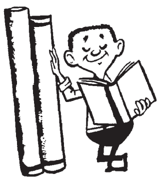
Error and Complementary Error Functions



Reading **Problems**

Outline

Background

The error function and the complementary error function are important special functions which appear in the solutions of diffusion problems in heat, mass and momentum transfer, probability theory, the theory of errors and various branches of mathematical physics. It is interesting to note that there is a direct connection between the error function and the Gaussian function and the normalized Gaussian function that we know as the "bell curve". The Gaussian function is given as

$$G(x) = Ae^{-x^2/(2\sigma^2)}$$

where σ is the standard deviation and A is a constant.

The Gaussian function can be normalized so that the accumulated area under the curve is unity, i.e. the integral from $-\infty$ to $+\infty$ equals 1. If we note that the definite integral

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

then the normalized Gaussian function takes the form

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)}$$

If we let

$$t^2 = \frac{x^2}{2\sigma^2} \qquad \text{and} \qquad dt = \frac{1}{\sqrt{2}\sigma}\, dx$$

then the normalized Gaussian integrated between $-x$ and $+x$ can be written as

$$\int_{-x}^{x} G(x)\, dx = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2}\, dt$$

or recognizing that the normalized Gaussian is symmetric about the $y-$axis, we can write

$$\int_{-x}^{x} G(x)\,dx = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\,dt = \text{erf}\ x = \text{erf}\ \left(\frac{x}{\sqrt{2}\sigma}\right)$$

and the complementary error function can be written as

$$\text{erfc}\ x = 1 - \text{erf}\ x = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2}\,dt$$

Historical Perspective

The normal distribution was first introduced by de Moivre in an article in 1733 (reprinted in the second edition of his Doctrine of Chances, 1738) in the context of approximating certain binomial distributions for large n. His result was extended by Laplace in his book Analytical Theory of Probabilities (1812), and is now called the Theorem of de Moivre-Laplace.

Laplace used the normal distribution in the analysis of errors of experiments. The important method of least squares was introduced by Legendre in 1805. Gauss, who claimed to have used the method since 1794, justified it in 1809 by assuming a normal distribution of the errors.

The name *bell curve* goes back to Jouffret who used the term *bell surface* in 1872 for a bivariate normal with independent components. The name *normal distribution* was coined independently by Charles S. Peirce, Francis Galton and Wilhelm Lexis around 1875 [Stigler]. This terminology is unfortunate, since it reflects and encourages the fallacy that “everything is Gaussian”.

Definitions

1. Gaussian Function

The normalized Gaussian curve represents the probability distribution with standard distribution σ and mean μ relative to the average of a random distribution.

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

This is the curve we typically refer to as the "bell curve" where the mean is zero and the standard distribution is unity.

2. Error Function

The error function equals twice the integral of a normalized Gaussian function between 0 and $x/\sigma\sqrt{2}$.

$$y = \text{erf } x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \qquad \text{for} \qquad x \geq 0, \quad y\ [0, 1]$$

where

$$t = \frac{x}{\sqrt{2}\,\sigma}$$

3. Complementary Error Function

The complementary error function equals one minus the error function

$$1 - y = \text{erfc } x = 1 - \text{erf } x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \qquad \text{for} \qquad x \geq 0, \quad y\ [0, 1]$$

4. Inverse Error Function

$$x = \text{inerf } y$$

inerf y exists for y in the range $-1 < y < 1$ and is an odd function of y with a Maclaurin expansion of the form

$$\mathrm{inverf}\ y = \sum_{n=1}^{\infty} c_n\ y^{2n-1}$$

5. Inverse Complementary Error Function

$$x = \mathrm{inerfc}\ (1 - y)$$

Theory

Gaussian Function

The Gaussian function or the Gaussian probability distribution is one of the most fundamental functions. The Gaussian probability distribution with mean μ and standard deviation σ is a normalized Gaussian function of the form

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \tag{1.1}$$

where $G(x)$, as shown in the plot below, gives the probability that a variate with a Gaussian distribution takes on a value in the range $[x, x + dx]$. Statisticians commonly call this distribution the normal distribution and, because of its shape, social scientists refer to it as the "bell curve." $G(x)$ has been normalized so that the accumulated area under the curve between $-\infty \leq x \leq +\infty$ totals to unity. A cumulative distribution function, which totals the area under the normalized distribution curve is available and can be plotted as shown below.

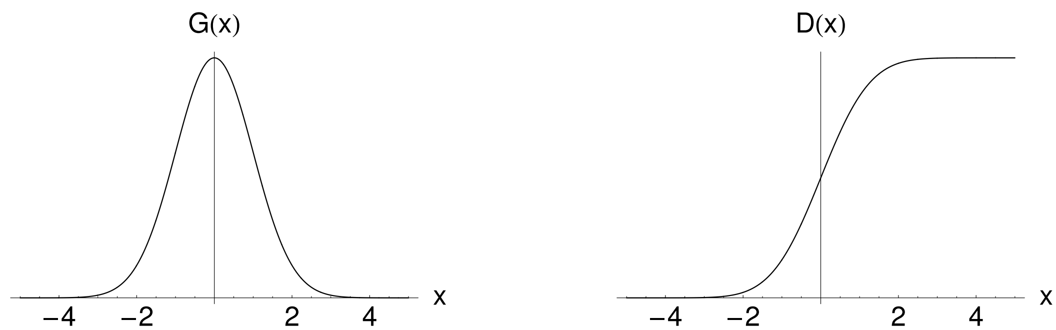


Figure 2.1: Plot of Gaussian Function and Cumulative Distribution Function

When the mean is set to zero ($\mu = 0$) and the standard deviation or variance is set to unity ($\sigma = 1$), we get the familiar normal distribution

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \tag{1.2}$$

which is shown in the curve below. The normal distribution function $N(x)$ gives the probability that a variate assumes a value in the interval $[0, x]$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\, dt \tag{1.3}$$

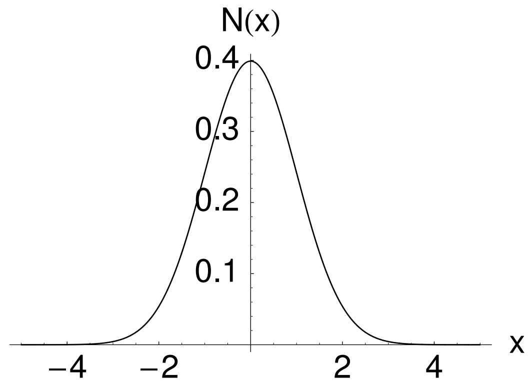


Figure 2.2: Plot of the Normalized Gaussian Function

Gaussian distributions have many convenient properties, so random variates with unknown distributions are often assumed to be Gaussian, especially in physics, astronomy and various aspects of engineering. Many common attributes such as test scores, height, etc., follow roughly Gaussian distributions, with few members at the high and low ends and many in the middle.

