Assignment 9

Section 7.4

1) $\mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\}$ $\Rightarrow$ Firstly, we know that $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$ So we need only consider the $\frac{6}{s^4}$ term.

$\Rightarrow$ $\frac{6}{s^4}$ is of the type $f(t)=t^n$ where $n=3$ because $\mathcal{L}\{t^3\} = \frac{6}{s^4}$.

$$\therefore \mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\} = e^t t^3$$

7) $\mathcal{L}^{-1}\left\{\frac{2s+16}{s^2+4s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{2s+16}{s^2+4s+4+9}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{2s}{(s+2)^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{16}{(s+2)^2+3^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s+2)}{(s+2)^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-4}{(s+2)^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{16}{(s+2)^2+3^2}\right\}$$

$$= 2e^{-2t}\cos 3t + \mathcal{L}^{-1}\left\{\frac{12}{(s+2)^2+3^2}\right\}$$

$$= 2e^{-2t}\cos 3t + 4e^{-2t}\sin(3t)$$

#21) $F(s) = \dfrac{6s^2 - 13s + 2}{s(s-1)(s-6)} = \dfrac{A}{s} + \dfrac{B}{s-1} + \dfrac{C}{s-6}$

$$A(s-1)(s-6) + B(s)(s-6) + C(s)(s-1) = 6s^2 - 13s + 2$$

$$As^2 - 7As + 6A + Bs^2 - 6Bs + Cs^2 - Cs = 6s^2 - 13s + 2$$

$$s^2(A+B+C) + s(-7A-6B-C) + 1(6A) = 6s^2 - 13s + 2$$

$$\left.\begin{aligned} A+B+C &= 6 \\ -7A-6B-C &= -13 \\ 6A &= 2 \end{aligned}\right\} \begin{aligned} A &= \tfrac{1}{3} \\ B &= 1 \\ C &= \tfrac{14}{3} \end{aligned}$$

$$F(s) = \frac{1}{3s} + \frac{1}{s-1} + \frac{14}{3(s-6)}$$

$$\boxed{f(t) = \frac{1}{3} + e^{t} + \frac{14}{3}e^{6t}}$$

25) $F_{(s)} = \dfrac{7s^2 + 23s + 30}{(s-2)(s^2+2s+5)} = \dfrac{7s^2+23s+30}{(s-2)\left[(s+1)^2+2^2\right]}$

Use form: $\dfrac{A(s-\alpha) + B(\beta)}{(s-\alpha)^2 + \beta^2}$ for $\left[(s+1)^2+2^2\right]$ term where $\alpha = -1$, $\beta = 2$

$\therefore \dfrac{A(s+1) + 2B}{(s+1)^2+2^2} + \dfrac{C}{s-2} = \dfrac{7s^2+23s+30}{(s-2)((s+1)^2+2^2)}$

$\therefore As^2 - As - 2A + 2Bs - 4B + Cs^2 + 2Cs + C + 4C = 7s^2 + 23s + 30$

$s^2(A+C) + s(-A+2B+2C) + 1(-2A-4B+5C) = 7s^2+23s+30$

$$\left.\begin{aligned} \therefore A+C&=7 \\ -A+2B+2C&=23 \\ -2A-4B+5C&=30 \end{aligned}\right\} \quad \begin{aligned} A&=-1 \\ B&=3 \\ C&=8 \end{aligned}$$

$\therefore F(s) = \dfrac{-(s+1)}{(s+1)^2+2^2} + \dfrac{3(2)}{(s+1)^2+2^2} + \dfrac{8}{s-2}$

$\therefore f(t) = -e^{-t}\cos 2t + 3e^{-t}\sin 2t + 8e^{2t}$

Section 7.5

#3) $y'' + 6y' + 9y = 0 \qquad y(0) = -1 \qquad y'(0) = 6$

$$\mathcal{L}\{y''\}(s) + 6\mathcal{L}\{y'\}(s) + 9\mathcal{L}\{y\} = 0$$

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] + 6\left[sY(s) - y(0)\right] + 9Y(s) = 0$$

$$s^2 Y + s - \cancel{6} + 6sY + \cancel{6} + 9Y = 0$$

$$Y(s^2 + 6s + 9) = -s$$

$$Y = \frac{-s}{(s+3)^2} = \frac{-\cancel{(s+3)}}{(s+3)^{\cancel{2}}} + \frac{3}{(s+3)^2} = \frac{-1}{s+3} + \frac{3}{(s+3)^2}$$

$$\boxed{y = \mathcal{L}^{-1}\{Y\} = -e^{-3t} + 3e^{-3t}t}$$

#5) $w'' + w = t^2 + 2 \qquad w(0) = 1 \qquad w'(0) = -1$

$$\mathcal{L}\{w''\} + \mathcal{L}\{w\} = \mathcal{L}\{t^2 + 2\}$$

$$s^2 W - s\,w(0) - w'(0) + W = \frac{2}{s^3} + \frac{2}{s}$$

$$W(s^2+1) - s + 1 = \frac{2 + 2s^2}{s^3} = \frac{2(s^2+1)}{s^3}$$

$$W = \frac{2\cancel{(s^2+1)}}{s^3\cancel{(s^2+1)}} + \frac{(s-1)}{(s^2+1)} = \frac{2}{s^3} + \frac{s}{s^2+1^2} - \frac{1}{s^2+1^2}$$

