Assignment #3

Section 2.5

#3) $(y^2+2xy)\,dx - x^2\,dy = 0$

⇒ not seperable due to $2xy$ term

⇒ not linear due to y^2 term

⇒ check exactness: $M = y^2+2xy$ $\quad N = -x^2$; $\frac{\partial M}{\partial y} = 2y+2x$ $\quad \frac{\partial N}{\partial x} = -2x$ $\Big\}$ since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ the equation is not exact

⇒ check integrating factor of either x or y alone:

check: $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{2y+2x+2x}{-x^2} = \dfrac{2y+4x}{-x^2}$ $\Big\}$ Since not only a fcn of x, integrating factor does not apply

check: $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \dfrac{-2x-2y-2x}{y^2+2xy} = \dfrac{-2(y+2x)}{y(y+2x)} = \underbrace{\dfrac{-2}{y}}$

Since this is a fcn of only y apply the integrating factor $u = \exp\left(\int \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\, dy\right)$

#7) $(3x^2+y)\,dx + (x^2y - x)\,dy = 0$

$M = 3x^2+y$ $\quad N = x^2y - x$; $\frac{\partial M}{\partial y} = 1$ $\quad \frac{\partial N}{\partial x} = 2xy - 1$ $\Big\}$ ∴ not exact.

Check: $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{1-2xy+1}{x^2y - x} = \dfrac{2(1-xy)}{-x(1-xy)} = \dfrac{-2}{x}$

∴ apply integrating factor $u = \exp\left(\int \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\, dx\right) = \exp\left(\int \frac{-2}{x}\, dx\right)$

$= e^{-2\ln x} = x^{-2}$

∴ $\left(3 + \frac{y}{x^2}\right)dx + \left(y - \frac{1}{x}\right)dy = 0$

$M = 3 + \frac{y}{x^2}$ $\quad N = y - \frac{1}{x}$; $\frac{\partial M}{\partial y} = \frac{1}{x^2}$ $\quad \frac{\partial N}{\partial x} = \frac{1}{x^2}$ $\Big\}$ ∴ exact

$$F = \int M\,dx + g(y) = \int 3 + \frac{y}{x^2}\,dx + g(y) = 3x - \frac{y}{x} + g(y)$$

$$F = \int N\,dy + h(x) = \int y - \frac{1}{x}\,dy + h(x) = \frac{y^2}{2} - \frac{y}{x} + h(y)$$

Combine: $F = -\frac{y}{x} + 3x + \frac{y^2}{2} = C$

$$\therefore\ y^2 - \frac{2}{x}y + 6x - 2C = 0$$

Solve the quadratic:

$$y = \frac{\frac{2}{x} \pm \sqrt{\frac{4}{x^2} - 4(6x - 2C)}}{2}$$

$$y = \frac{\frac{2}{x} \pm 2\sqrt{\frac{1 - 6x^3 + 2Cx^2}{x^2}}}{2}$$

$$\boxed{y = \frac{1}{x} \pm \frac{1}{x}\sqrt{1 - 6x^3 + 2Cx^2}}$$

Check: (use + case)

$$\frac{dy}{dx} = \frac{-1}{x^2} - \frac{\sqrt{1 - 6x^3 + 2Cx^2}}{x^2} + \frac{(-18x^2 + 4Cx)}{2x\sqrt{1 - 6x^3 + 2Cx^2}}$$

$$\frac{dy}{dx} = \frac{-2\sqrt{1 - 6x^3 + 2Cx^2} - 2(1 - 6x^3 + 2Cx^2) - 18x^3 + 4Cx^2}{2x^2\sqrt{1 - 6x^3 + 2Cx^2}}$$

$$\frac{dy}{dx} = \frac{-\sqrt{1 - 6x^3 + 2Cx^2} - 1 + 6x^3 - \cancel{2Cx^2} - 9x^3 + \cancel{2Cx^2}}{x^2\sqrt{1 - 6x^3 + 2Cx^2}}$$

$$\boxed{\frac{dy}{dx} = \frac{-\sqrt{1 - 6x^3 + 2Cx^2} - 1 - 3x^3}{x^2\sqrt{1 - 6x^3 + 2Cx^2}}}$$

