rule is to find the area under the curve using geometric panels that approximate the area. We know that exact value of the integral is

$$I_{exact} = \int_a^b f(x)\, dx$$

that can also be equated to the approximate area determined by the trapezoidal rule plus an error term

$$I_{exact} \quad = \quad I_{trap} + \epsilon + I_{trap} + \underbrace{C\,h^2 + D\,h^4 + E\,h^6 + \ldots}_{\text{error terms (unknown)}}$$

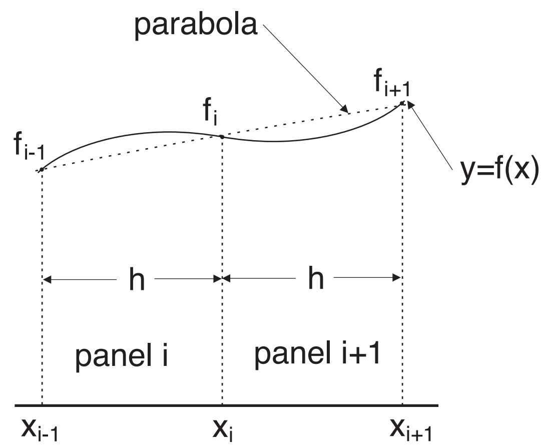
$$\boxed{I_{trap} = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f_i\right]}$$

where the panel width is h . The largest term in the error is $\epsilon \propto h^2$ indicating that the trapezoidal method is a 2nd order method, i.e. if we half the panel size we should see a decrease in the error of a factor of 4.

Simpson's Rule

Instead of using a simple linear approximation of the curve, as in the Trapezoidal rule, Simpson's rule uses a parabolic representation of the curve.

The area under the curve is now given by



$$Area = \int_{x_{i-1}}^{x_i} f(x)dx \approx \int_{x_{i-1}}^{x_i} (C_1x^2 + C_2x + C_3)dx$$

Parabolas are used for each 2 panel pair, therefore we require n, even number of panels with $(n+1)$ odd number of points.

$$I_{simp} = \frac{h}{3}\left[f(a) + f(b) + 4\sum_{i=1,2,3,\ldots}^{n-1} f_i + 2\sum_{i=2,4,6,\ldots}^{n-2} f_i\right]$$

and the error term is given by

$$\epsilon = C_1h^4 + D_1h^6 + E_1h^8 + \cdots$$

The largest error term is $\epsilon \propto h^4$ indicating a 4th order method.

Example: 3.5a

Find the value for

$$I = \int_0^2 x^4 dx$$

using Trapezoidal rule and Simpson's rule

Romberg Integration

Romberg integration is based on a procedure that eliminates the error term from the trapezoidal rule calculations. In essence it provides a procedure for getting an accurate answer without the need for a very small h. It is efficient and minimizes the potential for round off errors.

The exact value of the integral can be written as

$$I_{exact} = I_{trap} + \underbrace{C\,h^2 + D\,h^4 + E\,h^6 + \dots}_{\text{error terms (unknown)}} \qquad (1)$$

If we rewrite the trapezoidal rule with a panel width of $h/2$

$$I_{trap_2} = \frac{h/2}{2}\left[f(a) + f(b) + 2\sum_{i=2}^{2(n-1)} f(x_i)\right]$$

The exact value can then be written as

$$I_{exact} = I_{trap_2} + \underbrace{C\left(\frac{h}{2}\right)^2 + D\left(\frac{h}{2}\right)^4 + E\left(\frac{h}{2}\right)^6 + \dots}_{\text{error terms (unknown)}} \qquad (2)$$

If we multiply Eq. (2) by 4 and subtract Eq. (1), we get

$$\begin{aligned} 3\,I_{exact} &= (4I_{trap_2} - I_{trap_1}) + (\ \ )\Delta h^4 + (\ \ )\Delta h^6 + \dots \\ I_{exact} &= \frac{(4I_{trap_2} - I_{trap_1})}{3} + \frac{(\ \ )\Delta h^4}{3} + \frac{(\ \ )\Delta h^6}{3} + \dots \end{aligned}$$

The leading term gives a better estimate of I_{exact} because the error terms are proportionately smaller.