Computer Algebra Systems

Function	Maple	Mathematica
Probability Density Function - frequency of occurrence at x	`statevalf[pdf,dist](x)`	`PDF[`*dist, x*`]`
Cumulative Distribution Function - integral of probability density function up to x	`statevalf[cdf,dist](x)`	`CDF[`*dist, x*`]`
	dist = `normald[`$\boldsymbol{\mu}$, $\boldsymbol{\sigma}$`]` $\boldsymbol{\mu = 0}$ (mean) $\boldsymbol{\sigma = 1}$ (std. dev.)	*dist* = `NormalDistribution[`$\boldsymbol{\mu}$, $\boldsymbol{\sigma}$`]` $\boldsymbol{\mu = 0}$ (mean) $\boldsymbol{\sigma = 1}$ (std. dev.)

Potential Applications

1. *Statistical Averaging:*

Error Function

The error function is obtained by integrating the normalized Gaussian distribution.

$$\mathbf{erf}\ x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{1.4}$$

where the coefficient in front of the integral normalizes $\mathbf{erf}(\infty) = 1$. A plot of $\mathbf{erf}\ x$ over the range $-3 \leq x \leq 3$ is shown as follows.

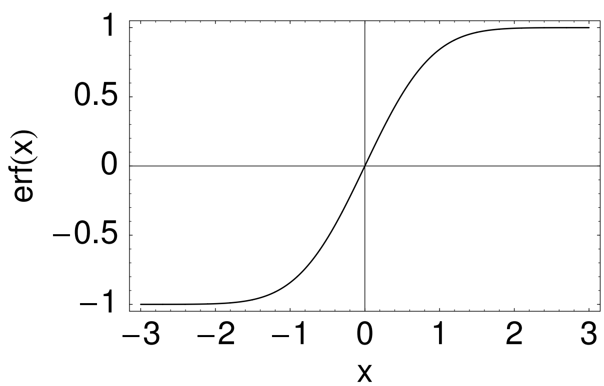


Figure 2.3: Plot of the Error Function

The error function is defined for all values of x and is considered an odd function in x since $\mathbf{erf}\ x = -\mathbf{erf}\ (-x)$.

The error function can be conveniently expressed in terms of other functions and series as follows:

$$\mathbf{erf}\ x = \frac{1}{\sqrt{\pi}}\, \gamma\left(\frac{1}{2}, x^2\right) \tag{1.5}$$

$$= \frac{2x}{\sqrt{\pi}}\, \mathrm{M}\left(\frac{1}{2}, \frac{3}{2}, -x^2\right) = \frac{2x}{\sqrt{\pi}}\, e^{-x^2}\, \mathrm{M}\left(1, \frac{3}{2}, x^2\right) \tag{1.6}$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \tag{1.7}$$

where $\gamma(\cdot)$ is the incomplete gamma function, $\mathbf{M}(\cdot)$ is the confluent hypergeometric function of the first kind and the series solution is a Maclaurin series.

Computer Algebra Systems

Function	Maple	Mathematica
Error Function	`erf(`x`)`	`Erf[`x`]`
Complementary Error Function	`erfc(`x`)`	`Erfc[`x`]`
Inverse Error Function	`fslove(erf(`x`)=`s`)`	`InverseErf[`s`]`
Inverse Complementary Error Function	`fslove(erfc(`x`)=`s`)`	`InverseErfc[`s`]`
	where $\boldsymbol{s}$ is a numerical value and we solve for $\boldsymbol{x}$	

Potential Applications

1. *Diffusion:* Transient conduction in a semi-infinite solid is governed by the diffusion equation, given as

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

where $\boldsymbol{\alpha}$ is thermal diffusivity. The solution to the diffusion equation is a function of either the **erf** $\boldsymbol{x}$ or **erfc** $\boldsymbol{x}$ depending on the boundary condition used. For instance, for constant surface temperature, where $\boldsymbol{T(0,t) = T_s}$

$$\frac{T(x,t) - T_s}{T_i - T_s} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

complementary Error Function

The complementary error function is defined as

$$\begin{aligned} \text{erfc}\, x &= 1 - \text{erf}\, x \\ &= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \end{aligned} \tag{1.8}$$

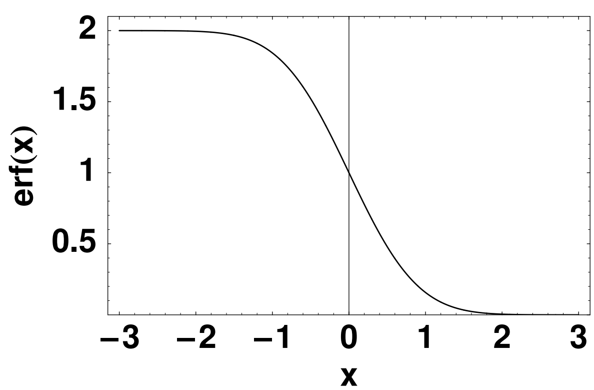


Figure 2.4: Plot of the complementary Error Function

and similar to the error function, the complementary error function can be written in terms of the incomplete gamma functions as follows:

$$\text{erfc}\, x = \frac{1}{\sqrt{\pi}}\, \Gamma\left(\frac{1}{2}, x^2\right) \tag{1.9}$$

As shown in Figure 2.5, the superposition of the error function and the complementary error function when the argument is greater than zero produces a constant value of unity.

Potential Applications

1. *Diffusion:* In a similar manner to the transient conduction problem described for the error function, the complementary error function is used in the solution of the diffusion equation when the boundary conditions are constant surface heat flux, where $q_s = q_0$

$$T(x,t) - T_i = \frac{2q_0(\alpha t/\pi)^{1/2}}{k}\, \exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q_0 x}{k}\, \text{erfc}\, \left(\frac{x}{2\sqrt{\alpha t}}\right)$$

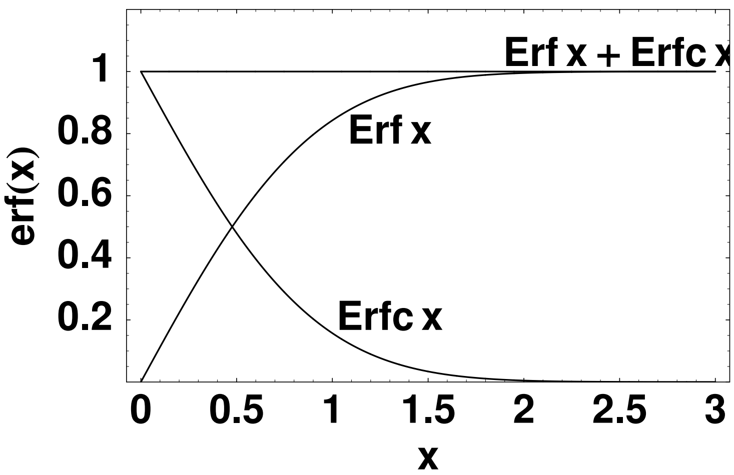


Figure 2.5: Superposition of the Error and complementary Error Functions

and surface convection, where $-k \left. \frac{\partial T}{\partial x} \right|_{x=0} = h[T_\infty - T(0,t)]$

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[\exp\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right)\right] \left[\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]$$

Relations and Selected Values of Error Functions

$$\mathbf{erf}\,(-x) = -\mathbf{erf}\,x \qquad \mathbf{erfc}\,(-x) = 2 - \mathbf{erfc}\,x$$

$$\mathbf{erf}\,0 = 0 \qquad \mathbf{erfc}\,0 = 1$$

$$\mathbf{erf}\,\infty = 1 \qquad \mathbf{erfc}\,\infty = 0$$

$$\mathbf{erf}\,(-\infty) = -1$$

$$\int_0^\infty \mathbf{erfc}\,x\,dx = 1/\sqrt{\pi}$$

$$\int_0^\infty \mathbf{erfc}^2\,x\,dx = (2-\sqrt{2})/\sqrt{\pi}$$

Ten decimal place values for selected values of the argument appear in Table 2.1.

