Assignment 7

Section 4.11

#5) $y''(t) + 10y'(t) + ky(t) = 0$ $\quad y(0)=1 \quad y'(0)=0$

$$r^2 + 10r + k = 0$$

Evaluate r: $r = \frac{-10 \pm \sqrt{100-4k}}{2}$

$\boxed{\text{if } k=20}$ $r = \frac{-10 \pm \sqrt{20}}{2} = -5 \pm \sqrt{5}$ $\therefore$ underdamped

$$\therefore y = Ae^{(-5+\sqrt{5})t} + Be^{(-5-\sqrt{5})t}$$

$y(0)=1 \quad \therefore 1 = A + B$

$y'(0)=0 \quad \therefore 0 = (-5+\sqrt{5})A + (-5-\sqrt{5})B$

$\Big\}$ $A = \frac{1+\sqrt{5}}{2}$, $B = \frac{1-\sqrt{5}}{2}$

$$\boxed{\therefore y = \frac{1+\sqrt{5}}{2} e^{(-5+\sqrt{5})t} + \frac{1-\sqrt{5}}{2} e^{(-5-\sqrt{5})t}}$$

$\boxed{\text{if } k=25}$ $r = \frac{-10 \pm \sqrt{100-100}}{2} = -5$ $\therefore$ critically damped

$$\therefore y = Ae^{-5t} + Bte^{-5t}$$

$y(0)=1 \quad \therefore 1 = A$

$y'(0)=0 \quad \therefore 0 = -5A + B$

$\Big\}$ $A = 1$, $B = 5$

$$\boxed{\therefore y = e^{-5t} + 5te^{-5t}}$$

$\boxed{\text{if } k=30}$ $r = \frac{-10 \pm \sqrt{100-120}}{2} = -5 \pm \sqrt{5}i$ $\therefore$ underdamped

$$\therefore y = e^{-5t}\left(A\cos(\sqrt{5}t) + B\sin(\sqrt{5}t)\right)$$

$y(0)=1 \quad \therefore 1 = A$

$y'(0)=0 \quad \therefore 0 = -5A + \sqrt{5}B$

$\Big\}$ $A = 1$, $B = \sqrt{5}$

$$\boxed{\therefore y = e^{-5t}\left(\cos(\sqrt{5}t) + \sqrt{5}\sin(\sqrt{5}t)\right)}$$

11) $m = 1\text{kg}$ $\qquad x(0) = 0$
$k = 100\ \text{N/m}$ $\qquad x'(0) = 1$
$b = 0.2\ \frac{\text{N}\cdot\text{sec}}{\text{m}}$

<u>check $b^2 - 4mk.$</u> $\qquad 0.2^2 - 4(1)(100) = -399.96 \qquad \therefore$ underdamped

$$\therefore \alpha = -\frac{b}{2m} = \frac{-0.2}{2(1)} = -0.1$$
$$\therefore \beta = \frac{1}{2m}\sqrt{4mk - b^2} = \frac{1}{2(1)}\sqrt{4(1)(100) - 0.2^2} = 9.9995 \approx 10$$

$$\therefore x = Ae^{-0.1t}\cos(10t) + Be^{-0.1t}\sin(10t)$$

$x(0) = 0 \qquad 0 = A \qquad \therefore A = 0$
$x'(0) = 1 \qquad 1 = -0.1A + 10B \qquad \therefore B = 0.1$

$$x = 0.1e^{-0.1t}\sin(10t)$$

<u>To find maximum displacement, examine x' and find critical point.</u>

$$0 = x' = -0.1^2 \cancel{e^{-0.1t}}\sin(10t) + \cancel{e^{-0.1t}}\cos(10t)$$
$$\cos(10t) = 0.1^2 \sin(10t)$$
$$100 = \tan(10t)$$
$$t = \frac{\tan^{-1}(100)}{10} \approx 0.156 \text{ seconds.}$$

