MODELING TRANSIENT CONDUCTION IN DOUBLY CONNECTED REGIONS BETWEEN ISOTHERMAL BOUNDARIES OF ARBITRARY SHAPE

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Abstract

Analytical models are developed to predict dimensionless heat flow rate for transient conduction in the doubly connected region formed between an arbitrarily shaped, isothermal inner body and its surrounding isothermal enclosure. This modeling procedure is based on limiting cases and trends identified in the exact solution for the concentric spheres. Asymptotic solutions for the limiting cases of small and large relative domain size, as well as short and long time, are combined to provide comprehensive models applicable to the full range of the independent parameters. Validation of the models by the spherical enclosure solution and numerical data for the concentric cubes has shown excellent agreement between the model and the data, within less than 3% RMS difference for most cases.

Nomenclature

a	=	inner sphere radius; m
A_i	=	inner body surface area; m^2
b	=	outer sphere radius; m
c_p	=	specific heat; J/kgK
Fo	=	Fourier number, $\equiv \alpha t/A_i$
k	=	thermal conductivity; W/mK
$\mathcal{L}$	=	arbitrary scale length; m
L	=	plane wall thickness; m
m, n, p	=	fitting parameters
Q	=	total heat flow rate; W
$Q^\star$	=	dimensionless heat flow rate, $\equiv Q/(k\sqrt{A_i}\theta_i)$
r	=	radial coordinate
$\vec{r}$	=	radius vector
s	=	cube side length; m
S	=	conduction shape factor; m
$S^\star$	=	dimensionless shape factor, $\equiv S/\sqrt(A_i)$
t	=	time; s
T	=	temperature; K
V	=	volume; m^3
x	=	Cartesian coordinate

Greek Symbols

α	=	thermal diffusivity; $k/\rho\, c_p$; m^2/s
β	=	relative domain size, Eq.(21)
δ	=	gap spacing, Eq.(19); m
Δ	=	conduction layer thickness, $\equiv \sqrt{\alpha t}$; m
ϕ	=	dimensionless temperature difference, $\equiv \theta/\theta_i$
ρ	=	density; kg/m^3
θ	=	temperature difference, $\equiv T - T_o$; K

Subscripts

e	=	effective, based on concentric spheres
i	=	inner boundary
L	=	based on plane wall thickness L
o	=	outer boundary
t	=	transient

Introduction

Complete full time solutions for transient conduction in the doubly connected region formed between an isothermal inner body and its surrounding isothermal enclosure are of interest to engineers for the development of heat transfer models for electronic equipment enclosures. These enclosures are typically sealed to prevent damage to the compo-

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nents and circuitry due to moisture or other contaminants. Although they provide effective protection from the elements, these enclosures prevent the transfer of heat by convection from the components directly to the surrounding environment. Instead, these equipment rely on conduction and convection within the enclosed fluid to transfer the heat from the heated inner bodies to the enclosure walls. Analytical models for the heat transfer in these enclosed environments that can be used during preliminary design to perform parametric studies are of particular interest to engineers in the microelectronics and telecommunications industries.

Models for transient conduction in the three-dimensional region from an isothermal body to its surrounding enclosure for the full range of time are presently not available for the general case involving arbitrary-shaped inner and outer boundaries. An exact solution exists for one simplified geometric case only, the concentric spherical enclosure, as presented by Carslaw and Jaeger[1]. The computation of the transient temperature distribution and subsequent calculation of the heat flow rate using this series solution may require a large number of terms for certain combinations of geometry and time. Exact solutions or approximate methods are currently not available for problems involving other boundary shapes, such as cuboids or spheroids, or for problems with different boundary shapes.

Hassani[2] and Hassani and Hollands[3] presented an analytical model for calculating the dimensionless conduction shape factor in three-dimensional enclosures with uniform gap spacing formed between boundaries of similar shape. This model is based on a combination of asymptotic solutions for small and large gap spacing, and a fitting parameter is correlated as a function of the aspect ratio of the inner body. The Hassani and Hollands[3] model is applicable only to steady state conduction in enclosures with uniform gap spacing, and transient conduction is not included in their analysis.

Yovanovich et al.[4] developed a general model for transient conduction for isothermal convex bodies of arbitrary shape in a full space valid for the full range of time. The model is based on a combination of the half-space, transient asymptote with the steady state conduction shape factor, where the fitting parameter is described as a function of aspect ratio. The model was shown to be in excellent agreement with numerical data for a variety of body shapes over the full range of time. It is anticipated that the model developed by Yovanovich et al.[4] can be used in the limiting case of large relative domain size.

