Fourier Law of Heat Conduction

We can examine a small representative element of the bar over the range \( x \rightarrow x + \Delta x \). Just as in the case of a thermodynamic closed system, we can treat this slice as a closed system where no mass flows through any of the six faces.

From a 1\(^{st}\) law energy balance:

\[
\frac{\partial E}{\partial t} = \dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{W}
\]

Using Fourier’s law of heat conduction \( \dot{Q}_x = -kA \frac{\partial T}{\partial x} \) we can write the general 1-D conduction equation as

\[
\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \dot{g} = \rho C \frac{\partial T}{\partial t}
\]

\( \text{longitudinal conduction} \)

\( \text{internal heat generation} \)

\( \text{thermal inertia} \)
Special Cases

1. Multidimensional Systems: The general conduction equation can be extended to three dimensional Cartesian systems as follows:

\[
\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + \dot{g} = \rho C \frac{\partial T}{\partial t}
\]

2. Constant Properties: If we assume that properties are independent of temperature, then the conductivity can be taken outside the derivative

\[\nabla^2 T + \frac{\dot{g}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}\]

3. Steady State: If \( t \to \infty \) then all terms \( \frac{\partial}{\partial t} \to 0 \)

\[\nabla^2 T = -\frac{\dot{g}}{k} \quad \Leftarrow \text{ Poisson’s Equation}\]

4. No Internal Heat Generation:

\[\nabla^2 T = 0 \quad \Leftarrow \text{ Laplace’s Equation}\]

Thermal Resistance Networks

Thermal circuits based on heat flow rate, \( \dot{Q} \), temperature difference, \( \Delta T \) and thermal resistance, \( R \), enable analysis of complex systems. This is analogous to current flow through electrical circuits where, \( I = \Delta V/R \)

<table>
<thead>
<tr>
<th>Thermal</th>
<th>Electrical</th>
<th>Role</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{Q} ), heat flow rate</td>
<td>( I ), current flow</td>
<td>transfer mechanism</td>
</tr>
<tr>
<td>( T_{in} - T_{out} ), temperature difference</td>
<td>( V_{in} - V_{out} ), voltage difference</td>
<td>driving force</td>
</tr>
<tr>
<td>( R ), thermal resistance</td>
<td>( R ), electrical resistance</td>
<td>impedance to flow</td>
</tr>
</tbody>
</table>

Resistances in Series

The total heat flow across the system can be written as

\[\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{total}} \quad \text{where} \quad R_{total} = \sum_{i=1}^{4} R_i\]
Resistances in Parallel

For systems of parallel flow paths as shown below

In general, for parallel networks we can use a parallel resistor network as follows:

\[
\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \quad \text{and} \quad \dot{Q} = \frac{T_1 - T_2}{R_{total}}
\]
**Thermal Contact Resistance**

- the thermal interface between contacting materials offers resistance to heat flow
- the total heat flow rate can be written as

\[ \dot{Q}_{\text{total}} = h_c A \Delta T_{\text{interface}} \]

The conductance can be written in terms of a resistance as

\[ h_c A = \frac{\dot{Q}_{\text{total}}}{\Delta T} = \frac{1}{R_c} \]

Table 10-2 can be used to obtain some representative values for contact conductance.

**Cylindrical Systems**

For steady, 1D heat flow from \( T_1 \) to \( T_2 \) in a cylindrical system as shown below

\[ \dot{Q}_r = \frac{T_1 - T_2}{\ln(r_2/r_1)} \]

where \( R = \left( \frac{\ln(r_2/r_1)}{2\pi k L} \right) \)
Composite Cylinders

\[ R_{\text{total}} = R_1 + R_2 + R_3 + R_4 \]
\[ = \frac{1}{h_1 A_1} + \frac{\ln(r_2/r_1)}{2\pi k_2 L} + \frac{\ln(r_3/r_2)}{2\pi k_3 L} + \frac{1}{h_4 A_4} \]

Example 5-1: Determine the temperature \((T_1)\) of an electric wire surrounded by a layer of plastic insulation with a thermal conductivity of 0.15 \(W/mK\) when the thickness of the insulation is a) 2 \(mm\) and b) 4 \(mm\), subject to the following conditions:

Given:

\( I = 10 \ A \)
\( \Delta \varepsilon = \varepsilon_1 - \varepsilon_2 = 8 \ V \)
\( D = 3 \ mm \)
\( L = 5 \ m \)
\( k = 0.15 \ W/mK \)
\( T_\infty = 30 ^\circ C \)
\( h = 12 \ W/m^2 \cdot K \)