$\boxed{\therefore \text{The block will attain its maximum velocity to the right when } t = 0.156 \text{ sec}}$

16) mass = m, period = 3 sec
mass = m + 2, period = 4 sec
Neglect damping and external forces. Find m.

$$\text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$\therefore \quad 3 = 2\pi \frac{\sqrt{m}}{\sqrt{k}} \quad ①$$
$$4 = 2\pi \frac{\sqrt{m+2}}{\sqrt{k}} \quad ②$$

① ÷ ②
$$0.75 = \frac{\sqrt{m}}{\sqrt{m+2}}$$
$$0.75^2 = \frac{m}{m+2}$$
$$(m+2)0.75^2 = m$$
$$1.125 = 0.4375m$$
$$m = 2.5714 \text{ kg.}$$

$\boxed{\therefore \text{The mass is } 2.5714 \text{ kg}}$

Section 4.12

3) $y'' + 9y = 2\cos 3t$ $\quad y(0) = 1 \quad y'(0) = 0$

homogenous: $y'' + 9y = 0$
$$r^2 + 9 = 0$$
$$r = \frac{\pm\sqrt{-4(9)}}{2} = \pm 3i$$

$$\therefore y_c = A\cos(3t) + B\sin(3t)$$

$y(0) = 1 \quad 1 = A$
$y'(0) = 0 \quad 0 = 3B$

$\left.\begin{matrix} \\ \end{matrix}\right\} \begin{matrix} \therefore A = 1 \\ \therefore B = 0 \end{matrix}$