The objective of the current study is to develop

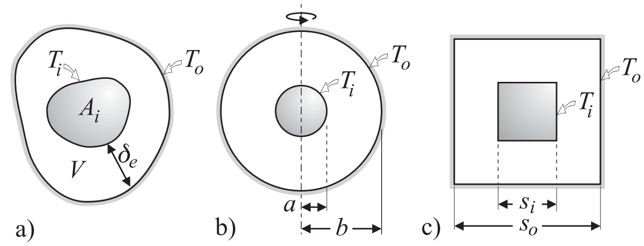

Figure 1: Schematic of Enclosure Geometries: a) arbitrarily-shaped boundaries; b) concentric spheres; c) concentric cubes

a simple and accurate modeling procedure for the dimensionless heat flow rate within doubly connected regions of arbitrary shape for the full range of time. The model will be developed using the available exact solution for transient conduction in concentric spheres and the plane wall, and will be validated using the exact solution for the spherical enclosure. Accurate numerical data will be generated for concentric cubes for a wide range of geometry and time and the model will validated using these data as well.

Problem Description

The problem of interest in the current study involves a region bounded internally and externally by arbitrarily-shaped, isothermal boundaries, as shown in Fig. 1 a). The initial temperature is isothermal, $T = T_o$, and at some given time the temperature on the inner boundary is raised to $T_i > T_o$, while the temperature on the outer boundary is maintained at its initial value. The governing equation for this problem is the diffusion equation:

$$\nabla^2 \theta = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}, \quad t > 0 \tag{1}$$

where the temperature difference θ is defined as a function of position and time by $\theta = T(\vec{r}, t) - T_o$. The initial condition is:

$$t = 0, \quad \theta = 0$$

everywhere inside the region of volume V. Boundary conditions are imposed at the inner and outer body surfaces as follows:

$$\begin{array}{lll} \text{Inner Boundary,} & t > 0, & \theta = \theta_i \\ \text{Outer Boundary,} & t > 0, & \theta = 0 \end{array}$$

For the special case of concentric spheres shown in Fig. 2 b), the diffusion equation is expressed in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}, \quad t > 0 \tag{2}$$

where $\theta = T(r,t) - T_o$. The initial and boundary conditions are:

$$\begin{aligned} a \le r \le b, \quad & t = 0, \quad \theta = 0 \\ r = a, \quad & t > 0, \quad \theta = \theta_i \\ r = b, \quad & t > 0, \quad \theta = 0 \end{aligned}$$

The instantaneous heat transfer rate into the enclosed region is determined based on the temperature gradient at the inner boundary:

$$Q = -k \iint_{A_i} \nabla\theta \cdot \vec{n}\, dA_i \tag{3}$$

where $\vec{n}$ represents an outward directed normal on the inner boundary surface. The instantaneous heat transfer rate is non-dimensionalized with respect to a general length scale $\mathcal{L}$ by:

$$Q^\star_{\mathcal{L}} = \frac{Q\mathcal{L}}{kA_i\theta_i} = \frac{\mathcal{L}}{A_i} \iint_{A_i} -\nabla\phi \cdot \vec{n}\, dA_i \tag{4}$$

where $\phi = \theta/\theta_i$ is the dimensionless temperature difference. In Yovanovich et al.[4] it was shown that the use of the square root of the active surface area as the characteristic length significantly reduces the dependence of the solution on geometry, making the model applicable to a variety of body shapes. Based on this previous work, a length scale based on the inner body surface area is chosen:

$$\mathcal{L} = \sqrt{A_i}$$

The dimensionless time based on this scale length is defined as:

$$Fo = \frac{\alpha t}{A_i} \tag{5}$$

The relative size of the inner and outer boundaries, sometimes called the aspect ratio of the enclosure, will be represented by the relative domain size parameter $\beta = f(V, A_i)$, where V is the volume of the enclosed region. A full definition of β will be presented during the model development, where it will be shown that for the limiting case of concentric spheres the relative domain size parameter reduces to the radii ratio, $\beta = b/a$.

Hollow Sphere: Exact Solution

In order to proceed with the model development, it is helpful to examine the exact solution for the hollow sphere, as presented by Carslaw and Jaeger[1], for the full range of the dimensionless parameters:

$$1 < \beta < \infty \qquad \text{and} \qquad 0 < Fo < \infty$$

The solution for the transient temperature distribution presented by Carslaw and Jaeger[1] can be simplified based on the initial and boundary conditions to the following:

$$\begin{aligned} \theta &= \theta_o \frac{a}{r}\left[\frac{b-r}{b-a}\right] \\ &\quad - \frac{2a}{\pi r}\sum_{n=1}^{\infty} \frac{\sin n\pi}{n} \frac{(r-a)}{(b-a)} \exp\left[-\frac{n^2\pi^2\alpha t}{(b-a)^2}\right] \end{aligned} \tag{6}$$

valid for all time $t > 0$. The instantaneous heat transfer rate is determined based on the temperature gradient at the inner boundary:

$$Q = -kA_i \left.\frac{\partial\theta}{\partial r}\right|_{r=a} \tag{7}$$

The dimensionless instantaneous heat transfer rate becomes:

$$Q^\star = \frac{2\sqrt{\pi}\beta}{\beta - 1} + \frac{4\sqrt{\pi}}{\beta - 1}\sum_{n=1}^{\infty} \exp\left[-\frac{4n^2\pi^3 Fo}{(\beta-1)^2}\right] \tag{8}$$

where $\beta = b/a$. This series solution was programmed using the computer algebra software Maple[5] and sufficient terms were included in the series to satisfy a convergence tolerance of 1×10^{-8} percent difference between subsequent values. Values for the dimensionless instantaneous heat flow rate were calculated for ten values of β in the range $1.1 \le \beta \le 50$ and for the range of dimensionless time $1 \times 10^{-5} \le Fo \le 50$. The results of the exact solution for the hollow sphere are presented in Fig. 2.