Rearrange original ODE: $\frac{dy}{dx} = \frac{-(3x^2 + y)}{(x^2 y - x)}$

LS	RS
$\Rightarrow \frac{dy}{dx}$	$\Rightarrow \frac{-(3x^2 + \frac{1}{x} + \frac{1}{x}\sqrt{1 - 6x^3 + 2Cx^2})}{x^2(\frac{1}{x} + \frac{1}{x}\sqrt{1 - 6x^3 + 2Cx^2}) - x}$
$\Rightarrow \frac{-\sqrt{1 - 6x^3 + 2Cx^2} - 1 - 3x^3}{x^2\sqrt{1 - 6x^3 + 2Cx^2}}$	$\Rightarrow \frac{-3x^2 - \frac{1}{x} - \frac{1}{x}\sqrt{1 - 6x^3 + 2Cx^2}}{x\sqrt{1 - 6x^3 + 2Cx^2}}$
	$\Rightarrow \frac{-3x^3 - 1 - \sqrt{1 - 6x^3 + 2x^2}}{x^2\sqrt{1 - 6x^3 + 2Cx^2}}$

LS = RS so the solution is correct

#13) $(2y^2 - 6xy)\,dx + (3xy - 4x^2)\,dy = 0$

Use integrating factor $x^n y^m$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}\left[(2y^2 - 6xy)(x^n y^m)\right] = \frac{\partial}{\partial x}\left[(3xy - 4x^2)(x^n y^m)\right]$$

$$\frac{\partial}{\partial y}\left(2x^n y^{m+2} - 6x^{n+1} y^{m+1}\right) = \frac{\partial}{\partial x}\left(3x^{n+1} y^{m+1} - 4x^{n+2} y^m\right)$$

$$(m+2)(2x^n y^{m+1}) - (m+1)(6x^{n+1} y^m) = (n+1)(3x^n y^{m+1}) - (n+2)(4x^{n+1} y^m)$$

$$\therefore \quad (m+2)(2x^n y^{m+1}) = (n+1)(3x^n y^{m+1})$$

$$x^n y^{m+1}(2m+4) = x^n y^{m+1}(3n+3)$$

$$2m+4 = 3n+3$$

$$2m = 3n - 1 \quad ①$$

$$\therefore \quad -(m+1)(6x^{n+1} y^m) = -(n+2)(4x^{n+1} y^m)$$

$$x^{n+1} y^m(-6m-6) = x^{n+1} y^m(-4n-8)$$

$$3m+3 = 2n+4$$

$$m = \frac{2}{3}n + \frac{1}{3} \quad ②$$

Sub ① → ②

$$2\left(\frac{2}{3}n + \frac{1}{3}\right) = 3n - 1$$

$$\frac{4}{3}n + \frac{2}{3} = 3n - 1$$

$$-\frac{5}{3}n = -\frac{5}{3}$$

$$\boxed{n = 1}$$

Sub $n=1$ into ② $\quad m = \frac{2}{3} + \frac{1}{3} = 1$

$$\boxed{\therefore m = 1}$$

$\therefore$ The integrating factor is $u = xy$.

Solve:

$M = 2xy^3 - 6x^2y^2$

$N = 3x^2y^2 - 4yx^3$

$F = \int M\,dx + g(y) = x^2y^3 - 2x^3y^2 = C$

$F = \int N\,dy + h(x) = x^2y^3 - 2x^3y^2 = C$

$\boxed{\therefore\ x^2y^3 - 2x^3y^2 = C}$ ← This can be solved for y and then checked, but it is long and tedious and will not be shown.

Section 2.6

#13) $\dfrac{dx}{dt} = \dfrac{x^2 + t\sqrt{t^2+x^2}}{tx}$

$\dfrac{dx}{dt} = \dfrac{x^2 + t\sqrt{t^2\left(1+\frac{x^2}{t^2}\right)}}{tx}$

$\dfrac{dx}{dt} = \dfrac{x^2 + t^2\sqrt{1+\left(\frac{x}{t}\right)^2}}{tx}$

$\dfrac{dx}{dt} = \dfrac{x}{t} + \dfrac{t}{x}\sqrt{1+\left(\frac{x}{t}\right)^2}$

Sub in $z = \frac{x}{t} \Rightarrow x = zt \quad \therefore \frac{dx}{dt} = z + t\frac{dz}{dt}$

$\cancel{z} + t\dfrac{dz}{dt} = \cancel{z} + \dfrac{\sqrt{1+z^2}}{z}$

$\displaystyle\int \frac{z}{\sqrt{1+z^2}}\,dz = \int \frac{1}{t}\,dt$

Sub in $w = 1+z^2 \quad \therefore dw = 2z\,dz \Rightarrow dz = \frac{1}{2z}\,dw$