Find:

\( T_1 = ??? \)

when:

\( \delta = 2 \ mm \)
\( \delta = 4 \ mm \)
Critical Thickness of Insulation

Although the purpose of adding more insulation is to increase resistance and decrease heat transfer, we can see from the resistor network that increasing $r_o$ actually results in a decrease in the convective resistance.

$$R_{total} = R_1 + R_2 = \frac{\ln(r_o/r_i)}{2\pi kL} + \frac{1}{h2\pi r_o L}$$

Could there be a situation in which adding insulation increases the overall heat transfer?

$$r_{cr, cyl} = \frac{k}{h} \quad [m]$$
Spherical Systems

We can write the heat transfer as

$$\dot{Q} = \frac{4\pi kr_i r_o}{r_0 - r_i} (T_i - T_o) = \frac{(T_i - T_o)}{R}$$

where $R = \frac{r_o - r_i}{4\pi kr_i r_o}$

Notes:

1. the critical thickness of insulation for a spherical shell is given as

$$r_{cr, sphere} = \frac{2k}{h} [m]$$

Heat Transfer from Finned Surfaces

The conduction equation for a fin with constant cross section is

$$k A_c \frac{d^2 T}{dx^2} - h P (T - T_\infty) = 0$$

The temperature difference between the fin and the surroundings (temperature excess) is usually expressed as

$$\theta = T(x) - T_\infty$$
which allows the 1-D fin equation to be written as

\[
\frac{d^2\theta}{dx^2} - m^2 \theta = 0
\]

where the fin parameter \( m \) is

\[
m = \left( \frac{hP}{kA_c} \right)^{1/2}
\]

The solution to the differential equation for \( \theta \) is

\[
\theta(x) = C_1 \sinh(mx) + C_2 \cosh(mx) \quad \text{[\( \equiv \theta(x) = C_1 e^{mx} + C_2 e^{-mx} \)]}
\]

Potential boundary conditions include:

Base: \( \Omega x = 0 \quad \theta = \theta_b \)

Tip: \( \Omega x = L \quad \theta = \theta_L \) \quad [\( T \)-specified tip]

\[
\theta = \left. \frac{d\theta}{dx} \right|_{x=L} = 0 \quad \text{[adiabatic (insulated) tip]}
\]

\[
\theta \to 0 \quad \text{[infinitely long fin]}
\]
**Fin Efficiency and Effectiveness**

The dimensionless parameter that compares the actual heat transfer from the fin to the ideal heat transfer from the fin is the *fin efficiency*

\[
\eta = \frac{\text{actual heat transfer rate}}{\text{maximum heat transfer rate when the entire fin is at } T_b} = \frac{\dot{Q}_b}{hP L \theta_b}
\]

An alternative figure of merit is the *fin effectiveness* given as

\[
\epsilon_{fin} = \frac{\text{total fin heat transfer}}{\text{the heat transfer that would have occurred through the base area in the absence of the fin}} = \frac{\dot{Q}_b}{hA_c \theta_b}
\]

**Transient Heat Conduction**

The Biot number can be obtained as follows:

\[
\frac{T_H - T_s}{T_s - T_\infty} = \frac{L/(k \cdot A)}{1/(h \cdot A)} = \frac{\text{internal resistance to H.T.}}{\text{external resistance to H.T.}} = \frac{hL}{k} = Bi
\]

\[R \text{ _int} \ll R \text{ _ext}: \text{ the Biot number is small and we can conclude} \]

\[T_H - T_s \ll T_s - T_\infty \quad \text{ and in the limit } \quad T_H \approx T_s\]

\[R \text{ _ext} \ll R \text{ _int}: \text{ the Biot number is large and we can conclude} \]

\[T_s - T_\infty \ll T_H - T_s \quad \text{ and in the limit } \quad T_s \approx T_\infty\]
Transience Conduction Analysis

- if the internal temperature of a body remains relatively constant with respect to time
  - can be treated as a lumped system analysis
  - heat transfer is a function of time only, $T = T(t)$

- typical criteria for lumped system analysis $\rightarrow Bi \leq 0.1$

Lumped System Analysis

For the 3-D body of volume $V$ and surface area $A$, we can use a lumped system analysis if

$$Bi = \frac{hV}{kA} < 0.1 \quad \Leftarrow \text{results in an error of less that 5\%}$$