$$\therefore y_c = \cos(3t)$$

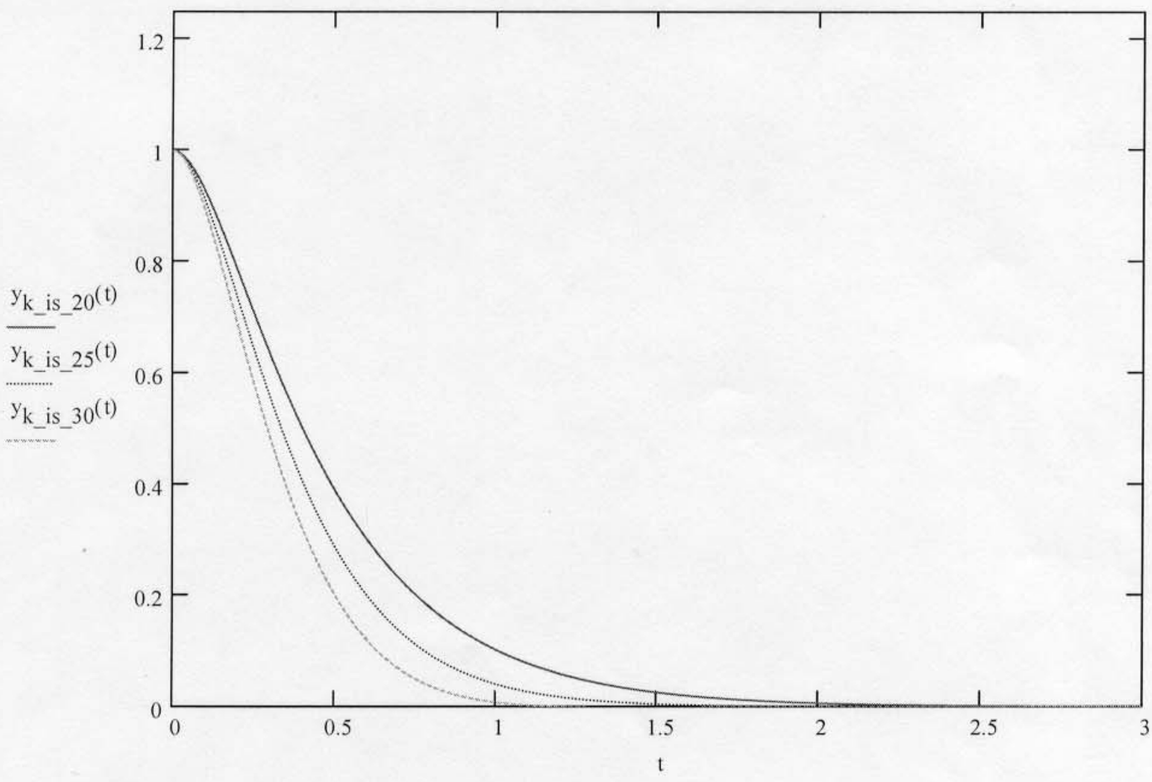
Section 4.11

Q5)

$$y_{k\_is\_20}(t) := \frac{\left(1+\sqrt{5}\right)}{2}\cdot e^{\left(-5+\sqrt{5}\right)\cdot t} + \frac{\left(1-\sqrt{5}\right)}{2}\cdot e^{\left(-5-\sqrt{5}\right)\cdot t}$$

$$y_{k\_is\_25}(t) := e^{-5t} + 5\cdot t\cdot e^{-5t}$$

$$y_{k\_is\_30}(t) := e^{-5t}\cdot\left(\cos\left(\sqrt{5}\,t\right) + \sqrt{5}\cdot\sin\left(\sqrt{5}\,t\right)\right)$$



Particular: $2\cos(3t)$ gives $y_p = C\cos 3t + D\sin 3t$ using the method of undetermined coefficients.

$\Rightarrow$ BUT... $C\cos(3t)$ is the same form as y_c !!

$$\therefore \quad y_p = t\,(C\cos 3t + D\sin 3t)$$
$$y_p' = C\cos 3t - 3Ct\sin 3t + D\sin 3t + 3Dt\cos 3t$$
$$y_p'' = -3C\sin 3t - 3C\sin 3t - 9Ct\cos 3t + 3D\cos 3t + 3D\cos 3t - 9Dt\sin 3t$$

Sub into $y'' + 9y = 2\cos 3t$

$$-6C\sin 3t + 6D\cos 3t - \cancel{9Ct\cos 3t} - \cancel{9Dt\sin 3t} + \cancel{9Ct\cos 3t} + \cancel{9Dt\sin 3t} = 2\cos 3t$$

$\therefore 6D = 2 \qquad D = \frac{1}{3}$

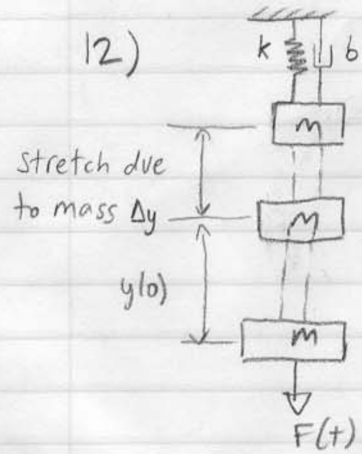
$\therefore -6C = 0 \qquad C = 0$

$$\therefore y_p = \frac{1}{3}t\sin 3t$$

$$\boxed{\therefore y = y_c + y_p = \cos(3t) + \frac{1}{3}t\sin(3t)}$$

This question is now fixed!

12)



$m = 2\,kg$

$\Delta y = 0.2\,m$

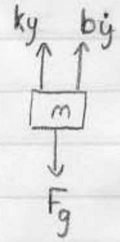
$y(0) = 0.05\,m$

$y'(0) = 0$

$F(t) = 0.3\cos t\ N$

$b = 5\ \frac{N\,sec}{m}$

Consider System before it is given 5 cm displacement, and before F(t) is applied.



→ System is at rest 0.2 m below height where the spring is unstretched.