Table 2.1 Ten decimal place values of **erf** x

x	**erf** x	x	**erf** x
0.0	0.00000 00000	2.5	0.99959 30480
0.5	0.52049 98778	3.0	0.99997 79095
1.0	0.84270 07929	3.5	0.99999 92569
1.5	0.96610 51465	4.0	0.99999 99846
2.0	0.99532 22650	4.5	0.99999 99998

Approximations

Power Series for Small x ($x < 2$)

Since

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\, dt \tag{1.10}$$

and the series is uniformly convergent, it may be integrated term by term. Therefore

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} \tag{1.11}$$

$$= \frac{2}{\sqrt{\pi}} \left\{ \frac{x}{1 \cdot 0!} - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \cdots \right\} \tag{1.12}$$

Asymptotic Expansion for Large x ($x > 2$)

Since

$$\text{erfc } x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \frac{1}{t}\, e^{-t^2}\, t\, dt$$

we can integrate by parts by letting

$$u = \frac{1}{t} \qquad dv = e^{-t^2}\, d\, dt$$

$$du = -t^{-2}\, dt \qquad v = -\frac{1}{2}\, e^{-t^2}$$

therefore

$$\int_x^{\infty} \frac{1}{t}\, e^{-t^2}\, t\, dt = \Big[uv \Big]_x^{\infty} - \int_x^{\infty} v\, du = \left[-\frac{1}{2t} e^{-t^2} \right]_x^{\infty} - \int_x^{\infty} \frac{1}{2} \frac{e^{-t^2}}{t^2}\, dt$$

Thus

$$\mathbf{erfc}\ x = \frac{2}{\sqrt{\pi}}\left\{\frac{1}{2x}e^{-x^2} - \frac{1}{2}\int_x^\infty \frac{e^{-t^2}}{t^2}dt\right\} \tag{1.13}$$

Repeating the process n times yields

$$\begin{aligned}\frac{\sqrt{\pi}}{2}\mathbf{erfc}\ x &= \frac{1}{2}e^{-x^2}\left(\frac{1}{x} - \frac{1}{2x^3} + \frac{1\cdot 3}{2^2x^5} - \cdots + (-1)^{n-1}\frac{1\cdot 3\cdots(2n-3)}{2^{n-1}x^{2n-1}}\right) + \\ &\quad +(-1)^n\frac{1\cdot 3\cdots(2n-1)}{2^n}\int_x^\infty \frac{e^{-t^2}}{t^{2n}}\ dt \end{aligned} \tag{1.14}$$

Finally we can write

$$\sqrt{\pi}xe^{x^2}\mathbf{erfc}\ x = 1 + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2x^2)^n} \tag{1.15}$$

This series does not converge, since the ratio of the n^{th} term to the $(n-1)^{th}$ does not remain less than unity as n increases. However, if we take n terms of the series, the remainder,

$$\frac{1\cdot 3\cdots(2n-1)}{2^n}\int_x^\infty \frac{e^{-t^2}}{t^{2n}}\ dt$$

is less than the n^{th} term because

$$\int_x^\infty \frac{e^{-t^2}}{t^{2n}}\ dt < e^{-x^2} < \int_0^\infty \frac{dt}{t^{2n}}$$

We can therefore stop at any term taking the sum of the terms up to this term as an approximation of the function. The error will be less in absolute value than the last term retained in the sum. Thus for large x, **erfc** x may be computed numerically from the asymptotic expansion.

$$\begin{aligned}\sqrt{\pi}xe^{x^2}\mathbf{erfc}\ x &= 1 + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2x^2)^n} \\ &= 1 - \frac{1}{2x^2} + \frac{1\cdot 3}{(2x^2)^2} - \frac{1\cdot 3\cdot 5}{(2x^2)^3} + \cdots \end{aligned} \tag{1.16}$$

Some other representations of the error functions are given below:

$$\text{erf } x = \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(3/2)^n} \tag{1.17}$$

$$= \frac{2x}{\sqrt{\pi}} \ \mathrm{M}\left(\frac{1}{2}, \frac{3}{2}, -x^2\right) \tag{1.18}$$

$$= \frac{2x}{\sqrt{\pi}} e^{-x^2} \ \mathrm{M}\left(1, \frac{3}{2}, x^2\right) \tag{1.19}$$

$$= \frac{1}{\sqrt{\pi}} \ \gamma\left(\frac{1}{2}, x^2\right) \tag{1.20}$$

$$\text{erfc } x = \frac{1}{\sqrt{\pi}} \ \Gamma\left(\frac{1}{2}, x^2\right) \tag{1.21}$$

The symbols γ and Γ represent the incomplete gamma functions, and **M** denotes the confluent hypergeometric function or Kummer's function.

Derivatives of the Error Function

$$\frac{d}{dx}\text{erf } x = \frac{2}{\sqrt{\pi}} e^{-x^2} = \frac{d}{dx}\left\{\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt\right\} \tag{1.22}$$

Use of "Leibnitz" rule of differentiation of integrals gives:

$$\frac{d^2}{dx^2}\text{erfc } x = \frac{d}{dx} \frac{2}{\sqrt{\pi}} e^{-x^2} = -\frac{2}{\sqrt{\pi}}(2x)e^{-x^2} \tag{1.23}$$

$$\frac{d^3}{dx^3}\text{erfc } x = \frac{d}{dx}\left\{-\frac{2}{\sqrt{\pi}}(2x)e^{-x^2}\right\} = \frac{2}{\sqrt{\pi}}(4x^2 - 2)e^{-x^2} \tag{1.24}$$

In general we can write

$$\frac{d^{n+1}}{dx^{n+1}}\text{erf } x = (-1)^n \frac{2}{\sqrt{\pi}} \mathrm{H}_n(x) \ e^{-x^2} \quad (n = 0, 1, 2 \ldots) \tag{1.25}$$

where $\mathbf{H}_n(x)$ are the Hermite polynomials.