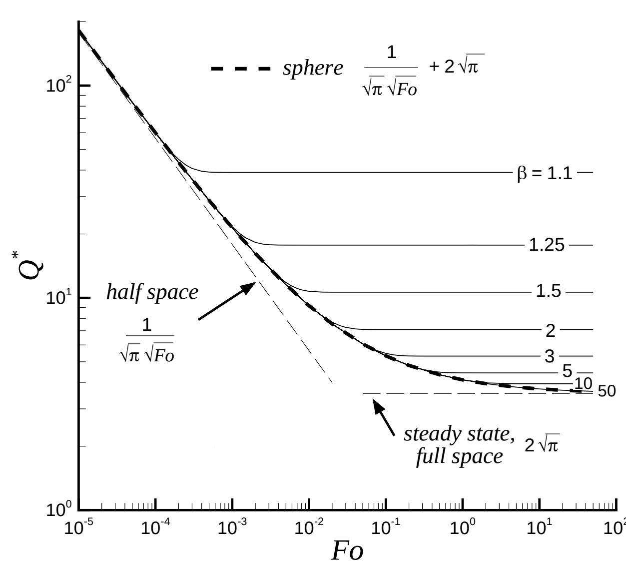

Figure 2: Exact Solution for Concentric Spheres

The results plotted in Fig. 2 clearly demonstrate that the exact solution possesses a number of limiting cases as a function of Fo and β. The short and long time limits can be defined by comparing the temperature field penetration depth, $\Delta = \sqrt{\alpha t}$, with the gap spacing of the enclosure, $\delta = b - a$. Converting Δ and δ in terms of the previously defined dimensionless parameters yields:

$$\Delta \to \sqrt{Fo}\ , \qquad \delta \to \frac{\beta - 1}{2\sqrt{\pi}}$$

For short time, when $\sqrt{Fo} << (\beta - 1)/2\sqrt{\pi}$, the solutions for all β approach the dimensionless heat flow rate for the semi-infinite solid, or half-space solution:

$$Q^\star \to \frac{1}{\sqrt{\pi}\sqrt{Fo}} \tag{9}$$

For long time, when $\sqrt{Fo} >> (\beta - 1)/2\sqrt{\pi}$, the limiting case of $\beta \to \infty$ approaches the steady state conduction shape factor for the sphere in a full space:

$$S^\star_\infty = 2\sqrt{\pi} \tag{10}$$

For intermediate values of Fo and β, the exact solution follows the transient, full space sphere solution for $\sqrt{Fo} < (\beta - 1)/2\sqrt{\pi}$ before undergoing a rapid transition to a asymptote corresponding to steady state conduction in the spherical annulus:

$$S^\star_0 = \frac{2\sqrt{\pi}}{\beta - 1} \tag{11}$$

Model Development

Given the solution for the concentric spherical enclosure, a modeling procedure is proposed for double connected regions of arbitrary shape based on the assumption that similar solution trends and limits exist for enclosures with non-spherical boundaries. The model is based on a combination of asymptotic solutions for small and large relative values of Fo and β, as presented graphically in Fig. 3. Each of the four "corners" of the schematic in Fig. 3 represent a different limit for the solution, as identified for the concentric spheres, and each of these limits will be incorporated into the proposed model.

In order to combine the first pair of asymptotic solutions, the short and long time limits, the composite solution technique of Churchill and Usagi[5] is used:

$$Q^\star = \left[(Q^\star_t)^n + (S^\star)^n\right]^{1/n}, \quad t > 0 \tag{12}$$

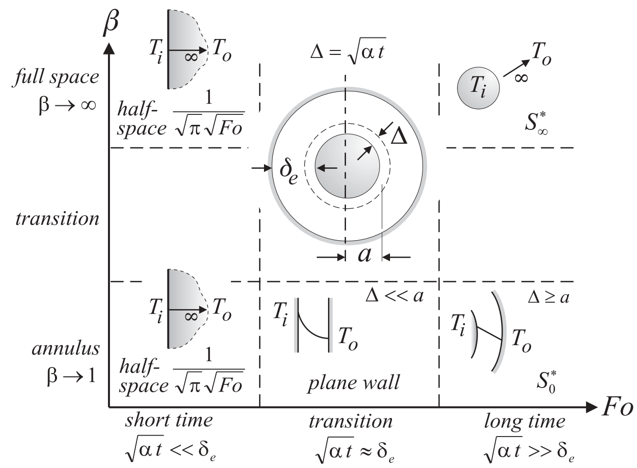

Figure 3: Schematic of Solution Limits

where $Q^\star_t$ and $S^\star$ are the short time transient solution and the steady state conduction shape factor, respectively. This combination of the transient half-space solution with the steady state conduction shape factor formulation is similar to that presented by Yovanovich et al.[4] for the full space problem, $\beta \to \infty$. However, in this case the asymptotic solutions in Eq. (12) are functions of the geometric parameter β, as detailed in the following sections.