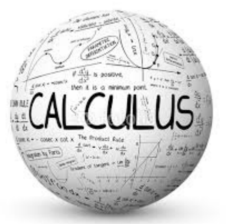
Example: 3.5b

Find the value for

$$I = \int_0^2 x^4 dx$$

using Romberg integration

Double Integrals



Reading

Trim 13.1 ⟶ *Double Integrals and Double Iterated Integrals*

13.2 ⟶ *Eval. of Double Integrals by Double Iterated Integrals*

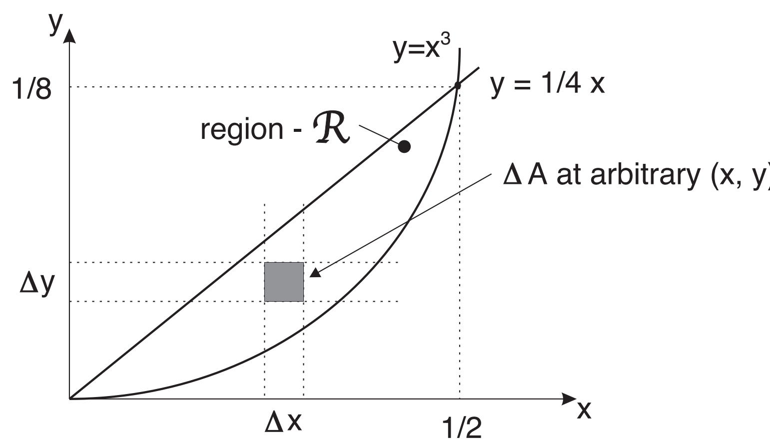
13.7 ⟶ *Double Iterated Integrals in Polar Coordinates*

Assignment

web page ⟶ *assignment #7*

Cartesian Coordinates

Find the area in the +'ve quadrant bounded by $y = \frac{1}{4}x$ and $y = x^3$.



The basic area element in 2D is

$$\Delta A = \Delta x \cdot \Delta y$$

We can build this area up into a strip by summing over Δy, keeping x fixed.

$$\Delta A_{strip} = \left(\sum_{y=x^3}^{1/4x} \Delta y \right)_{fixed\ x} \Delta x$$

Sum up all Δx strips to get the total area

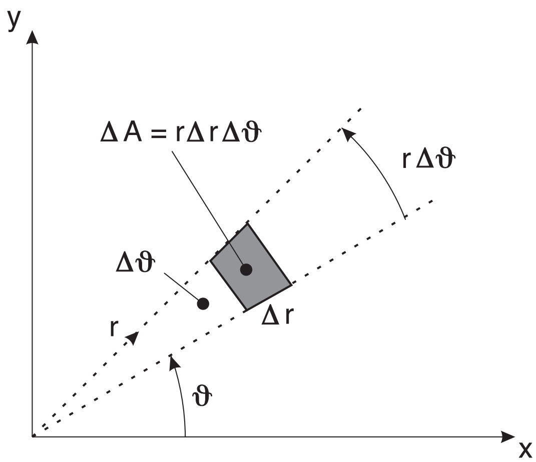
$$A = \sum_{x=0}^{1/2} \left[\sum_{y=x^3}^{1/4x} \Delta y \right] \Delta x$$

In the limit as $\Delta x \rightarrow dx$ and $\Delta y \rightarrow dy$ we get a double integral as follows

$$A = \int_{x=0}^{1/2} \left(\int_{y=x^3}^{1/4x} dy \right) dx$$

Polar Coordinates

In Cartesian coordinates our area element was $\Delta A = \Delta x \Delta y$, which in differential form gave us



$$A = \int\int_{\mathcal{R}} dxdy$$

We can change the principal coordinates into polar coordinates by transforming x and y into r and θ.