Repeated Integrals of the Complementary Error Function

$$i^n\text{erfc } x = \int_x^\infty i^{n-1}\text{erfc } t\, dt \qquad\qquad n = 0, 1, 2, \ldots \tag{1.26}$$

where

$$i^{-1}\text{erfc } x = \frac{2}{\sqrt{\pi}} e^{-x^2} \tag{1.27}$$

$$i^0\text{erfc } x = \text{erfc } x \tag{1.28}$$

$$\begin{aligned} i^1\text{erfc } x &= i\text{erfc } x = \int_x^\infty \text{erfc } t\, dt \\ &= \frac{1}{\sqrt{\pi}} \exp(-x^2) - x \text{ erfc } x \end{aligned} \tag{1.29}$$

$$\begin{aligned} i^2\text{erfc } x &= \int_x^\infty i \text{ erfc } t\, dt \\ &= \frac{1}{4}\left[(1+2x^2) \text{ erfc } x - \frac{2}{\sqrt{\pi}}\, x \exp(-x^2)\right] \\ &= \frac{1}{4}\left[\text{erfc } x - 2x \cdot \text{ ierfc } x\right] \end{aligned} \tag{1.30}$$

The general recurrence formula is

$$2ni^n \text{ erfc } x = i^{n-2} \text{ erfc } x - 2xi^{n-1} \text{ erfc } x \quad (n = 1, 2, 3, \ldots) \tag{1.31}$$

Therefore the value at $x = 0$ is

$$i^n\text{erfc } 0 - \frac{1}{2^n\, \Gamma\left(1 + \dfrac{n}{2}\right)} \qquad (n = -1, 0, 1, 2, 3, \ldots) \tag{1.32}$$

It can be shown that $y = i^n \text{ erfc } x$ is the solution of the differential equation

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2ny = 0 \tag{1.33}$$

The general solution of

$$y'' + 2xy' - 2ny = 0 \qquad -\infty \le x \le \infty \tag{1.34}$$

is of the form

$$y = Ai^n\text{erfc } x + Bi^n\text{erfc } (-x) \tag{1.35}$$

Derivatives of Repeated Integrals of the Complementary Error Function

$$\frac{d}{dx}\left[i^n\text{erfc } x\right] = (-1)^{n-1}\text{erfc } x \qquad (n = 0, 1, 2, 3\ldots) \tag{1.36}$$

$$\frac{d^n}{dx^n}\left[e^{x^2}\text{erfc } x\right] = (-1)^n 2^n n! e^{x^2} i^n\text{erfc } x \qquad (n = 0, 1, 2, 3\ldots) \tag{1.37}$$

Some Integrals Associated with the Error Function

$$\int_0^{x^2} \frac{e^{-t}}{\sqrt{t}}\, dt = \sqrt{\pi}\ \mathrm{erf}\ x \tag{1.38}$$

$$\int_0^{x} e^{-t\ y}\, dt = \frac{\sqrt{\pi}}{2y}\ \mathrm{erf}\ x \tag{1.39}$$

$$\int_0^{1} \frac{e^{-t^2\ x^2}}{1+t^2}\, dt = \frac{\pi}{2}\ e^{x^2}\left[1-\{\mathrm{erf}\ x\}^2\right] \tag{1.40}$$

$$\int_0^{\infty} \frac{e^{-t\ x}}{\sqrt{y+t}}\, dt = \frac{\sqrt{\pi}}{\sqrt{x}} e^{xy} \mathrm{erfc}\ (\sqrt{xy}) \qquad x>0 \tag{1.41}$$

$$\int_0^{\infty} \frac{e^{-t^2 x}}{t^2+y^2}\, dt = \frac{\pi}{2y} e^{xy^2}\ \mathrm{erfc}\ (\sqrt{x}y) \qquad x>0,\ y>0 \tag{1.42}$$

$$\int_0^{\infty} \frac{e^{-tx}}{(t+y)\sqrt{t}}\, dt = \frac{\pi}{\sqrt{y}} e^{xy}\ \mathrm{erfc}\ (xy) \qquad x>0, y\neq 0 \tag{1.43}$$

$$\int_0^{\infty} e^{-t\ x} \mathrm{erf}\ (\sqrt{yt})\, dt = \frac{\sqrt{y}}{x}(x+y)^{-1/2} \qquad (x+y)>0 \tag{1.44}$$

$$\int_0^{\infty} e^{-t\ x} \mathrm{erf}\ (\sqrt{y/t}\, dt = \frac{1}{x} e^{-2\sqrt{xy}} \qquad x>0, y>0 \tag{1.45}$$

$$\int_{-a}^{\infty} \mathrm{erfc}\ (t)\, dt = \mathrm{ierfc}\ (a) + 2a = \mathrm{ierfc}\ (-a) \tag{1.46}$$

$$\int_{-a}^{a} \mathrm{erf}\ (t)\, dt = 0 \tag{1.47}$$

$$\int_{-a}^{a} \mathrm{erfc}(t)\, dt = 2a \tag{1.48}$$

$$\int_{-a}^{\infty} \mathrm{ierfc}\ (t)\, dt = i^2\ \mathrm{erfc}\ (-a) = \frac{1}{2} + a - i^2\ \mathrm{erfc}\ (a) \tag{1.49}$$

$$\int_a^{\infty} i^n\ \mathrm{erfc}\ \left(\frac{t+c}{b}\right)\, dt = b i^{n+1}\ \mathrm{erfc}\ \left(\frac{a+c}{b}\right) \tag{1.50}$$

Numerical Computation of Error Functions

The power series form of the error function is not recommended for numerical computations when the argument approaches and exceeds the value $x = 2$ because the large alternating terms may cause cancellation, and because the general term is awkward to compute recursively. The function can, however, be expressed as a confluent hypergeometric series.

$$\text{erf}\ x = \frac{2}{\sqrt{\pi}} x\ e^{-x^2}\ \text{M}\left(1, \frac{3}{2}, x^2\right) \tag{1.51}$$

in which all terms are positive, and no cancellation can occur. If we write

$$\text{erf}\ x = b \sum_{n=0}^{\infty} a_n \qquad 0 \leq x \leq 2 \tag{1.52}$$

with

$$b = \frac{2x}{\sqrt{\pi}}\ e^{-x^2} \qquad a_0 = 1 \qquad a_n = \frac{x^2}{(2n+1)/2}\ a_{n-1} \quad n \geq 1$$

then **erf** x can be computed very accurately (e.g. with an absolute error less that 10^{-9}). Numerical experiments show that this series can be used to compute **erf** x up to $x = 5$ to the required accuracy; however, the time required for the computation of **erf** x is much greater due to the large number of terms which have to be summed. For $x \geq 2$ an alternate method that is considerably faster is recommended which is based upon the asymptotic expansion of the complementary error function.

$$\begin{aligned} \text{erfc}\ x &= \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\ dt \\ &= \frac{e^{-x^2}}{\sqrt{\pi} x} 2\ F_o\left(\frac{1}{2}, 1, -\frac{1}{x^2}\right) \qquad x \to \infty \end{aligned} \tag{1.53}$$

which cannot be used to obtain arbitrarily accurate values for any x. An expression that converges for all $x > 0$ is obtained by converting the asymptotic expansion into a continued fraction

$$\sqrt{\pi}e^{x^2}\mathbf{erfc}\ x = \cfrac{1}{x + \cfrac{1/2}{x + \cfrac{1}{x + \cfrac{3/2}{x + \cfrac{2}{x + \cfrac{5/2}{x + \ldots}}}}}} \qquad x > 0 \tag{1.54}$$

which for convenience will be written as

$$\mathbf{erfc}\ x = \frac{e^{-x^2}}{\sqrt{\pi}} \left\{ \frac{1}{x+}\ \frac{1/2}{x+}\ \frac{1}{x+}\ \frac{3/2}{x+}\ \frac{2}{x+}\ \cdots \right\} \qquad x > 0 \tag{1.55}$$

It can be demonstrated experimentally that for $x \geq 2$ the 16th approximant gives **erfc** x with an absolute error less that 10^{-9}. Thus we can write

$$\mathbf{erfc}\ x = \frac{e^{-x^2}}{\sqrt{\pi}} \left\{ \frac{1}{x+}\ \frac{1/2}{x+}\ \frac{1}{x+}\ \frac{3/2}{x+}\ \cdots\ \frac{8}{x} \right\} \qquad x \geq 2 \tag{1.56}$$

Using a fixed number of approximants has the advantage that the continued fraction can be evaluated rapidly beginning with the last term and working upward to the first term.