Short Time Asymptote

For short time, when the temperature field penetration depth Δ is much smaller than the dimensions of the inner body, the problem is equivalent to transient conduction into a semi-infinite solid, also called the half-space solution. From the conduction texts[1,6] the transient temperature distribution in a half space with an isothermal boundary condition is:

$$\phi = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{13}$$

where x represents distance in the direction normal to the surface. Solving for the dimensionless heat flow rate gives:

$$Q^\star = \frac{1}{\sqrt{\pi}\sqrt{Fo}} \tag{14}$$

From the concentric sphere solution in Fig. 2 it can be shown that Eq. (14) is the limit for *very* short time for all β. However, as β increases the solution tends to follow the full space transient solution. From Yovanovich et al.[4] this limit is shown to be:

$$Q^\star_t = \left[\left(\frac{1}{\sqrt{\pi}\sqrt{Fo}}\right)^m + (S^\star_\infty)^m\right]^{1/m} \tag{15}$$

where $S^\star_\infty$ is the steady state conduction shape factor for the inner body. For the limit of the spherical enclosure with $\beta \to \infty$, Eq. (15) is identical to the available exact solution (Yovanovich et al.[4]).

Steady State Asymptote

Exact solutions for the dimensionless conduction shape factor $S^\star$ are available for a limited set of problems, often involving geometrically similar bodies with boundaries conforming to the given coordinate system. Hassani[2] and Hassani and Hollands[3] present an analytical model for the dimensional shape factor for enclosures with uniform gap spacing:

$$S = ({S_0}^p + {S_\infty}^p)^{1/p} \tag{16}$$

where the small and large gap asymptotes are given by:

$$S_0 = \frac{\sqrt{A_i}}{\delta}\ , \qquad S_\infty = 3.51\sqrt{A_i}$$

The fitting parameter p is determined from a correlation of numerical results as a function of the inner body surface area, volume and aspect ratio. Although applicable for problems with uniform gap spacing, it is unclear how to define the gap spacing δ when the boundaries are non-conforming.

The proposed model for the conduction shape factor is based on the available solution for the concentric spheres:

$$S = \frac{4\pi}{\dfrac{1}{a} - \dfrac{1}{b}} \tag{17}$$

Substituting the gap spacing parameter $\delta = b - a$ recasts the expression into a form that illustrates the two limiting cases:

$$S = \frac{4\pi a^2}{\delta} + 4\pi a$$

When the shape factor is non-dimensionalized using $\sqrt{A_i}$, the resulting expression is:

$$S^\star = \frac{2\sqrt{\pi}a}{\delta} + 2\sqrt{\pi} \tag{18}$$

In order to apply this solution as a general model for a wide range of geometries, it is necessary to introduce the effective gap spacing δ_e, a single parameter that quantifies the average gap spacing over the entire enclosed region. For the enclosure with arbitrarily shaped boundaries shown in Fig. 2 a), the relative domain size δ_e is determined from the equivalent spherical enclosure by the following procedure. First, the surface area of the inner body is maintained to determine the equivalent inner radius:

$$a = \frac{\sqrt{A_i}}{2\sqrt{\pi}}$$

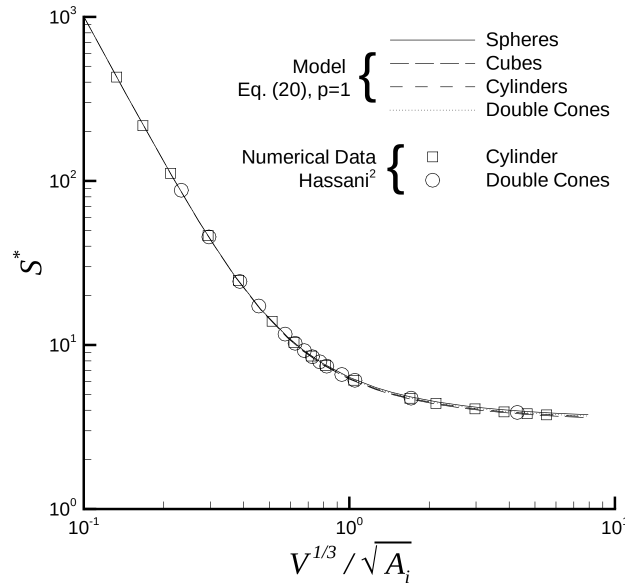

Figure 4: Shape Factor Model Validation

Then δ_e is determined as a function of the equivalent outer radius when the total enclosed volume V is maintained:

$$\begin{aligned} \delta_e &= b - a \\ &= \left(\frac{3}{4\pi}\right)^{1/3}\left(V + \frac{4\pi a^3}{3}\right)^{1/3} - a \\ &= \frac{1}{2}\left\{\left[\frac{6V}{\pi} + \left(\frac{A_i}{\pi}\right)^{3/2}\right]^{1/3} - \left(\frac{A_i}{\pi}\right)^{1/2}\right\} \end{aligned} \tag{19}$$