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

The Polar coordinate area element becomes

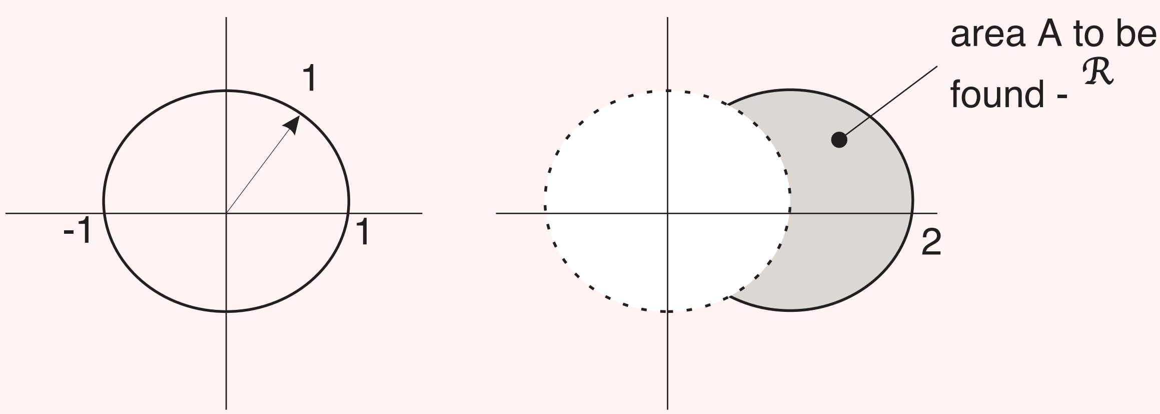
$$\Delta A = r \Delta r \Delta \theta$$

when integrated becomes

$$A = \int\int_{\mathcal{R}} r dr d\theta$$

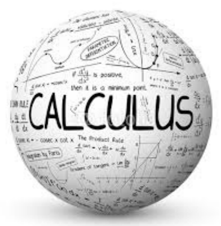
Example: 3.6

Find the area in the +'ve quadrant bounded by 2 circles



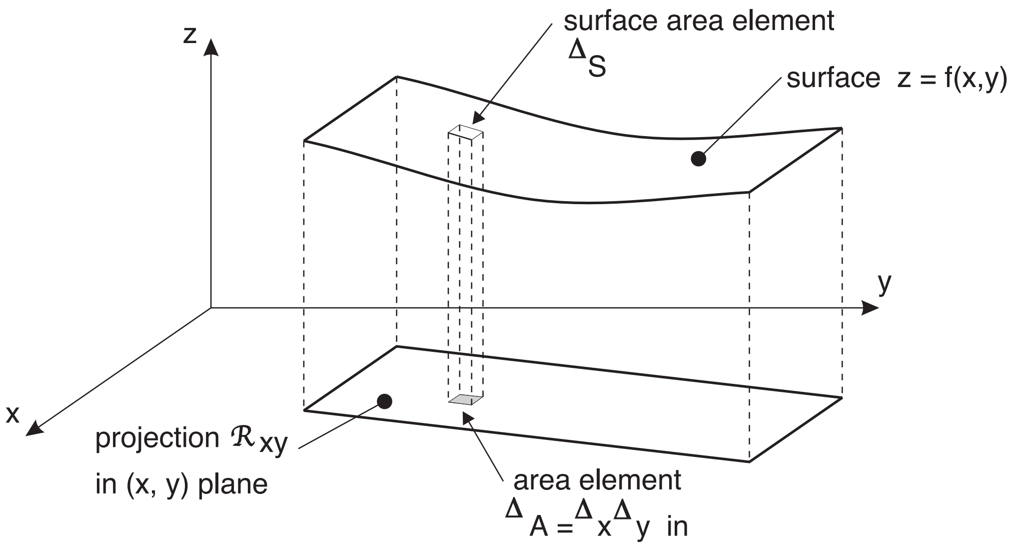
$$\begin{aligned} x^2 + y^2 &= 1 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

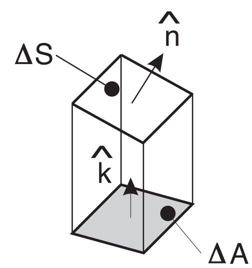
Surface Areas from Double Integrals



Reading
Trim 13.3 $\longrightarrow$ *Areas and Volumes of Solids of Revolution*
13.6 $\longrightarrow$ *Surface Area*

Assignment
web page $\longrightarrow$ *assignment #7*





How is ΔS related to ΔA? Imagine shining a light vertically down through ΔS to get ΔA.

1. the surface is defined as $z = f(x, y)$

2. redefine as $F = z - f(x, y)$ where the surface is given as $F = 0$
   - $F > 0$ and $F < 0$ will the regions above and below the surface, respectively

3. the gradient of the function F is given as

$$\nabla F = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

   ∇F is the perpendicular to the surface and the perpendicular to the tangent planes

$$\vec{n} = \nabla F$$

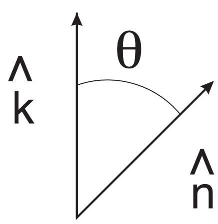
4. get the unit normal vector as follows

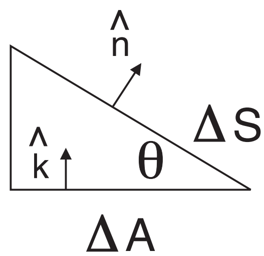
$$\hat{n} = \frac{\nabla F}{|\nabla F|} = \frac{-\hat{i}\dfrac{\partial f}{\partial x} - \hat{j}\dfrac{\partial f}{\partial y} + \hat{k}}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 + 1}}$$

5. find the component of the ΔS surface projected onto $\hat{k}$
from Trim 12.5 we know that

$$\Delta A = \cos\theta \Delta S$$

Note, when $\theta = 0 \quad \Rightarrow \quad \Delta A = \Delta S$ (this is the surface parallel to the xy plane.





In general,

$$\Delta A = \underbrace{\cos\theta}_{\hat{n}\cdot\hat{k}} \Delta S$$

$$\hat{n} \cdot \hat{k} = |\hat{n}||\hat{k}| \cos\theta = \cos\theta$$

$$\Delta A = \Delta S(\hat{n} \cdot \hat{k}) = \Delta S \frac{1}{|\nabla F|}$$

since $\hat{n} \cdot \hat{k}$ produces a numerator of $\hat{k} \cdot \hat{k} = 1$ and a denominator of $|\nabla F|$

Rearranging the above equation, we can solve for ΔS. In the limit

$$dS = \underbrace{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}_{|\nabla F|} \underbrace{dxdy}_{dA}$$