$\therefore \dot{y} = 0$

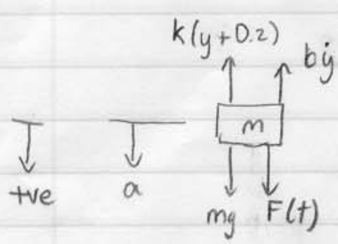
$\therefore y = 0.2$

$mg = ky$

$(2)(9.8) = k(0.2)$

$k = 98\ N/m$

Consider Full System



$mg - k(y+0.2) - b\dot{y} = ma + F(t)$

$ma + ky + 0.2k - mg + b\dot{y} = -F(t)$

$2\ddot{y} + 98y + \cancel{0.2(98)} - \cancel{2(9.8)} + 5\dot{y} = -0.3\cos t$

Examine homogeneous: $2y'' + 5y' + 98y = 0$

$2r^2 + 5r + 98 = 0$

$r = \left(-5 \pm \sqrt{25 - 4(98)(2)}\right) / 4$

$r = (-5 \pm 27.550i)/4$

$\therefore r = -1.25 \pm 6.8875i$

$Y_c = Ae^{-1.25t}\cos(6.8875t) + Be^{-1.25t}\sin(6.8875t)$

NOTE:
Only sub in initial conditions into the full Y solution! $(Y = Y_c + Y_p)$

~~$y(0) = 0.05 \qquad 0.05 = A$~~

~~$y'(0) = 0 \qquad 0 = -1.25A + 6.8875B$~~

~~$\therefore A = 0.05, \quad B = 0.009$~~

~~$\therefore Y_c = 0.05e^{-1.25t}\cos(6.8875t) + 0.009e^{-1.25t}\sin(6.8875t)$~~

Particular Solution $\qquad RHS = -0.3\cos t$

$$\therefore y_p = C\cos t + D\sin t$$
$$y_p' = -C\sin t + D\cos t$$
$$y_p'' = -C\cos t - D\sin t$$

Sub into $\qquad 2y'' + 5y' + 98y = -0.3\cos t$

$$2(-C\cos t - D\sin t) + 5(-C\sin t + D\cos t) + 98(C\cos t + D\sin t) = -0.3\cos t$$
$$(-2C + 5D + 98C)\cos t + (-2D - 5C + 98D)\sin t = -0.3\cos t$$

$\therefore 96C + 5D = -0.3 \quad (1)$

$\therefore 96D - 5C = 0 \quad (2)$

$D = -1.62\times10^{-4}$, $C = -3.12\times10^{-3}$

$$\boxed{\therefore Y_p = -3.12\times10^{-3}\cos t - 1.62\times10^{-4}\sin t}$$

$$\therefore Y = Y_p + Y_c = -3.12\times10^{-3}\cos t - 1.62\times10^{-4}\sin t + Ae^{-1.25t}\cos(6.8875t) + Be^{-1.25t}\sin(6.8875t)$$

Sub in initial conditions:

$y(0) = 0.05 \qquad 0.05 = -3.12\times10^{-3} - \cancel{1.62\times10^{-4}(0)} + A + \cancel{B(0)}$

$0.05312 = A$

$y'(0) = 0 \qquad 0 = 3.12\times10^{-3}\sin(0) - 1.62\times10^{-4}\cos(0) - 1.25Ae^{0}\cos(0) - 6.8875Ae^{0}\sin(0) - 1.25Be^{0}\sin(0) + 6.8875Be^{0}\cos(0)$