In addition to substituting δ_e and the inner body surface area into the first term in Eq. (18), the second term is also modified. Recognizing that the second term is equivalent to the conduction shape factor, the shape factor for the inner body in a full space, $S^\star_\infty$, is substituted for this limit. The resulting expression is:

$$S^\star = \left\{\left[\frac{2\sqrt{\pi}}{\left(1 + 6\sqrt{\pi}\,V/{A_i}^{3/2}\right)^{1/3} - 1}\right]^p + (S^\star_\infty)^p\right\}^{1/p} \tag{20}$$

The fitting parameter p is included to improve the accuracy of model for enclosures with large variations in the local gap spacing. However, for the two cases examined in this work a superposition solution using $p = 1$ provided acceptable accuracy.

Figure 4 compares the predictions of the model for a wide range of conforming boundaries with the numerical data of Hassani[2] for concentric cylinders and base-attached double cones of aspect ratio 1.

This plot clearly demonstrates the excellent agreement between the model and the data, with a maximum difference of less than 2% occurring at the limit of large $V^{1/3}/\sqrt{A_i}$.

It is convenient at this time to derive the effective relative domain size β_e for enclosures with arbitrarily shaped boundaries. From the definition of the effective gap spacing, β_e is determined:

$$\begin{aligned}\beta_e &= \frac{b}{a} = \frac{\delta_e + a}{a} \\ &= \left(6\sqrt{\pi}\frac{V}{{A_i}^{3/2}} + 1\right)^{1/3}\end{aligned} \quad (21)$$

Based on this definition for the effective relative domain size, the conduction shape factor expression can be rewritten as:

$$S^\star = \left[\left(\frac{2\sqrt{\pi}}{\beta_e - 1}\right)^p + (S_\infty^\star)^p\right]^{1/p} \quad (22)$$

Values for Fitting Parameters

Combining the expressions for the short and long time asymptotes developed in the previous sections, Eqs. (15) and (22), with the general form of the composite model, Eq. (12), gives the following:

$$\begin{aligned}Q^\star &= \left\{\left[\left(\frac{1}{\sqrt{\pi}\sqrt{Fo}}\right)^m + (S_\infty^\star)^m\right]^{n/m}\right. \\ &\quad \left. + \left[\left(\frac{2\sqrt{\pi}}{\beta_e - 1}\right)^p + (S_\infty^\star)^p\right]^{n/p}\right\}^{1/n}\end{aligned} \quad (23)$$

For the purposes of the current study, the fitting parameters on the short and long time asymptotes, m and p, will be set to 1. In the case of the short time limit, Yovanovich et al.[4] demonstrated that the maximum range of the fitting parameter was $0.9 < m < 1.1$, and that the use of a superposition solution, $m = 1$, for bodies with aspect ratio of approximately unity results in a maximum error of less than 4%. For the long time limit, both Hassani[2] and Hassani and Hollands[3] fitting parameters in the range $1 \leq p \leq 1.12$, and Teertstra[7] confirmed that a maximum error of less than 5% results from the use of superposition solution, $p = 1$, for conforming geometries. It should be noted that changes may be required to these fitting parameter values in the case of enclosures with large variations in the local gap thickness or bodies with small or large aspect ratio.

Using superposition for the short and long time asymptotes yields:

$$Q^\star = \left[\left(\frac{1}{\sqrt{\pi}\sqrt{Fo}} + S_\infty^\star\right)^n + \left(\frac{2\sqrt{\pi}}{\beta_e - 1} + S_\infty^\star\right)^n\right]^{1/n} \quad (24)$$

In order to determine values of the fitting parameter n, the two limiting cases of small and large enclosures, $\beta_e \to 1$ and $\beta_e \to \infty$, are examined separately. For small relative domain size, $\beta \to 1$, the gap thickness is assumed to be small compared to the inner body dimensions and the problem approaches that of transient conduction in a plane wall. The diffusion equation for the plane wall in Cartesian coordinates is:

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \quad (25)$$

with initial and boundary conditions:

$$\begin{aligned} 0 \leq x \leq L, &\quad t = 0, \quad \theta = 0 \\ x = 0, &\quad t > 0, \quad \theta = \theta_i \\ x = L, &\quad t > 0, \quad \theta = 0 \end{aligned}$$

Based on the solution for the transient temperature distribution the dimensionless heat flow rate at the heated wall, $x = 0$, can be determined:

$$Q_L^\star = 1 + \sum_{n=1}^{\infty} 2\exp\left(-n^2\pi^2 Fo_L\right) \quad (26)$$

where the wall thickness L is the scale length for the dimensionless parameters.