Given the surface $z = f(x, y)$, the surface area is

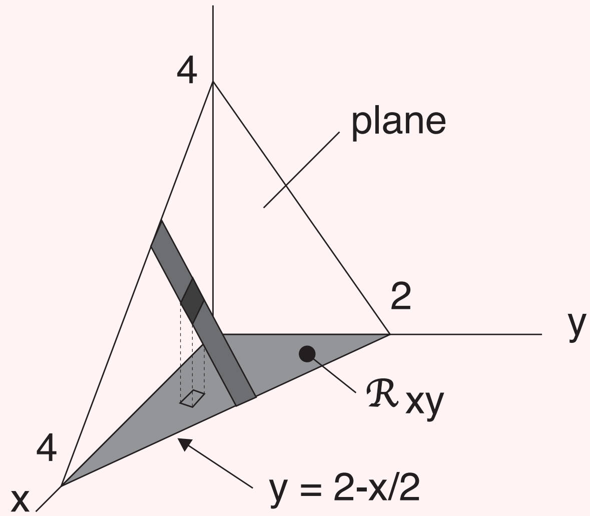
$$\boxed{S = \int\int_{\mathcal{R}_{xy}} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy}$$

where $\mathcal{R}_{xy}$ is the projection of the $f(x, y)$ surface down onto the (x, y) plane.

While this is the most common form of the equation, we could also find S by projecting onto another coordinate plane. Sometimes it is more convenient to do it this way. See Trim 14.6 for applicable equations.

Example: 3.7

Find the surface area in the +'ve octant for $z = f(x, y) = 4 - x - 2y$.



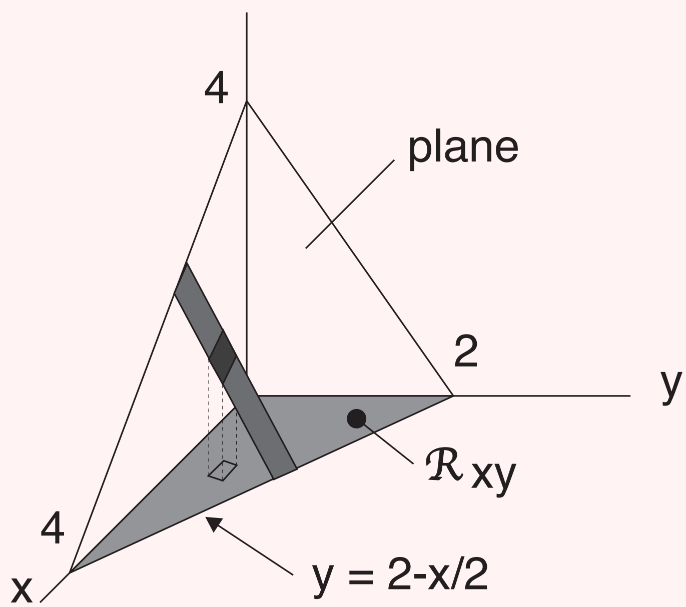
Example: 3.8

Given the sphere, $x^2 + y^2 + z^2 = a^2$, derive the formula for surface area.

Example: 3.9

Find the volume formed in the +'ve octant between the coordinate planes and the surface

$$z = f(x, y) = 4 - x - 2y$$

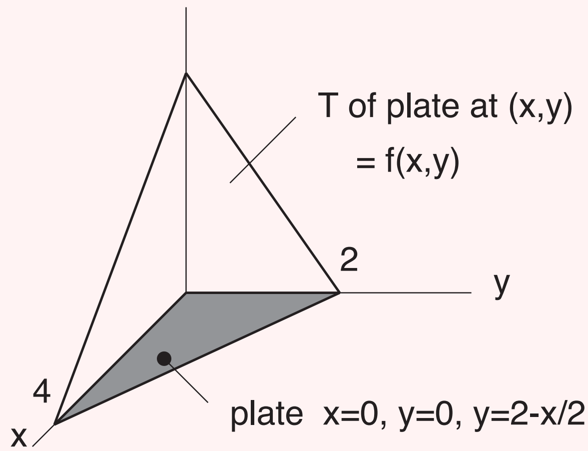


Example: 3.10a

Find the mean value of $y = f(x) = \sin x$ in the domain $x = 0$ to $x = \pi$.