Rational Approximations of the Error Functions ($0 \le x < \infty$)

Numerous rational approximations of the error functions have been developed for digital computers. The approximations are often based upon the use of economized Chebyshev polynomials and they give values of **erf** $\boldsymbol{x}$ from 4 decimal place accuracy up to 24 decimal place accuracy.

Two approximations by Hastings et al.[11] are given below.

$$\text{erf}\ x = 1 - [t(a_1 + t(a_2 + a_3 t))]\, e^{-x^2} + \epsilon(x) \qquad 0 \le x \tag{1.57}$$

where

$$t = \frac{1}{1 + px}$$

and the coefficients are

$$\begin{aligned} p &= 0.47047 \\ a_1 &= 0.3480242 \\ a_2 &= -0.0958798 \\ a_3 &= 0.7478556 \end{aligned}$$

This approximation has a maximum absolute error of $|\epsilon(x)| < 2.5 \times 10^{-5}$.

Another more accurate rational approximation has been developed for example

$$\text{erf}\ x = 1 - [t(a_1 + t(a_2 + t(a_3 + t(a_4 + a_5 t))))]\, e^{-x^2} + \epsilon(x) \tag{1.58}$$

where

$$t = \frac{1}{1 + px}$$

and the coefficients are

$$\begin{aligned}
p &= 0.3275911 \\
a_1 &= 0.254829592 \\
a_2 &= -0.284496736 \\
a_3 &= 1.421413741 \\
a_4 &= -1.453152027 \\
a_5 &= 1.061405429
\end{aligned}$$

This approximation has a maximum absolute error of $|\epsilon(x)| < 1.5 \times 10^{-7}$.

Assigned Problems

Problem Set for Error and Complementary Error Function **Due Date: February 12, 2004**

1. Evaluate the following integrals to four decimal places using either power series, asymptotic series or polynomial approximations:

a) $\displaystyle\int_0^2 e^{-x^2}\, dx$ b) $\displaystyle\int_{0.001}^{0.002} e^{-x^2}\, dx$

c) $\displaystyle\frac{2}{\sqrt{\pi}}\int_{1.5}^{\infty} e^{-x^2}\, dx$ d) $\displaystyle\frac{2}{\sqrt{\pi}}\int_{5}^{10} e^{-x^2}\, dx$

e) $\displaystyle\int_1^{1.5} \left(\frac{1}{2}e^{-x^2}\right) dx$ f) $\displaystyle\sqrt{\frac{2}{\pi}}\int_1^{\infty} \left(\frac{1}{2}e^{-x^2}\right) dx$

2. The value of **erf 2** is **0.995** to three decimal places. Compare the number of terms required in calculating this value using:

a) the convergent power series, and
b) the divergent asymptotic series.

Compare the approximate errors in each case after two terms; after ten terms.

3. For the function $\mathbf{ierfc}(x)$ compute to four decimal places when $x = 0,\ 0.2,\ 0.4,\ 0.8,$ and 1.6.

4. Prove that

i) $\displaystyle\sqrt{\pi}\ \mathrm{erf}(x) = \gamma\left(\frac{1}{2}, x^2\right)$

ii) $\displaystyle\sqrt{\pi}\ \mathrm{erfc}(x) = \Gamma\left(\frac{1}{2}, x^2\right)$

where $\gamma\left(\frac{1}{2}, x^2\right)$ and $\Gamma\left(\frac{1}{2}, x^2\right)$ are the incomplete Gamma functions defined as:

$$\gamma(a, y) = \int_0^y e^{-u} u^{a-1} \, du$$

and

$$\Gamma(a, y) = \int_y^\infty e^{-u} u^{a-1} \, du$$

5. Show that $\theta(x, t) = \theta_0 \ \mathrm{erfc}(x/2\sqrt{\alpha t})$ is the solution of the following diffusion problem:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \theta}{\partial t} \qquad\qquad x \geq 0, \ t > 0$$

and

$$\begin{aligned} \theta(0, t) &= \theta_0, \quad \text{constant} \\ \theta(x, t) &\to 0 \quad\quad \text{as} \quad x \to \infty \end{aligned}$$

6. Given $\theta(x, t) = \theta_0 \ \mathrm{erf} \ x/2\sqrt{\alpha t}$:

   i) Obtain expressions for $\dfrac{\partial \theta}{\partial t}$ and $\dfrac{\partial \theta}{\partial x}$ at any x and all $t > 0$

   ii) For the function $\qquad \dfrac{\sqrt{\pi}}{2} \dfrac{x}{\theta_0} \dfrac{\partial \theta}{\partial x}$

   show that it has a maximum value when $x/2\sqrt{\alpha t} = 1/\sqrt{2}$ and the maximum value is $1/\sqrt{2e}$.

7. Given the transient point source solution valid within an isotropic half space

$$T = \frac{q}{2\pi k r} \ \mathrm{erfc}(r/2\sqrt{\alpha t}), \quad dA = r \ dr \ d\theta$$

derive the expression for the transient temperature rise at the centroid of a circular area (πa^2) which is subjected to a uniform and constant heat flux q. Superposition of point source solutions allows one to write

$$T_0 = \int_0^a \int_0^{2\pi} T \, dA$$

8. For a dimensionless time $Fo < 0.2$ the temperature distribution within an infinite plate $-L \leq x \leq L$ is given approximately by

$$\frac{T(\zeta, Fo) - T_s}{T_0 - T_s} = 1 - \left\{ \text{erfc} \, \frac{1-\zeta}{2\sqrt{Fo}} + \text{erfc} \, \frac{1+\zeta}{2\sqrt{Fo}} \right\}$$

for $0 \leq \zeta \leq 1$ where $\zeta = x/L$ and $Fo = \alpha t/L^2$.

Obtain the expression for the mean temperature $(\overline{T}(Fo) - T_s)/(T_0 - T_s)$ where

$$\overline{T} = \int_0^1 T(\zeta, Fo) \, d\zeta$$

The initial and surface plate temperature are denoted by T_0 and T_s, respectively.