$1.62\times10^{-4} + 1.25(0.05312) = 6.8875B$

$B = 0.00966$

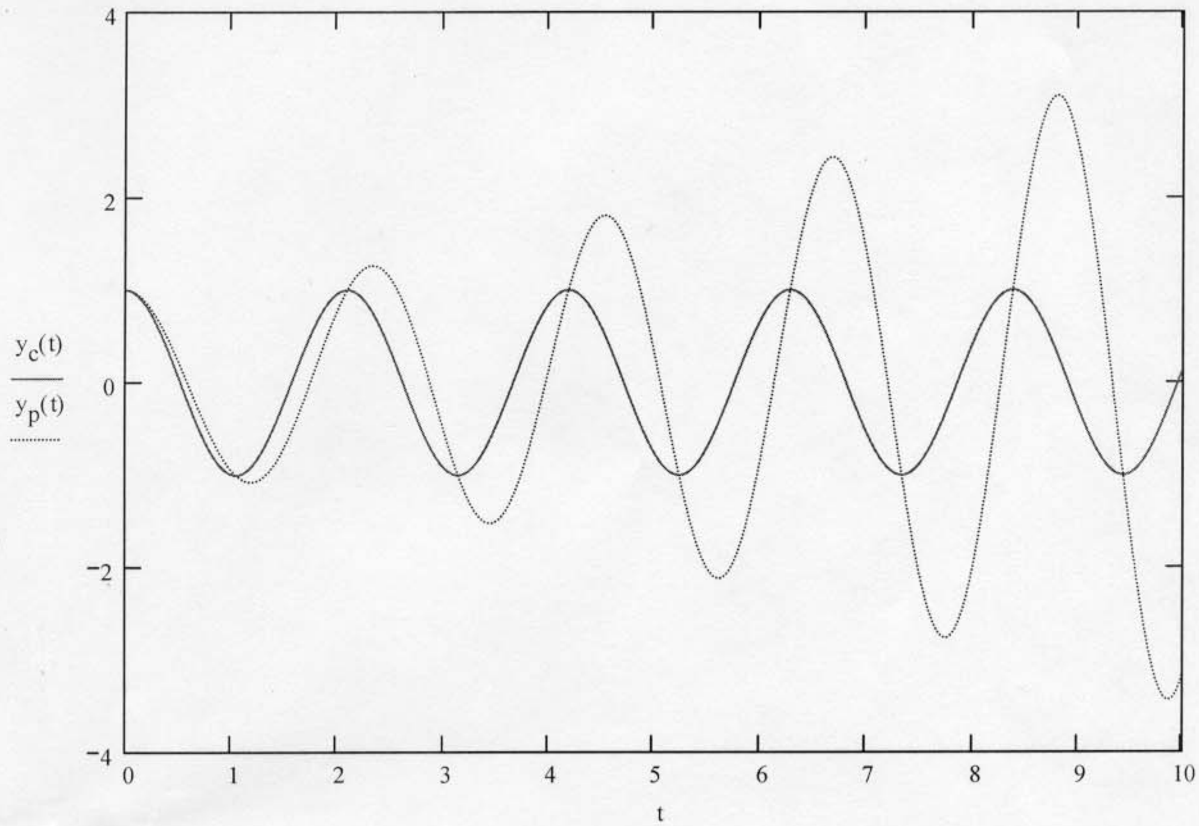
$$\boxed{\therefore\ Y = -3.12\times10^{-3}\cos t - 1.62\times10^{-4}\sin t + 0.05312\,e^{-1.25t}\cos(6.8875t) + 0.00966\,e^{-1.25t}\sin(6.8875t)}$$

Resonant Frequency $= \dfrac{\gamma_r}{2\pi} = \dfrac{\sqrt{\frac{k}{m} - \frac{b^2}{2m}}}{2\pi} = \dfrac{\sqrt{\frac{98}{2} - \frac{5^2}{2(2)^2}}}{2\pi} = 1.149$

Section 4.12

Q3) $y_c(t) := \cos(3t)$

$$y_p(t) := \cos(3t) + \frac{1}{3}\cdot t\cdot \sin(3t)$$



Q12) $y(t) := 0.05312\cdot e^{-1.25t}\cdot\cos(6.8875t) + 0.00966\cdot e^{-1.25t}\cdot\sin(6.8875t) - 0.00312\cdot\cos(t) - 0.000162\cdot\sin(t)$