The best value of the fitting parameter n for this limiting case of $\beta \to 1$ is determined by comparing the plane wall solution with a two-term model composed of half-space and steady state asymptotes:

$$Q_L^\star = \left[\left(\frac{1}{\sqrt{\pi}\sqrt{Fo_L}}\right)^n + 1\right]^{1/n} \quad (27)$$

A fitting parameter of $n = 8.5$ was found to minimize the maximum percent difference between the model and the exact solution in the comparison shown in Fig. 5.

For the limiting case of large relative domain size, $\beta_e \to \infty$, Yovanovich et al.[4] showed that a superposition solution of the half space and steady state asymptotes provides excellent agreement with the numerical data for the full space. However, if $n = 1$ is used in the formulation of the model shown in Eq. (24), the long time results will be error at the limit of large β_e. To correct this, an alternate model formulation is recommended for large values of the relative domain size, $\beta_e > 10$:

$$Q^\star = \left[\left(\frac{1}{\sqrt{\pi}\sqrt{Fo}}\right)^n + \left(\frac{2\sqrt{\pi}}{\beta_e - 1} + S_\infty^\star\right)^n\right]^{1/n} \quad (28)$$

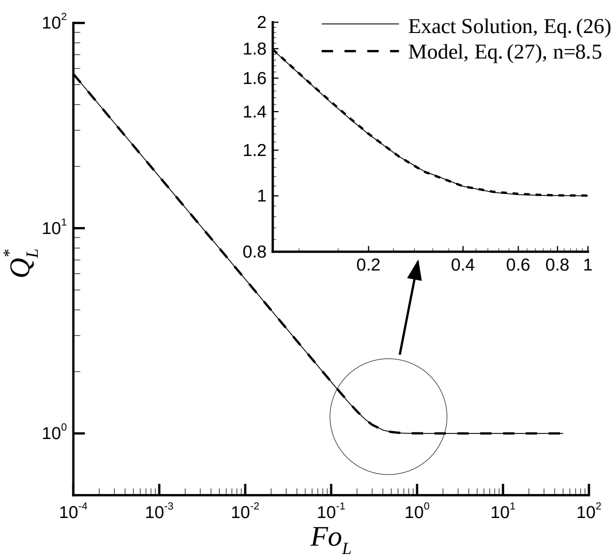

Figure 5: Plane Wall Model Validation

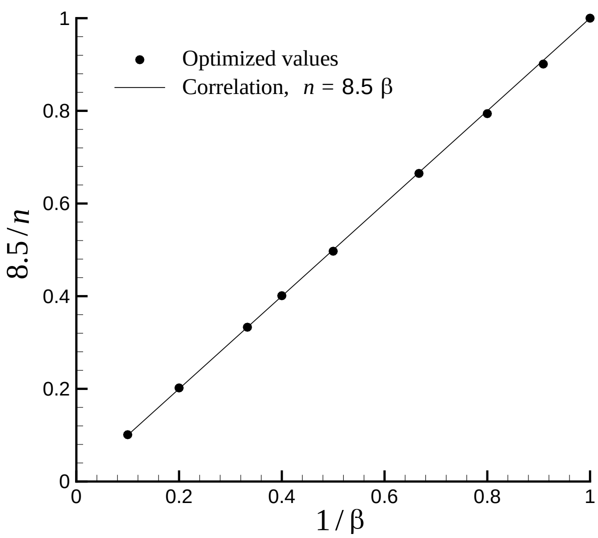

Figure 6: Fitting Parameter Correlation for $1 < \beta \leq 10$

For the remaining values of the relative domain size, $1 < \beta_e \leq 10$, optimized values of the fitting parameter n are determined in the following manner. The model is applied to the concentric spherical enclosure for each of the values of β shown in Fig. 2, and the results are compared with the exact solution. Values of the fitting parameter are then determined that minimize the maximum difference between the model predictions and the exact solution for each case. The resulting values, including the limit for $\beta_e \to 1$, are normalized using the plane wall fitting parameter $n = 8.5$. The inverse of the normalized fitting parameter, $8.5/n$, is correlated as a function of $1/\beta_e$, as shown in Fig. 6, and the resulting linear relationship can be used to predict n for the range of relative domain size $1 < \beta_e \leq 10$:

$$n = 8.5\beta_e \tag{29}$$

Model Summary

The models proposed in this work for transient dimensionless heat flow rate within doubly connected regions of arbitrary shape for the full range of time are summarized as follows:

$$Q^\star = \left[\left(\frac{1}{\sqrt{\pi}\sqrt{Fo}} + S^\star_\infty\right)^{8.5\beta_e} + \left(\frac{2\sqrt{\pi}}{\beta_e - 1} + S^\star_\infty\right)^{8.5\beta_e}\right]^{1/(8.5\beta_e)} \tag{30}$$