9. Compare the approximate short time ($Fo < 0.2$) solution:

$$\theta(\zeta, Fo) = 1 - \sum_{n=1}^{3} (-1)^{n+1} \left\{ \text{erfc} \frac{(2n-1) - \zeta}{2\sqrt{Fo}} + \text{erfc} \frac{(2n-1) + \zeta}{2\sqrt{Fo}} \right\}$$

and the approximate long time ($Fo > 0.2$) solution

$$\theta(\zeta, Fo) = \sum_{n=1}^{3} \frac{2(-1)^{n+1}}{\delta_n} \, e^{-\delta_n^2 Fo} \, \cos(\delta_n \zeta)$$

with $\delta_n = (2n-1)\pi/2$.

For the centerline ($\zeta = 0$) compute to four decimal places $\theta(0, Fo)_{ST}$ and $\theta(0, Fo)_{LT}$ for $Fo = 0.02, 0.06, 0.1, 0.4, 1.0$ and 2.0 and compare your values with the "exact" values given in Table 1.

Table 1: Exact values of $\boldsymbol{\theta(0, Fo)}$ for the Infinite Plate

$\boldsymbol{Fo}$	$\boldsymbol{\theta(0, Fo)}$
0.02	1.0000
0.06	0.9922
0.10	0.9493
0.40	0.4745
1.0	0.1080
2.0	0.0092

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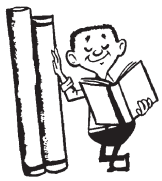
Chebyshev Polynomials



Reading **Problems**

Differential Equation and Its Solution

The Chebyshev differential equation is written as

$$(1 - x^2)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + n^2\,y = 0 \qquad\qquad n = 0, 1, 2, 3, \ldots$$

If we let $x = \cos t$ we obtain

$$\frac{d^2y}{dt^2} + n^2 y = 0$$

whose general solution is

$$y = A\cos nt + B\sin nt$$

or as

$$y = A\cos(n\cos^{-1}x) + B\sin(n\cos^{-1}x) \qquad\qquad |x| < 1$$

or equivalently

$$y = A\mathrm{T}_n(x) + B\mathrm{U}_n(x) \qquad\qquad |x| < 1$$

where $\mathrm{T}_n(x)$ and $\mathrm{U}_n(x)$ are defined as Chebyshev polynomials of the first and second kind of degree n, respectively.

If we let $x = \cosh t$ we obtain

$$\frac{d^2y}{dt^2} - n^2 y = 0$$

whose general solution is

$$y = A\cosh nt + B\sinh nt$$

or as

$$y = A\cosh(n\cosh^{-1} x) + B\sinh(n\cosh^{-1} x) \qquad |x| > 1$$

or equivalently

$$y = A\mathrm{T}_n(x) + B\mathrm{U}_n(x) \qquad |x| > 1$$

The function $\mathrm{T}_n(x)$ is a polynomial. For $|x| < 1$ we have

$$\begin{aligned} \mathrm{T}_n(x) + i\mathrm{U}_n(x) &= (\cos t + i\sin t)^n = \left(x + i\sqrt{1-x^2}\right)^n \\ \mathrm{T}_n(x) - i\mathrm{U}_n(x) &= (\cos t - i\sin t)^n = \left(x - i\sqrt{1-x^2}\right)^n \end{aligned}$$

from which we obtain

$$\mathrm{T}_n(x) = \frac{1}{2}\left[\left(x + i\sqrt{1-x^2}\right)^n + \left(x - i\sqrt{1-x^2}\right)^n\right]$$

For $|x| > 1$ we have

$$\begin{aligned} \mathrm{T}_n(x) + \mathrm{U}_n(x) &= e^{nt} = \left(x \pm \sqrt{x^2-1}\right)^n \\ \mathrm{T}_n(x) - \mathrm{U}_n(x) &= e^{-nt} = \left(x \mp \sqrt{x^2-1}\right)^n \end{aligned}$$

The sum of the last two relationships give the same result for $\mathrm{T}_n(x)$.

Chebyshev Polynomials of the First Kind of Degree n

The Chebyshev polynomials $\mathbf{T}_n(x)$ can be obtained by means of Rodrigue's formula

$$\mathbf{T}_n(x) = \frac{(-2)^n n!}{(2n)!}\sqrt{1-x^2}\,\frac{d^n}{dx^n}\,(1-x^2)^{n-1/2} \qquad n = 0, 1, 2, 3, \ldots$$

The first twelve Chebyshev polynomials are listed in Table 1 and then as powers of x in terms of $\mathbf{T}_n(x)$ in Table 2.

Table 1: Chebyshev Polynomials of the First Kind

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x \\
T_{10}(x) &= 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1 \\
T_{11}(x) &= 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x
\end{aligned}
$$

Table 2: Powers of x as functions of $T_n(x)$

$$
\begin{aligned}
1 &= T_0 \\
x &= T_1 \\
x^2 &= \frac{1}{2}(T_0 + T_2) \\
x^3 &= \frac{1}{4}(3T_1 + T_3) \\
x^4 &= \frac{1}{8}(3T_0 + 4T_2 + T_4) \\
x^5 &= \frac{1}{16}(10T_1 + 5T_3 + T_5) \\
x^6 &= \frac{1}{32}(10T_0 + 15T_2 + 6T_4 + T_6) \\
x^7 &= \frac{1}{64}(35T_1 + 21T_3 + 7T_5 + T_7) \\
x^8 &= \frac{1}{128}(35T_0 + 56T_2 + 28T_4 + 8T_6 + T_8) \\
x^9 &= \frac{1}{256}(126T_1 + 84T_3 + 36T_5 + 9T_7 + T_9) \\
x^{10} &= \frac{1}{512}(126T_0 + 210T_2 + 120T_4 + 45T_6 + 10T_8 + T_{10}) \\
x^{11} &= \frac{1}{1024}(462T_1 + 330T_3 + 165T_5 + 55T_7 + 11T_9 + T_{11})
\end{aligned}
$$

Generating Function for $T_n(x)$

The Chebyshev polynomials of the first kind can be developed by means of the generating function

$$\frac{1 - tx}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} T_n(x) t^n$$

Recurrence Formulas for $T_n(x)$

When the first two Chebyshev polynomials $T_0(x)$ and $T_1(x)$ are known, all other polynomials $T_n(x),\ n \geq 2$ can be obtained by means of the recurrence formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

The derivative of $T_n(x)$ with respect to x can be obtained from

$$(1 - x^2)T_n'(x) = -nxT_n(x) + nT_{n-1}(x)$$

Special Values of $T_n(x)$

The following special values and properties of $T_n(x)$ are often useful:

$$T_n(-x) = (-1)^n T_n(x)$$

$$T_{2n}(0) = (-1)^n$$

$$T_n(1) = 1$$

$$T_{2n+1}(0) = 0$$

$$T_n(-1) = (-1)^n$$

Orthogonality Property of $T_n(x)$

We can determine the orthogonality properties for the Chebyshev polynomials of the first kind from our knowledge of the orthogonality of the cosine functions, namely,

$$\int_0^{\pi} \cos(m\theta)\ \cos(n\ \theta)\ d\theta = \begin{cases} 0 & (m \neq n) \\ \pi/2 & (m = n \neq 0) \\ \pi & (m = n = 0) \end{cases}$$