For an incompressible substance specific heat is constant and we can write

$$mC \frac{dT}{dt} = -Ah_1 \frac{1}{R_{th}} (T - T_{\infty})$$

where $C_{th} = \text{lumped capacitance}$ and $R_{th} = \text{thermal resistance}$

We can integrate and apply the initial condition, $T = T_i \ @ t = 0$ to obtain

$$\frac{T(t) - T_{\infty}}{T_i - T_{\infty}} = e^{-t/(R_{th}C_{th})} = e^{-t/\tau} = e^{-bt}$$
where the time constant can be written as

\[ \frac{1}{b} = \tau = R_{th} \cdot C_{th} = \text{thermal time constant} = \frac{mC}{Ah} = \frac{\rho VC}{Ah} \]

**Example 5-2:** Determine the time it takes a fuse to melt if a current of 3 A suddenly flows through the fuse subject to the following conditions:

**Given:**
- \( D = 0.1 \text{ mm} \)
- \( T_{\text{melt}} = 900 \, ^\circ C \)
- \( k = 20 \, W/mK \)
- \( L = 10 \text{ mm} \)
- \( T_\infty = 30 \, ^\circ C \)
- \( \alpha = 5 \times 10^{-5} \, m^2/s \equiv \frac{k}{\rho C_p} \)

**Assume:**
- constant resistance \( R = 0.2 \, \text{ohms} \)
- the overall heat transfer coefficient is \( h = h_{\text{conv}} + h_{\text{rad}} = 10 \, W/m^2K \)
- neglect any conduction losses to the fuse support
Approximate Analytical and Graphical Solutions (Heisler Charts)

The lumped system analysis can be used if \( Bi = hL/k \leq 0.1 \) but what if \( Bi > 0.1 \) we must find a solution to the PDE where temperature is \( T(x, t) = f(x, L, t, k, \alpha, h, T_i, T_\infty) \)

\[
\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}
\]

The analytical solution to this equation takes the form of a series solution

\[
\frac{T(x, t) - T_\infty}{T_i - T_\infty} = \sum_{n=1,3,5\ldots}^{\infty} A_n e^{-\frac{n\lambda L^2}{\alpha t}} \cos\left(\frac{n\lambda x}{L}\right)
\]

By using dimensionless groups, we can reduce the temperature dependence to 3 dimensionless parameters

**Dimensionless Group** | **Formulation**
---|---
temperature | \( \theta(x, t) = \frac{T(x, t) - T_\infty}{T_i - T_\infty} \)
position | \( x = x/L \)
heat transfer | \( Bi = hL/k \) Biot number

\( Fo = \alpha t/L^2 \) Fourier number

note: Cengel uses \( \tau \) instead of \( Fo \).

With this, two approaches are possible

1. use the first term of the infinite series solution. This method is only valid for \( Fo > 0.2 \)
2. use the Heisler charts for each geometry as shown in Figs. 11-15, 11-16 and 11-17
First term solution: $Fo > 0.2 \rightarrow$ error about 2% max.

Plane Wall: 
\[
\theta_{\text{wall}}(x, t) = \frac{T(x, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \cos(\lambda_1 x/L)
\]

Cylinder:  
\[
\theta_{\text{cyl}}(r, t) = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} J_0(\lambda_1 r/r_o)
\]

Sphere: 
\[
\theta_{\text{sph}}(r, t) = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}
\]

$\lambda_1, A_1$ can be determined from Table 11-2 based on the calculated value of the Biot number (will likely require some interpolation). The Bessel function, $J_0$ can be calculated using Table 11-3.

Heisler Charts

- find $T_0$ at the center for a given time (Table 11-15 a, Table 11-16 a or Table 11-17 a)
- find $T$ at other locations at the same time (Table 11-15 b, Table 11-16 b or Table 11-17 b)
- find $Q_{\text{tot}}$ up to time $t$ (Table 11-15 c, Table 11-16 c or Table 11-17 c)

**Example 5-3:** An aluminum plate made of Al 2024-T6 with a thickness of 0.15 m is initially at a temperature of 300 K. It is then placed in a furnace at 800 K with a convection coefficient of 500 W/m²K.

Find:

i) the time (s) for the plate midplane to reach 700 K

ii) the surface temperature at this condition. Use both the Heisler charts and the approximate analytical, first term solution.