Example: 3.10b

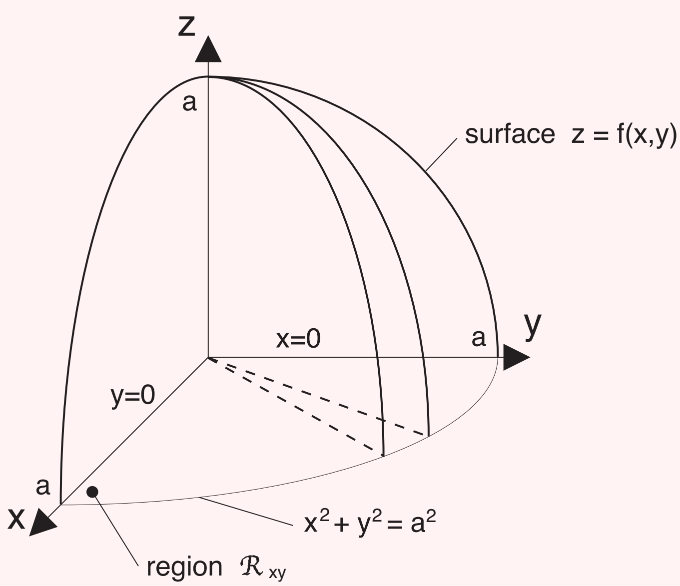
Find the mean value of temperature for $T = f(x, y) = 4 - x - 2y$.



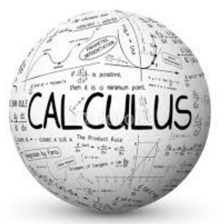
Example: 3.10c

Derive the formula for the volume of revolution. for the following sphere:

$x^2 + y^2 + z^2 = a^2$.



Triple Integrals



Reading
Trim 13.8 ⟶ *Triple Integrals and Triple Iterated Integrals*
13.9 ⟶ *Volumes*

Assignment
web page ⟶ *assignment #8*

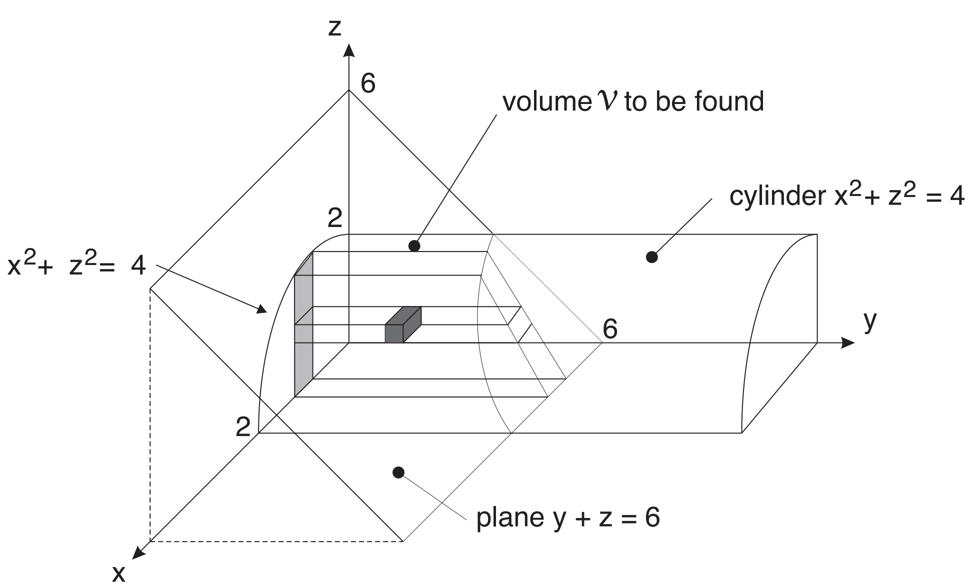
Volume Calculations in Cartesian Coordinates

The triple integral can be identified as

$$\int\int\int_{\mathcal{V}} \underbrace{dx\,dy\,dz}_{d\mathcal{V}\,-\,volume\ element} \qquad \text{or} \qquad \int\int\int_{\mathcal{V}} f\,dx\,dy\,dz$$

add up the $d\mathcal{V}$ elements in x, y, z directions, i.e. a triple sum.

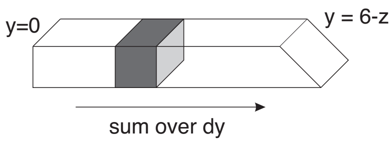
Consider the solid defined by $x^2 + z^2 = 4$ in the positive octant. Find the volume of this solid between the coordinate planes and the plane $y + z = 6$.



Start with a volume element at arbitrary (x, y, z) in space inside $d\mathcal{V}$

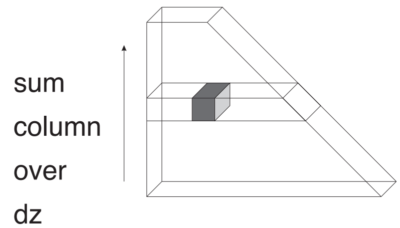
$$d\mathcal{V} = dx\, dy\, dz$$

Build up a column - sum over y keeping x, z constant.



$$\text{column volume} = \left(\int_{y=0}^{6-z} dy\right) dxdz$$

Build up a slice - sum columns over z, keeping y, x fixed.



$$\text{slice volume} = \left[\int_{z=0}^{\sqrt{4-x^2}} \left(\int_{y=0}^{6-z} dy\right) dz\right] dx$$