Then substituting

$$\begin{aligned} T_n(x) &= \cos(n\theta) \\ \cos\theta &= x \end{aligned}$$

to obtain the orthogonality properties of the Chebyshev polynomials:

$$\int_{-1}^{1} \frac{T_m(x)\ T_n(x)\ dx}{\sqrt{1-x^2}} = \begin{cases} 0 & (m \neq n) \\ \pi/2 & (m = n \neq 0) \\ \pi & (m = n = 0) \end{cases}$$

We observe that the Chebyshev polynomials form an orthogonal set on the interval $-1 \leq x \leq 1$ with the weighting function $(1 - x^2)^{-1/2}$

Orthogonal Series of Chebyshev Polynomials

An arbitrary function $f(x)$ which is continuous and single-valued, defined over the interval $-1 \leq x \leq 1$, can be expanded as a series of Chebyshev polynomials:

$$\begin{aligned} f(x) &= A_0T_0(x) + A_1T_1(x) + A_2T_2(x) + \ldots \\ &= \sum_{n=0}^{\infty} A_nT_n(x) \end{aligned}$$

where the coefficients A_n are given by

$$A_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)\ dx}{\sqrt{1-x^2}} \qquad\qquad n = 0$$

and

$$A_n = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)\ T_n(x)\ dx}{\sqrt{1-x^2}} \qquad\qquad n = 1, 2, 3, \ldots$$

The following definite integrals are often useful in the series expansion of $f(x)$:

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi \qquad\qquad \int_{-1}^{1} \frac{x^3\ dx}{\sqrt{1-x^2}} = 0$$

$$\int_{-1}^{1} \frac{x\ dx}{\sqrt{1-x^2}} = 0 \qquad\qquad \int_{-1}^{1} \frac{x^4\ dx}{\sqrt{1-x^2}} = \frac{3\pi}{8}$$

$$\int_{-1}^{1} \frac{x^2\ dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \qquad\qquad \int_{-1}^{1} \frac{x^5\ dx}{\sqrt{1-x^2}} = 0$$

Chebyshev Polynomials Over a Discrete Set of Points

A continuous function over a continuous interval is often replaced by a set of discrete values of the function at discrete points. It can be shown that the Chebyshev polynomials $T_n(x)$ are orthogonal over the following discrete set of $N+1$ points x_i, equally spaced on θ,

$$\theta_i = 0,\ \frac{\pi}{N},\ \frac{2\pi}{N}, \ldots (N-1)\ \frac{\pi}{N},\ \pi$$

where

$$x_i = \arccos\ \theta_i$$

We have

$$\frac{1}{2}T_m(-1)T_n(-1)+\sum_{i=2}^{N-1}T_m(x_i)T_n(x_i)+\frac{1}{2}T_m(1)T_n(1) = \begin{cases} 0 & (m \neq n) \\ N/2 & (m = n \neq 0) \\ N & (m = n = 0) \end{cases}$$

The $T_m(x)$ are also orthogonal over the following N points t_i equally spaced,

$$\theta_i = \frac{\pi}{2N}, \frac{3\pi}{2N}, \frac{5\pi}{2N}, \ldots, \frac{(2N-1)\pi}{2N}$$

and

$$t_i = \arccos \theta_i$$

$$\sum_{i=1}^{N} T_m(t_i)T_n(t_i) = \begin{cases} 0 & (m \neq n) \\ N/2 & (m = n \neq 0) \\ N & (m = n = 0) \end{cases}$$

The set of points t_i are clearly the midpoints in θ of the first case. The unequal spacing of the points in $x_i(Nt_i)$ compensates for the weight factor

$$W(x) = (1 - x^2)^{-1/2}$$

in the continuous case.

Additional Identities of Chebyshev Polynomials

The Chebyshev polynomials are both orthogonal polynomials and the trigonometric $\cos nx$ functions in disguise, therefore they satisfy a large number of useful relationships.

The differentiation and integration properties are very important in analytical and numerical work. We begin with

$$T_{n+1}(x) = \cos[(n+1)\ \cos^{-1} x]$$

and

$$T_{n-1}(x) = \cos[(n-1)\ \cos^{-1} x]$$

Differentiating both expressions gives

$$\frac{1}{(n+1)} \frac{d[T_{n+1}(x)]}{dx} = \frac{-\sin[(n+1)\ \cos^{-1} x}{-\sqrt{1-x^2}}$$

and

$$\frac{1}{(n-1)} \frac{d[T_{n-1}(x)]}{dx} = \frac{-\sin[(n-1)\ \cos^{-1} x}{-\sqrt{1-x^2}}$$

Subtracting the last two expressions yields

$$\frac{1}{(n+1)} \frac{d[T_{n+1}(x)]}{dx} - \frac{1}{(n-1)} \frac{d[T_{n-1}(x)]}{dx} = \frac{\sin(n+1)\theta - \sin(n-1)\theta}{\sin\theta}$$

or

$$\frac{T'_{n+1}(x)}{(n+1)} - \frac{T'_{n-1}(x)}{(n-1)} = \frac{2\cos n\theta\ \sin\theta}{\sin\theta} = 2T_n(x) \qquad n \geq 2$$

Therefore

$$\begin{aligned} T_2'(x) &= 4T_1 \\ T_1'(x) &= T_0 \\ T_0'(x) &= 0 \end{aligned}$$

We have the formulas for the differentiation of Chebyshev polynomials, therefore these formulas can be used to develop integration for the Chebyshev polynomials:

$$\begin{aligned} \int T_n(x)dx &= \frac{1}{2}\left[\frac{T_{n+1}(x)}{(n+1)} - \frac{T_{n-1}(x)}{(n-1)}\right] + C \qquad n \geq 2 \\ \int T_1(x)dx &= \frac{1}{4}T_2(x) + C \\ \int T_0(x)dx &= T_1(x) + C \end{aligned}$$

The Shifted Chebyshev Polynomials

For analytical and numerical work it is often convenient to use the half interval $0 \leq x \leq 1$ instead of the full interval $-1 \leq x \leq 1$. For this purpose the shifted Chebyshev polynomials are defined:

$$T_n^*(x) = T_n * (2x - 1)$$

Thus we have for the first few polynomials

$$\begin{aligned} T_0^* &= 1 \\ T_1^* &= 2x - 1 \\ T_2^* &= 8x^2 - 8x + 1 \\ T_3^* &= 32x^3 - 48x^2 + 18x - 1 \\ T_4^* &= 128x^4 - 256x^3 + 160x^2 - 32x + 1 \end{aligned}$$

and the following powers of x as functions of $T_n^*(x)$;

$$\begin{aligned}
1 &= T_0^* \\
x &= \frac{1}{2}(T_0^* + T_1^*) \\
x^2 &= \frac{1}{8}(3T_0^* + 4T_1^* + T_2^*) \\
x^3 &= \frac{1}{32}(10T_0^* + 15T_1^* + 6T_2^* + T_3^*) \\
x^4 &= \frac{1}{128}(35T_0^* + 56T_1^* + 28T_2^* + 8T_3^* + T_4^*)
\end{aligned}$$