$\displaystyle\int \frac{\cancel{z}}{\sqrt{w}}\,\frac{1}{2\cancel{z}}\,dw = \ln|t| + C$

$\frac{1}{2}\left(2\sqrt{w}\right) = \ln|t| + C$

$\sqrt{w} = \ln|t| + C$

Sub back in for $w = 1+z^2$

$$\sqrt{1+z^2} = \ln|t| + c$$

Sub back in for $z = \frac{x}{t}$

$$\boxed{\sqrt{1+\left(\frac{x}{t}\right)^2} = \ln|t| + c}$$

← difficult to check, so will not be shown.

#23) $\frac{dy}{dx} = \frac{2y}{x} - x^2y^2$

$$\frac{dy}{dx} - \frac{2}{x}y = -x^2y^2$$

Sub in $v = y^{-1} = \frac{1}{y}$

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx} \qquad \frac{dy}{dx} = -y^2\frac{dv}{dx} = -\left(\frac{1}{v}\right)^2\frac{dv}{dx}$$

$$\therefore -\left(\frac{1}{v}\right)^2\frac{dv}{dx} - \frac{2}{x}\frac{1}{v} = -x^2\left(\frac{1}{v}\right)^2$$

$$-\frac{dv}{dx} - \frac{2}{x}v = -x^2$$

$$\frac{dv}{dx} + \frac{2}{x}v = x^2$$

Use integrating factor: $u = \exp\int\frac{2}{x}dx = \exp^{2\ln x} = x^2$

$$\therefore \frac{d}{dx}\left(x^2v\right) = x^2\left(x^2\right)$$

$$x^2v = \int x^4\,dx$$

$$x^2v = \frac{1}{5}x^5 + C$$

$$\frac{x^2}{y} = \frac{1}{5}x^5 + C$$

$$y = \frac{x^2}{\frac{1}{5}x^5 + c} = \frac{x^2}{\frac{x^5+5c}{5}} = \frac{5x^2}{x^5+5c}$$

$$\boxed{\therefore y = \frac{5x^2}{x^5+5c}}$$

$$\frac{dy}{dx} = \frac{10x}{x^5+5c} - \frac{25x^6}{(x^5+5c)^2}$$

$$\frac{dy}{dx} = \frac{10x^6 + 50xc - 25x^6}{(x^5+5c)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-15x^6 + 50xC}{(x^5+5c)^2}}$$

check:

LS

$\Rightarrow \frac{dy}{dx}$

$\Rightarrow \frac{-15x^6 + 50xC}{(x^5+5C)^2}$

RS

$\Rightarrow \frac{2y}{x} - x^2y^2$

$\Rightarrow \frac{10x}{x^5+5C} - \frac{25x^6}{(x^5+5C)^2}$

$\Rightarrow \frac{10x(x^5+5C) - 25x^6}{(x^5+5C)^2}$

$\Rightarrow \frac{-15x^6 + 50Cx}{(x^5+5C)^2}$

$\boxed{\therefore LS = RS}$

#42) $\frac{dy}{dx} = \frac{2y}{x} + \cos\left(\frac{y}{x^2}\right)$

sub in $y = vx^2$

$\frac{dy}{dx} = 2xv + x^2\frac{dv}{dx}$

$\cancel{2xv} + x^2\frac{dv}{dx} = \cancel{2vx} + \cos(v)$

$\int \frac{dv}{\cos v} = \int \frac{dx}{x^2}$

$\int \sec v \, dv = \frac{-1}{x} + C$

$\ln|\sec v + \tan v| = -\frac{1}{x} + C$

$\boxed{\ln\left|\sec\left(\frac{y}{x^2}\right) + \tan\left(\frac{y}{x^2}\right)\right| = -\frac{1}{x} + C}$

Assignment #3 Q3

$$\frac{dy}{dx} = \frac{2y - x + 5}{2x - y - 4}$$

Sub in $y = y^* - k$ and $x = x^* - h$

$dy = dy^*$ $\qquad dx = dx^*$

$$\frac{dy^*}{dx^*} = \frac{2y^* - 2k - x^* + h + 5}{2x^* - 2h - y^* + k - 4}$$

to make the equation homogeneous, make the RHS a function of only $\frac{y^*}{x^*}$

$$\frac{dy^*}{dx^*} = \frac{\frac{2y^*}{x^*} - 1 + \frac{h + 5 - 2k}{x^*}}{2 - \frac{y^*}{x^*} + \frac{k - 4 - 2h}{x^*}}$$