Finally sum the slices over x

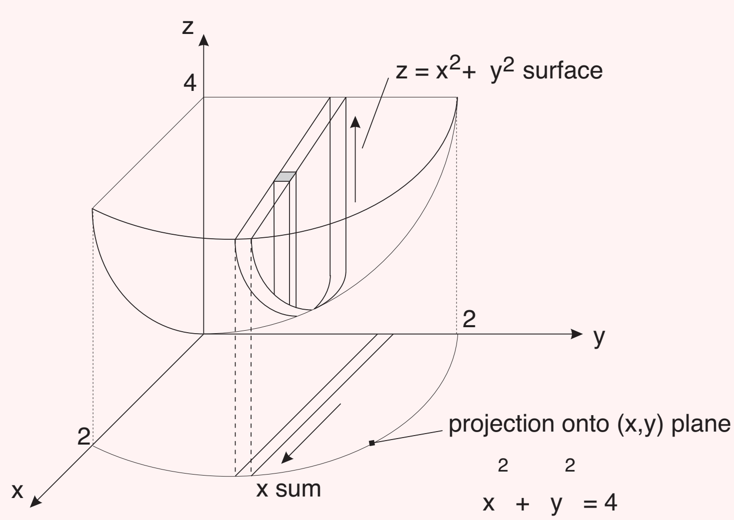
$$\mathcal{V} = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{6-z} dydzdx$$

Evaluation of the integral gives

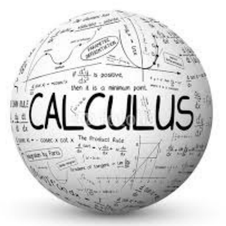
$$\begin{aligned}\mathcal{V} &= \int_0^2 \int_0^{\sqrt{4-x^2}} (6-z)dzdx = \int_0^2 6\sqrt{4-x^2} - \frac{(\sqrt{4-x^2})^2}{2} dx \\ &= 6\int_0^2 \sqrt{4-x^2}dx - \frac{1}{2}\int_0^2 (4-x^2)dx = 6\pi - \frac{8}{3} \approx 16.18 \qquad \text{use tables if necessary}\end{aligned}$$

Example: 3.11

Find the volume of the paraboloid, $z = x^2 + y^2$ for $0 \leq z \leq 4$. Consider only the +'ve octant, i.e. $1/4$ of the volume.



Volume Calculations in Cylindrical and Spherical Coordinates



Reading

Trim 13.11 $\longrightarrow$ *Triple Iterated Integrals in Cylindrical Coordinates*

13.12 $\longrightarrow$ *Triple Iterated Integrals in Spherical Coordinates*

Assignment

web page $\longrightarrow$ *assignment #8*

Cylindrical Coordinates

point: $\boldsymbol{P(r,\theta,z)}$ i.e. polar in $\boldsymbol{x,y}$ plane plus $\boldsymbol{z}$

volume element: $\boldsymbol{d\mathcal{V} = r\ dr\ d\theta\ dz}$

based on links to Cartesian coordinates

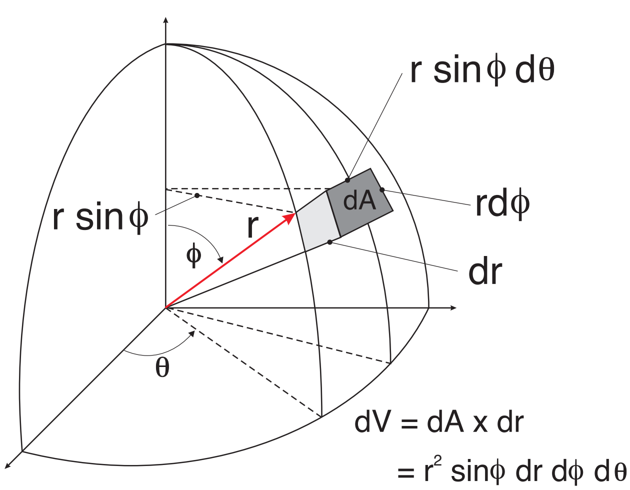
$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}(y/x) \\ z &= z \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

where $\boldsymbol{0 \le r, z \le \infty}$ and $\boldsymbol{0 \le \theta \le 2\pi}$

Typically we build up column, wedge slice and then the total volume, given as $\int\int\int \boldsymbol{r dr d\theta dz}$

The math operations are easier when we have axi-symmetric systems, i.e. cylinders and cones

Spherical Coordinates



point: $\boldsymbol{P(r,\theta,\phi)}$

volume element: $\boldsymbol{d\mathcal{V} = \underbrace{(r\sin\phi d\phi)}_{height}\underbrace{rdrd\theta}_{area}}$

based on links to Cartesian coordinates

$$
\begin{aligned}
r &= \sqrt{x^2+y^2+z^2} \\
\theta &= \tan^{-1}(y/x) \\
\phi &= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)
\end{aligned}
\quad \text{or} \quad
\begin{aligned}
x &= r\sin\phi\cos\theta \\
y &= r\sin\phi\sin\theta \\
z &= r\cos\phi
\end{aligned}
$$

where $0 \le r \le \infty$; $0 \le \theta \le 2\pi$; $0 \le \phi \le \pi$. Note: for $0 \le \phi \le \pi$ the $sin\phi$ is always +'ve for $d\mathcal{V}$ +'ve.