The recurrence relationship for the shifted polynomials is:

$$T_{n+1}^*(x) = (4x - 2)T_n^*(x) - T_{n-1}^*(x) \qquad T_0^*(x) = 1$$

or

$$xT_n^*(x) = \frac{1}{4}T_{n+1}^*(x) + \frac{1}{2}T_n^*(x) + \frac{1}{4}T_{n-1}^*(x)$$

where

$$T_n^*(x) = \cos\left[n\cos^{-1}(2x-1)\right] = T_n(2x-1)$$

Expansion of x^n in a Series of $T_n(x)$

A method of expanding x^n in a series of Chebyshev polynomials employes the recurrence relation written as

$$\begin{aligned}
xT_n(x) &= \frac{1}{2}\left[T_{n+1}(x) + T_{n-1}(x)\right] \qquad n = 1, 2, 3 \ldots \\
xT_0(x) &= T_1(x)
\end{aligned}$$

To illustrate the method, consider x^4

$$\begin{aligned} x^4 &= x^2(xT_1) = \frac{x^2}{2}[T_2 + T_0] = \frac{x}{4}[T_1 + T_3 + 2T_1] \\ &= \frac{1}{4}[3xT_1 + xT_3] = \frac{1}{8}[3T_0 + 3T_2 + T_4 + T_2] \\ &= \frac{1}{8}T_4 + \frac{1}{2}T_2 + \frac{3}{8}T_0 \end{aligned}$$

This result is consistent with the expansion of x^4 given in Table 2.

Approximation of Functions by Chebyshev Polynomials

Sometimes when a function $f(x)$ is to be approximated by a polynomial of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n + E_N(x) \qquad |x| \leq 1$$

where $|E_n(x)|$ does not exceed an allowed limit, it is possible to reduce the degree of the polynomial by a process called economization of power series. The procedure is to convert the polynomial to a linear combination of Chebyshev polynomials:

$$\sum_{n=0}^{N} a_n x^n = \sum_{n=0}^{N} b_n T_n(x) \qquad n = 0, 1, 2, \ldots$$

It may be possible to drop some of the last terms without permitting the error to exceed the prescribed limit. Since $|T_n(x)| \leq 1$, the number of terms which can be omitted is determined by the magnitude of the coefficient b.

The Chebyshev polynomials are useful in numerical work for the interval $-1 \leq x \leq 1$ because

1. $|T_n(x)] \leq 1$ within $-1 \leq x \leq 1$
2. The maxima and minima are of comparable value.

3. The maxima and minima are spread reasonably uniformly over the interval $\boldsymbol{-1 \leq x \leq 1}$

4. All Chebyshev polynomials satisfy a three term recurrence relation.

5. They are easy to compute and to convert to and from a power series form.

These properties together produce an approximating polynomial which minimizes error in its application. This is different from the least squares approximation where the sum of the squares of the errors is minimized; the maximum error itself can be quite large. In the Chebyshev approximation, the average error can be large but the maximum error is minimized. Chebyshev approximations of a function are sometimes said to be mini-max approximations of the function.

The following table gives the Chebyshev polynomial approximation of several power series.

Table 3: Power Series and its Chebyshev Approximation

1. $f(x) = a_0$

   $f(x) = a_0 T_0$

2. $f(x) = a_0 + a_1 x$

   $f(x) = a_0 T_0 + a_1 T_1$

3. $f(x) = a_0 + a_1 x + a_2 x^2$

   $$f(x) = \left(a_0 + \frac{a_2}{2}\right) T_0 + a_1 T_1 + \left(\frac{a_2}{2}\right) T_2$$

4. $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

   $$f(x) = \left(a_0 + \frac{a_2}{2}\right) T_0 + \left(a_1 + \frac{3a_3}{4}\right) T_1 + \left(\frac{a_2}{2}\right) T_2 + \left(\frac{a_3}{4}\right) T_3$$

5. $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

   $$f(x) = \left(a_0 + \frac{a_2}{2} + \frac{a_3}{8}\right) T_0 + \left(a_1 + \frac{3a_3}{4}\right) T_1 + \left(\frac{a_2}{2} + \frac{a_4}{2}\right) T_2 + \left(\frac{a_3}{8}\right) T_3 + \left(\frac{a_4}{8}\right) T_4$$

6. $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

   $$f(x) = \left(a_0 + \frac{a_2}{2} + \frac{3a_4}{8}\right) T_0 + \left(a_1 + \frac{3a_3}{4} + \frac{5a_5}{8}\right) T_1 + \left(\frac{a_2}{2} + \frac{a_4}{2}\right) T_2 + \left(\frac{a_3}{4} + \frac{5a_5}{16}\right) T_3 + \left(\frac{a_4}{8}\right) T_4 + \left(\frac{a_5}{16}\right) T_5$$

Table 4: Formulas for Economization of Power Series

$$
\begin{aligned}
x &= T_1 \\
x^2 &= \frac{1}{2}(1 + T_2) \\
x^3 &= \frac{1}{4}(3x + T_3) \\
x^4 &= \frac{1}{8}(8x^2 - 1 + T_4) \\
x^5 &= \frac{1}{16}(20x^3 - 5x + T_5) \\
x^6 &= \frac{1}{32}(48x^4 - 18x^2 + 1 + T_6) \\
x^7 &= \frac{1}{64}(112x^5 - 56x^3 + 7x + T_7) \\
x^8 &= \frac{1}{128}(256x^6 - 160x^4 + 32x^2 - 1 + T_8) \\
x^9 &= \frac{1}{256}(576x^7 - 432x^5 + 120x^3 - 9x + T_9) \\
x^{10} &= \frac{1}{512}(1280x^8 - 1120x^6 + 400x^4 - 50x^2 + 1 + T_{10}) \\
x^{11} &= \frac{1}{1024}(2816x^9 - 2816x^7 + 1232x^5 - 220x^3 + 11x + T_{11})
\end{aligned}
$$

For easy reference the formulas for economization of power series in terms of Chebyshev are given in Table 4.

Assigned Problems

Problem Set for Chebyshev Polynomials

1. Obtain the first three Chebyshev polynomials $T_0(x),\ T_1(x)$ and $T_2(x)$ by means of the Rodrigue's formula.

2. Show that the Chebyshev polynomial $T_3(x)$ is a solution of Chebyshev's equation of order 3.

3. By means of the recurrence formula obtain Chebyshev polynomials $T_2(x)$ and $T_3(x)$ given $T_0(x)$ and $T_1(x)$.

4. Show that $T_n(1) = 1$ and $T_n(-1) = (-1)^n$

5. Show that $T_n(0) = 0$ if n is odd and $(-1)^{n/2}$ if n is even.

6. Setting $x = \cos\theta$ show that

$$T_n(x) = \frac{1}{2}\left[\left(x + i\sqrt{1-x^2}\right)^n + \left(x - i\sqrt{1-x^2}\right)^n\right]$$

where $i = \sqrt{-1}$.

7. Find the general solution of Chebyshev's equation for $n = 0$.

8. Obtain a series expansion for $f(x) = x^2$ in terms of Chebyshev polynomials $T_n(x)$,

$$x^2 = \sum_{n=0}^{3} A_n T_n(x)$$

9. Express x^4 as a sum of Chebyshev polynomials of the first kind.