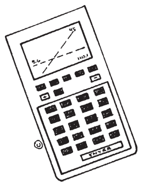

$\mathcal{ME}$ 201 ADVANCED CALCULUS

Assignment 11: *Surface Integrals, Divergence and Stoke's Theorem*

March 28, 2018

1. Evaluate the surface integrals of the following scalar functions:

   (a) $\displaystyle\int\int_S (x^2+y^2)z\,dS$ where S is that part of $z = x+y$ cut out by $x=0$, $y=0,\ x+y=1$.

   ANSWER: $2\sqrt{3}/15$

   (b) $\displaystyle\int\int_S xy\,dS$ where S is the first octant part of $z=\sqrt{x^2+y^2}$ cut out by $x^2+y^2=1$.

   ANSWER: $\sqrt{2}/8$

   (c) $\displaystyle\int\int_S z\,dS$ where S is that part of the surface $x^2+y^2-z^2=1$ between the planes $z=0$ and $z=1$.

   ANSWER: $\left(\dfrac{\pi}{3}\right)\left(3\sqrt{3}-1\right)$

2. Evaluate the surface integrals of the following vector functions:

   (a) $\displaystyle\int\int_S (yz^2\hat{i}+ye^x\hat{j}+x\hat{k})\cdot\hat{n}\,dS$ where S is defined by $y=x^2,\ 0\le y\le 4$, $0\le z\le 1$ and $\hat{n}$ is the unit vector to S with positive $y-$component.

   ANSWER: $2e^2-10e^{-2}$

   (b) $\displaystyle\int\int_S (x\hat{i}+y\hat{j})\cdot\hat{n}\,dS$ where S is that part of the surface $z=\sqrt{x^2+y^2}$ below $z=1$ and $\hat{n}$ is the unit normal to S with negative $z-$component.

   ANSWER: $2\pi/3$

   (c) $\displaystyle\int\int_S (x\hat{i}+y\hat{j})\cdot\hat{n}\,dS$ where S is the surface $x^2+y^2+z^2=4,\ z\ge 1$ and $\hat{n}$ is the unit upper normal to S.

   ANSWER: $10\pi/3$

3. A circular tube $S\ :\ x^2+z^2=1,\ 0\le y\le 2$ is a model for an artery. Blood flows through the artery and the force per unit area at any point on the arterial wall is given by:

$$\vec{F}=e^{-y}\hat{n}+\frac{1}{y^2+1}\hat{j}$$

where $\hat{n}$ is the unit outer normal to the arterial wall. Blood diffuses through the wall in such a way that if dS is a small area on S, the amount of diffusion through dS in 1 $second$ is $\vec{F}\cdot\hat{n}\ dS$. Find the total amount of blood leaving the entire wall per second.

ANSWER: $2\pi(1-e^{-2})$

4. Use the divergence theorem to evaluate the following surface integrals:

   (a) $\oint\oint_S (x^2\hat{i}\ +\ y^2\hat{j}\ +\ z^2\hat{k})\cdot\hat{n}\ dS$ where S is the sphere $x^2+y^2+z^2=a^2$ and $\hat{n}$ is the unit outer normal to S.

   ANSWER: 0

   (b) $\oint\oint_S (x\hat{i}+y\hat{j}+2z\hat{k})\cdot\hat{n}\ dS$ where S is the surface bounded by the volume defined by the surfaces $z=2x^2+y^2,\ x^2+y^2=3,\ z=0$ and $\hat{n}$ is the unit outer normal to S.

   ANSWER: 27π

   (c) $\oint\oint_S (xy\hat{i}\ +\ z^2\hat{k})\cdot\hat{n}\ dS$ where S is the surface enclosing the volume in the first octant bounded by the planes $z=0,\ y=x,\ y=2x, x+y+z=6$ and $\hat{n}$ is the unit outer normal to S.

   ANSWER: $57/2$

5. Use Stoke's theorem to evaluate the following integrals:

   (a) $\oint_C y^2\ dx\ +\ xy\ dy\ +\ xz\ dz$ where C is the curve $x^2+y^2=2y,\ y=z$ directed so that y increases when x is positive.

   ANSWER: 0

   (b) $\oint_C y\ dx+z\ dy+x\ dz$ where C is the curve $x+y=2b,\ x^2+y^2+z^2=2b(x+y)$ directed clockwise as viewed from the origin.

   ANSWER: $-2\sqrt{2}\pi b^2$

   (c) $\oint_C -2y^3x^2\ dx\ +\ x^3y^2\ dy\ +\ z\ dz$ where C is the curve $x^2+y^2+z^2=4,\ x^2+4y^2=4$ directed so that x decreases along that part of the curve in the first octant.

   ANSWER: 3π

6. Use the divergence theorem to evaluate

$$\oint\oint_S [xz^2\hat{\imath} \;+\; (x^2y - z^3)\hat{\jmath} \;+\; (2xy + y^2z)\hat{k}] \cdot \hat{n}\, dS$$

where S is the entire surface of the hemispherical region bounded by:

$$z = \sqrt{a^2 - x^2 - y^2}, \;\; z = 0$$

and $\hat{n}$ is the outward directed normal. ANSWER: $\dfrac{2\pi a^5}{5}$

7. Let C be the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z^2 = x^2 + y^2$ directed in the clockwise sense around the $z-$axis. For the vector field:

$$\vec{F} = (x^2 + z)\hat{\imath} \;+\; (y^2 + 2x)\hat{\jmath} \;+\; (z^2 - y)\hat{k}$$

Use Stoke's theorem to evaluate the line integral

$$\oint_C \vec{F} \cdot d\vec{r}$$

ANSWER: π