The solution procedure involves building up columns, slices as before to obtain the total volume, given as

$$\int\int\int r^2 \ \sin\ \phi \, dr \, d\theta \, d\phi$$

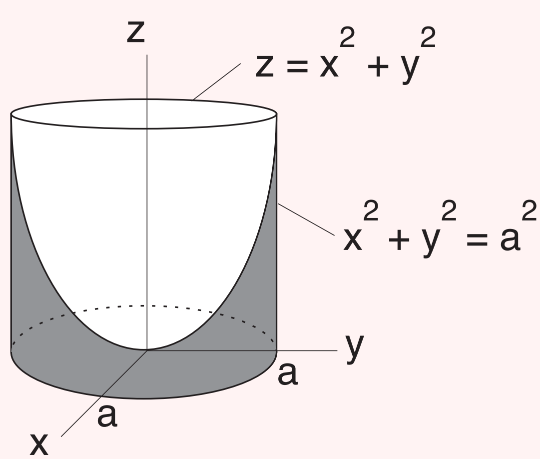
Example: 3.12

Find the volume bounded by a cylinder,

$$x^2 + y^2 = a^2$$

and a paraboloid,

$$z = x^2 + y^2$$



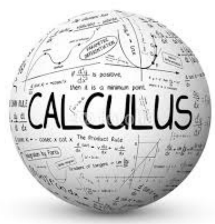
Spherical Coordinate Example

Example: 3.13

Derive a formula for the volume of a sphere with radius, a

$$x^2 + y^2 + z^2 = a^2$$

Moments of Area/ Mass / Volume



Reading

Trim 13.5 ⟶ *Centres of Mass and Moments of Inertia*

13.10 ⟶ *Centres of Mass and Moments of Inertia*

Assignment

web page ⟶ *assignment #9*

Centroids, Centers of Mass etc.

2-D case: thin plate of constant thickness

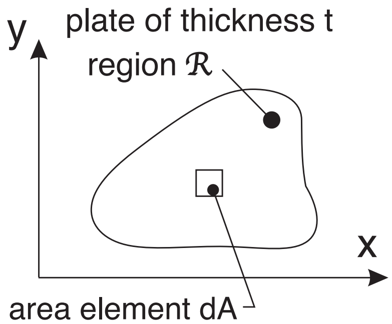
Sometimes, single integrals work, as in a 2-D case, where the thickness is given as t and is constant or a function of position as $t(x, y)$. The material density is given as $\rho\ (kg/m^3)$, again constant or a function of position as $\rho(x, y)$. We sometimes use the mass per unit area of the plate, $\rho^* = \rho \cdot t\ (kg/m^2)$.

	area	**mass**
basic element	$dA = dx\ dy$	$dM = \rho\, t\ dx\ dy$ or $\rho^*\ dx\ dy$
total area	$A = \int\int_{\mathcal{R}} dx\ dy$	$M = \int\int_{\mathcal{R}} dM = \int\int_{\mathcal{R}} \rho\, t\ dx\ dy$
	first moment of area	**first moment of mass**
	(*weight by distance from axis*)	
about $y-$ axis	$x\ dA = x\ dx\ dy$	$x\ dM = x\rho\, t\ dx\ dy$
total	$F_y = \int\int_{\mathcal{R}} x\ dA$	$\int\int_{\mathcal{R}} x\ dM$
about $x-$ axis	$F_x = \int\int_{\mathcal{R}} y\ dA$	$\int\int_{\mathcal{R}} y\ dM$
	centroid coordinates	**center of mass coordinates**
	$\overline{x} = \dfrac{\int\int_{\mathcal{R}} x\ dA}{A}$	$\overline{x_c} = \dfrac{\int\int_{\mathcal{R}} x\ dM}{M}$
	$\overline{y} = \dfrac{\int\int_{\mathcal{R}} y\ dA}{A}$	$\overline{y_c} = \dfrac{\int\int_{\mathcal{R}} y dM}{M}$
second moments		
	$\int\int_{\mathcal{R}} x^2\ dA$	$\int\int_{\mathcal{R}} x^2\ dM$
	$\int\int_{\mathcal{R}} y^2\ dA$	$\int\int_{\mathcal{R}} y^2\ dM$

3-D case:

We use the same basic ideas but the basic element is now $\mathcal{V} = dxdydz$

2-D Objects



Quantities of interest in applications such as dynamics.
Area: $A = \int\int_{\mathcal{R}} dA \; (Volume = tA)$