$\therefore h + 5 - 2k = 0 \Rightarrow h = 2k - 5$ ①

$\therefore k - 4 - 2h = 0$ ②

Sub ① → ② $\quad k - 4 - 2(2k - 5) = 0$

$k - 4 - 4k + 10 = 0$

$-3k = -6$

$$\boxed{\begin{aligned} &\therefore k = 2 \\ &\therefore h = -1 \end{aligned}}$$

check homogeneous:

$$\frac{dy^*}{dx^*} = \frac{2y^* - 2(2) - x^* - 1 + 5}{2x^* - 2(-1) - y^* + 2 - 4} = \frac{2y^* - x^*}{2x^* - y^*} = \frac{2\frac{y^*}{x^*} - 1}{2 - \frac{y^*}{x^*}}$$

$\boxed{\therefore \text{ The choice of constants is okay}}$

Assignment #3 Q 4

$$(x^2y + y^3)\,dy + x\,dx = 0$$

$M = x \qquad N = x^2y + y^3$

$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = 2xy$

Check $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{-2xy}{y(x^2+y^2)} = \dfrac{-2x}{x^2+y^2}$

check $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \dfrac{2xy}{x} = 2y$ $\qquad \therefore$ use $u = \exp\left(\int \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,dy\right)$

$$u = \exp\int 2y\,dy = e^{y^2}$$

rewrite original equation using integrating factor

$$(x^2ye^{y^2} + y^3e^{y^2})\,dy + xe^{y^2}\,dx = 0$$

$M = xe^{y^2} \qquad N = x^2ye^{y^2} + y^3e^{y^2}$

$\frac{\partial M}{\partial y} = 2xye^{y^2} \qquad \frac{\partial N}{\partial x} = 2xye^{y^2} \qquad \therefore$ exact

$$F = \int M\,dx + g(y) = \int xe^{y^2}\,dx + g(y) = \underline{\frac{x^2e^{y^2}}{2} + g(y)}$$

$$F = \int N\,dy + h(x) = \int x^2ye^{y^2} + y^3e^{y^2}\,dy + h(x)$$

$\left\{ \text{Sub in } z = y^2,\quad dz = 2y\,dy,\quad \therefore dy = \dfrac{dz}{2y} \right\}$

$$\therefore F = \int x^2\cancel{y}e^z\frac{dz}{2\cancel{y}} + \int z\cancel{y}e^z\frac{dz}{2\cancel{y}} + h(x)$$

$$F = \tfrac{1}{2}x^2\int e^z\,dz + \int ze^z\,dz + h(x)$$

$$F = \frac{x^2}{2}e^z + (z-1)e^z + h(x)$$

sub back in for y's. $\quad F = \dfrac{x^2e^{y^2}}{2} + (y^2-1)e^{y^2} + h(x)$

Combine F's...

$$\boxed{\therefore \frac{x^2e^{y^2}}{2} + (y^2-1)e^{y^2} = C}$$

Hilary

Assignment #3 Q5

$$x^2 \frac{dy}{dx} = (x^2+y^2)$$

$$\frac{dy}{dx} = 1 + \left(\frac{y}{x}\right)^2$$

$$\left\{ \begin{array}{l} \text{Sub in } v = \frac{y}{x} \\ y = xv \\ \frac{dy}{dx} = v + x\frac{dv}{dx} \end{array} \right\}$$

$$v + x\frac{dv}{dx} = 1 + v^2$$

$$\int \frac{dv}{1+v^2-v} = \int \frac{dx}{x}$$

$$\int \frac{dv}{v^2 - v + \frac{1}{4} - \frac{1}{4} + 1} = \ln|x| + C$$ (completing the square)

$$\int \frac{dv}{(v-\frac{1}{2})^2 + \frac{3}{4}} = \ln|x| + c$$ $\left(\text{this is of the form } \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)\right)$

$$\therefore \frac{4}{3}\tan^{-1}\left(\frac{(v-\frac{1}{2})3}{4}\right) = \ln|x| + c$$

$$\boxed{\frac{4}{3}\tan^{-1}\left(\frac{3}{4}\frac{y}{x} - \frac{3}{8}\right) = \ln|x| + c}$$