valid for $1 < \beta_e \leq 10$

$$Q^\star = \frac{1}{\sqrt{\pi}\sqrt{Fo}} + \frac{2\sqrt{\pi}}{\beta_e - 1} + S^\star_\infty \tag{31}$$

valid for $\beta_e > 10$

where:

$$\beta_e = \left(6\sqrt{\pi}\frac{V}{A_i^{3/2}} + 1\right)^{1/3} \tag{32}$$

Model Validation

The proposed models for dimensionless transient heat flow rate from Eqs. (30) and (31) are validated using the exact solution for the concentric spheres and numerical data from finite element CFD simulations of a concentric, cubical enclosure.

Figure 7 a) presents a comparison of the model and the exact solution for the spherical enclosure for relative domain size in the range $1.1 \leq \beta \leq 10$. This plot shows the excellent agreement between the model and the exact solution for the full range of Fo and $\beta < 10$, with maximum percent differences and RMS differences of less than 0.3%.

The superposition solution recommended for $\beta > 10$, Eq. (31), is compared with the exact solution for the concentric spheres in Fig. 7 b) for the full range of Fo for $\beta = 10$ and 50. From this plot it can be shown that the trends of the superposition model for $\beta = 10$ differ from the exact solution in the transition region, yielding a maximum percent difference of 9%. However, for the relative domain size $\beta = 50$, the maximum percent difference is reduced to less than 2%, with an RMS difference of approximately 1%.

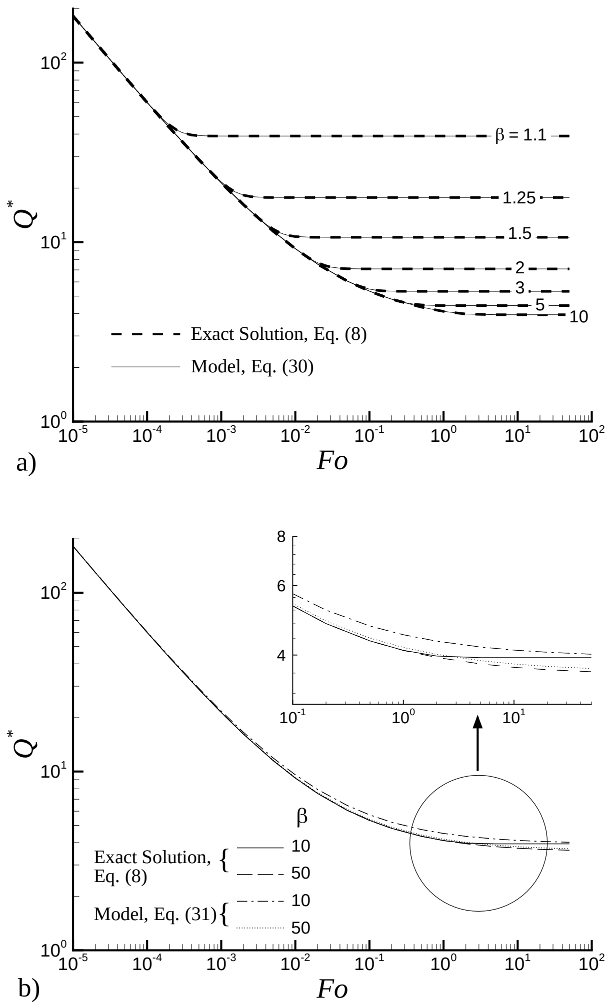

Figure 7: Model Validation: Concentric Spheres a) $1.1 < \beta \leq 10$; b) $\beta > 10$

In the second phase of the validation, the model is compared with numerical simulation results for the concentric cubical enclosure shown in Fig. 1 c). These numerical simulations were performed using the commercial, finite element based CFD package Icepak[8] for the range of relative domain size $1.15 \leq \beta_e \leq 44.9$. The solution domain for the numerical simulations utilizes 1/8 symmetry to reduce the number of elements required and improve solution time. Adiabatic boundary conditions are imposed at the 3 symmetry planes, while the remaining domain boundaries are isothermal at the sink temperature, $T_o = 20^oC$. An isothermal boundary condition of $T_i = 40^oC$ is applied on the inner cube. The size of the outer boundary was fixed, $s_o = 0.1\,m$, while s_i was varied in six steps to achieve the size length ratio values:

$$s_o/s_i = 1.2,\ 1.5,\ 2,\ 5,\ 10 \text{ and } 50$$

Constant property values for air at standard temperature and pressure were specified throughout the solution domain. Transient solutions were performed for 1000 iterations for 3 different time intervals, $\Delta t = 1, 10$ and $100\,s$, to achieve a range of Fo spanning three decades. A grid convergence study was performed for the limiting case of steady-state conduction, and the resulting discretization levels were maintained throughout all subsequent calculations. The results of the numerical simulations were reported by the software in terms of total heat flow rate, which was non-dimensionalized by:

$$Q^\star = \frac{Q}{k\sqrt{6}s_i\,(T_i - T_o)} \qquad (33)$$

Figures 8 a), b) and c) compare the predictions of the models for the concentric cubes with the numerical data for six β_e values. The proposed model for the range $1 < \beta_e \leq 10$ from Eq. (30) is used in all cases with the exception of $\beta_e = 44.9$ in Fig. 8 c), where the superposition solution, Eq. (31), is used instead. These plots demonstrate the excellent agreement between the proposed models and the numerical data for all values of Fo and β_e examined in this work, with maximum percent differences of less than 5%, and RMS differences of less than 3%.

Summary and Conclusions

Models have been developed to predict dimensionless transient heat flow rate within doubly connected regions of arbitrary shape for the full range of time. These models are based on trends and limits noted in the exact solution of the concentric spherical enclosure, and combine asymptotes for limiting cases of geometry and time, non-dimensionalized by β_e and Fo, respectively. Validation of the models with both the exact solution for the spherical enclosure and numerical data for the concentric cubes shows excellent agreement of less than 3% RMS for most cases.

A number of simplifications have been made in the assumption of fitting parameter values $m = p = 1$ for the transient and steady state asymptotes in the model. Additional numerical studies are clearly warranted to validate the effect of the model for non-conforming boundary shapes, inner bodies with aspect ratios substantially different from unity, and enclosures with a large variation in the local gap spacing.

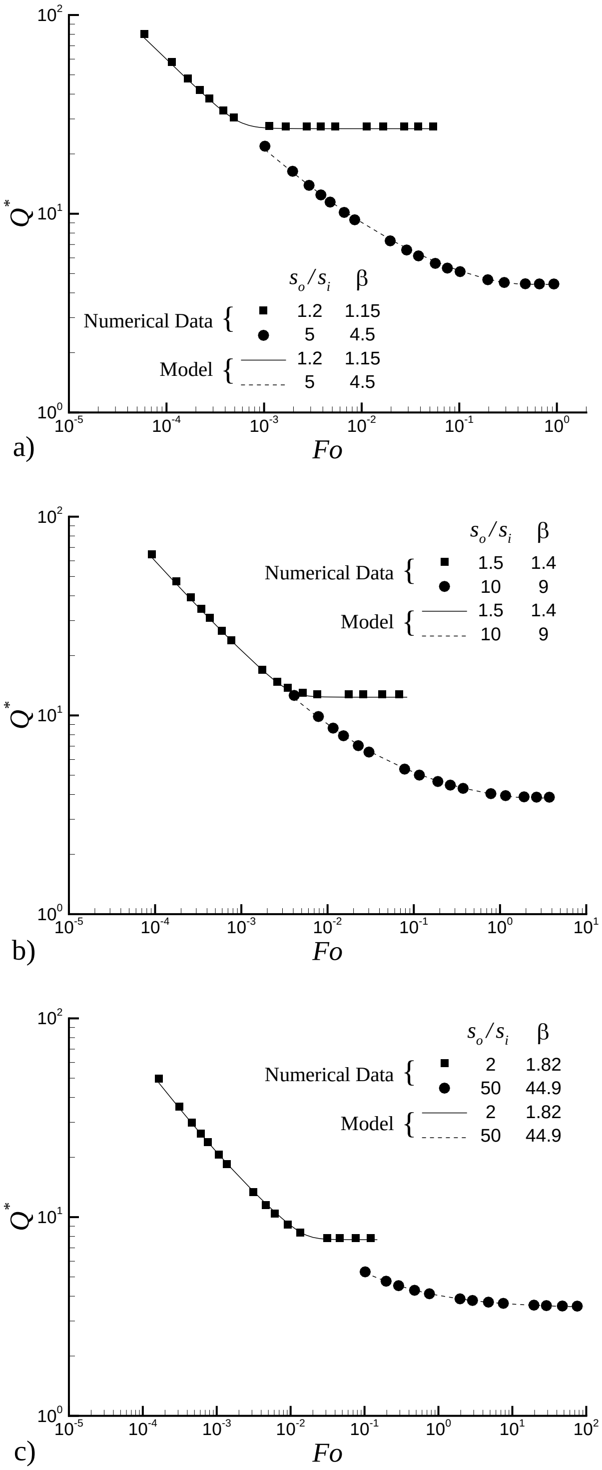

Figure 8: Model Validation: Concentric Cubes
a) $\beta_e = 1.15, 4.5$
b) $\beta_e = 1.4, 9$
c) $\beta_e = 1.82, 44.9$

Acknowledgments

The authors gratefully acknowledge the continued financial support of the National Sciences and Engineering Research Council of Canada and Materials and Manufacturing Ontario. The assistance of Prof. J. R. Culham in the model development and the technical support engineers at Fluent during the numerical simulations is also greatly appreciated.

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