Mass: $M = \int\int_{\mathcal{R}} \rho(x,y) t dA$
where $\rho(x,y)$ = **density of material** in (kg/m^3) at point (x,y)

Centroid = "geometrical center" of object

$\overline{x} = \frac{\int\int_{\mathcal{R}} x dA}{A}$ **1st moment of area** about $y-$ axis

$\overline{y} = \frac{\int\int_{\mathcal{R}} y dA}{A}$ **1st moment of area** about $x-$ axis

---

Center of Mass: useful in dynamics problems

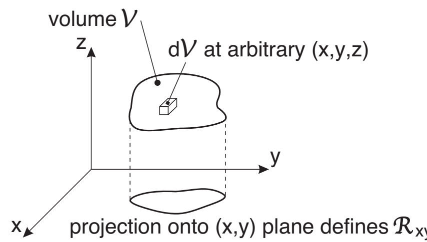
$$\overline{x_c} = \frac{\int\int_{\mathcal{R}} x dm}{M} = \frac{\int\int_{\mathcal{R}} x\rho(x,y) t dA}{M}$$

$$\overline{y_c} = \frac{\int\int_{\mathcal{R}} y dm}{M} = \frac{\int\int_{\mathcal{R}} y\rho(x,y) t dA}{M}$$

Note: that if the object density is uniform, then the centroid and center of mass are the same.

2nd Moments of Area and Mass:
⟶ Moments of Inertia
2nd moment of area about: $y - axis$
$I_y = \int\int_{\mathcal{R}} x^2 dA$
2nd moment of mass about: $y - axis$
$I_y = \int\int_{\mathcal{R}} x^2 \rho(x,y) t dA$
(similar formulas for I_x about the $x - axis$)

3-D Objects



Quantities of interest in applications such as dynamics.
Volume: $\mathcal{V} = \int\int\int_{\mathcal{V}} d\mathcal{V}$

Mass: $M = \int\int\int_{\mathcal{V}} \rho(x,y,z) d\mathcal{V}$
where $\rho(x,y,z)$ = **density of material** in (kg/m^3) at point (x,y,z)

Centroid = "geometrical center" of object

$\overline{x} = \frac{\int\int\int_{\mathcal{V}} x d\mathcal{V}}{\mathcal{V}}$ **1st moment of volume** about $y - z$ plane

$\overline{y} = \frac{\int\int\int_{\mathcal{V}} y d\mathcal{V}}{\mathcal{V}}$ **1st moment of volume** about $x - z$ plane

$\overline{z} = \frac{\int\int\int_{\mathcal{V}} z d\mathcal{V}}{\mathcal{V}}$ **1st moment of volume** about $x - y$ plane

---

Center of Mass: useful in dynamics problems

$$\overline{x_c} = \frac{\int\int\int_{\mathcal{V}} x\rho(x,y,z) d\mathcal{V}}{M}$$

similar formulas for $\overline{y}_c$ and $\overline{z}_c$

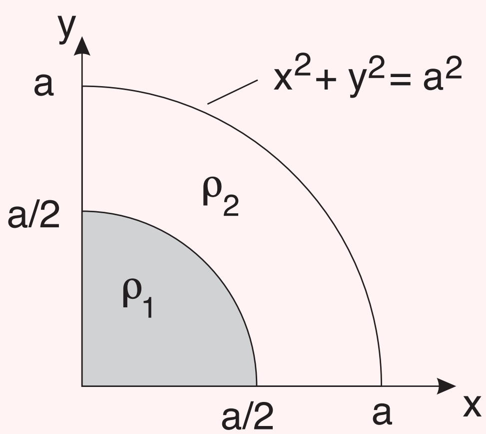
2nd Moments of Area and Mass:
⟶ Polar Moments of Inertia
volume moment about: $y - axis$
$J_y = \int\int\int_{\mathcal{V}} (x^2 + z^2) d\mathcal{V}$
mass moment about: $y - axis$
$J_y = \int\int\int_{\mathcal{V}} (x^2 + z^2)\rho(x,y,z) d\mathcal{V}$
(similar formulas for J_x about the $x - axis$)

and

(similar formulas for J_z about the $z - axis$)

Example: 3.14

Find the centroid, center of mass and the 1st moment of mass for a quarter circle of radius a with an inner circle of radius $a/2$ made of lead with a density of $\rho_1 = 11,000\ kg/m^3$ and an outer circle of radius a made aluminum with a density of $\rho_2 = 2,500\ kg/m^3$. The thickness is uniform throughout at $t = 10\ mm$.



$$\rho_1^* = 11\ g/cm^2 = 110\ kg/m^2$$
$$\rho_2^* = 2.5\ g/cm^2 = 25\ kg/m^2$$

Example: 3.15

Find the area of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$

Example: 3.16

Find the moment of inertia about the $y-$ axis of the area enclosed by the cardioid $r = a(1 - \cos\theta)$

Example: 3.17

Find the center of gravity of a homogeneous solid hemisphere of radius a