$$\boxed{w = t^2 + \cos t - \sin t}$$

25) $y''' - y'' + y' - y = 0 \qquad y(0)=1 \quad y'(0)=1 \quad y''(0)=3$

$$\left[s^3Y - s^2y(0) - sy'(0) - y''(0)\right] - \left[s^2Y - sy(0) - y'(0)\right] + \left[sY - y(0)\right] - Y = 0$$

$$Y\left(s^3 - s^2 + s - 1\right) = s^2 + \cancel{s} + 3 - \cancel{s} \cancel{-1} \cancel{+1}$$

$$Y = \frac{s^2+3}{s^3 - s^2 + s - 1}$$

Since $(1)^3 - (1)^2 + (1) - 1 = 0$, $(s-1)$ is a factor

$$\begin{array}{r|l} & s^2 \quad +1 \\ (s-1) & s^3 - s^2 + s - 1 \\ & s^3 - s^2 \\ \hline & 0 + s - 1 \\ & s - 1 \\ \hline & 0 \end{array}$$

$$\therefore\ Y = \frac{s^2+3}{(s-1)(s^2+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+1)} = \frac{As^2 + A + Bs^2 + Cs - Bs - C}{(s-1)(s^2+1)}$$

$$\therefore\ s^2(A+B) + s(C-B) + 1(A-C) = s^2 + 3$$

$$\left.\begin{array}{l} A+B=1 \\ C-B=0 \\ A-C=3 \end{array}\right\} \begin{array}{l} A=2 \\ B=-1 \\ C=-1 \end{array}$$

$$\therefore\ Y = \frac{2}{s-1} - \frac{s}{s^2+1^2} - \frac{1}{s^2+1^2}$$

$$\boxed{y = 2e^t - \cos t - \sin t}$$

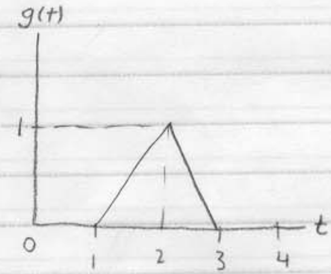
Section 1.6

#5) $g(t) = \begin{cases} 0 & 0<t<1 \\ 2 & 1<t<2 \\ 1 & 2<t<3 \\ 3 & 3<t \end{cases}$

$$g(t) = 2H(t-1) - H(t-2) + 2H(t-3)$$

$$G(s) = \mathcal{L}\{g(t)\} = \frac{2e^{-s}}{s} - \frac{e^{-2s}}{s} + \frac{2e^{-3s}}{s}$$

#9)



$g(t) = \begin{cases} 0 & 0<t<1 \\ t-1 & 1<t<2 \\ 1-(t-2) = 3-t & 2<t<3 \\ 0 & 3<t \end{cases}$

$$\begin{aligned} g(t) &= (t-1)H(t-1) + \left[(3-t)-(t-1)\right]H(t-2) - (3-t)H(t-3) \\ &= (t-1)H(t-1) + (-2t+4)H(t-2) + (t-3)H(t-3) \\ &= (t-1)H(t-1) - 2(t-2)H(t-2) + (t-3)H(t-3) \end{aligned}$$

$$G(s) = \mathcal{L}\{g(t)\} = \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$

#17) $F(s) = \dfrac{e^{-3s}(s-5)}{(s+1)(s+2)}$

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \mathcal{L}^{-1}\{e^{-3s}G(s)\} = g(t-3)\,H(t-3)$$

$\therefore$ only look at $\dfrac{(s-5)}{(s+1)(s+2)}$ part.

$$G(s) = \frac{(s-5)}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{As+2A+Bs+B}{(s+1)(s+2)}$$

$s(A+B) = s \implies A = 1-B$ ①

$2A+B = -5$ ②

$\left.\right\}$ $A=-6$, $B=7$

$$\therefore G(s) = \frac{-6}{s+1} + \frac{7}{s+2}$$

$$g(t) = -6e^{-t} + 7e^{-2t}$$

$$\therefore \mathcal{L}^{-1}\{F(s)\} = \left(-6e^{-(t-3)} + 7e^{-2(t-3)}\right)H(t-3)$$

$$= \left(-6e^{3-t} + 7e^{6-2t}\right)H(t-3)$$

#29) $y'' + y = H(t-3) \qquad y(0)=0 \qquad y'(0)=1$

$$s^2 Y - sy(0) - y'(0) + Y = \frac{e^{-3s}}{s}$$

$$Y(s^2+1) = \frac{e^{-3s}+s}{s}$$

$$Y = \frac{e^{-3s}+s}{s(s^2+1)} = \frac{e^{-3s}}{s(s^2+1)} + \frac{\cancel{s}}{\cancel{s}(s^2+1)}$$

Notice that $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s^2+1)}\right\}(t) = f(t-3)\,H(t-3)$

$\left(\text{where } f(t-3) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}(s-3)\right)$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{As^2+A+Bs^2+Cs}{(s)(s^2+1)}$$

$$\left.\begin{aligned} \therefore A+B&=0 \\ C&=0 \\ A&=1 \end{aligned}\right\} \quad \begin{aligned} \therefore A&=1 \\ B&=-1 \\ C&=0 \end{aligned}$$

$$Y = e^{-3s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{1}{s^2+1}$$

$$y = H(t-3)\left[1 - \cos(t-3)\right] + \sin t$$

Assignment 9 - Section 7.6

#29)

$$H(t) := \begin{cases} 1 & \text{if } t \geq 3 \\ 0 & \text{if } t < 3 \end{cases}$$

$$y(t) := H(t)\cdot(1 - \cos(t-3)) + \sin(t)$$