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Thermal Spreading Resistances in Compound Annular Sectors

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A general solution, based on the separation of variables method, for the thermal spreading resistance in compound annular sectors, is presented. Results are given for three heat flux distributions: inverse parabolic, uniform, and parabolic profiles. The general solution can be used to model any number of equally spaced heat sources on a compound or isotropic annulus. Graphical results are presented for a variety of parameter combinations. Finally, it is shown that the general solution for a thin annulus reduces to that of a flux channel when curvature effects are negligible.

Nomenclature

A_s	=	heat source area, m^2
a, b, c	=	radial dimensions, m
a, c	=	half-lengths of flux channel, m
Bi	=	Biot number, $h\mathcal{L}/k$
h	=	contact conductance or film coefficient, $W/m^2 \cdot K$
i	=	index denoting layers 1 and 2
$J_\nu(x)$	=	Bessel function of first kind, order ν
K	=	heat flux coefficient, Eq. (14)
k, k_1, k_2	=	thermal conductivities, $W/m \cdot K$
L	=	length of flux channel, m
$\mathcal{L}$	=	arbitrary length scale, m
m, n	=	indices for summations
N	=	number of heat sources
Q	=	heat flow rate, W
q	=	heat flux, W/m^2
R	=	thermal resistance, K/W
R_s	=	spreading resistance, K/W
R_T	=	total resistance, K/W
R_{1D}	=	one-dimensional resistance, K/W
R^*	=	dimensionless thermal resistance, $k_2 R \mathcal{L}$
r	=	radial coordinate, m
$\bar{T}_s$	=	mean source temperature, K
T_1, T_2	=	layer temperatures, K
T_∞	=	sink temperature, K
t, t_1, t_2	=	total and layer thicknesses, m
α, β	=	angular measure, rad
$\Gamma(x)$	=	gamma function
δ_m	=	eigenvalues, $m\pi/c$
ϵ	=	relative contact size, a/c, β/α
θ	=	temperature excess, $T - T_\infty$
κ	=	relative conductivity, k_2/k_1
λ_n	=	eigenvalues, $n\pi/\alpha$
μ	=	heat flux shape parameter, $-\frac{1}{2}$, 0, $\frac{1}{2}$
ρ	=	radii ratio
τ	=	relative thickness, t/c
ϕ_m	=	two-dimensional spreading function
φ_n, ϕ_n	=	two-dimensional spreading function
ψ	=	angular coordinate, rad
ψ_s	=	dimensionless spreading resistance, $R_s k_2 \mathcal{L}$

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Introduction

THE general solution for the spreading resistance of a flux specified heat source on a compound annular sector with convective or conductive cooling at one boundary will be presented (Fig. 1). In this system, heat flows through a portion of the outer surface through two layers, having different thermal conductivities, to the interior surface, which is convectively cooled by a uniform film conductance. It is assumed that perfect contact is achieved at the interface of the two materials. A review of the literature[1] reveals that this particular configuration has not been analyzed. Solutions to this problem for both the isotropic and compound configurations will be presented. A particular application of this solution is in tube heat exchangers where several cylindrical tubes or longitudinal fins are attached at equal spacing to a larger cylindrical tube (Fig. 2). The solution for the overall thermal resistance in an isotropic or compound annulus would allow the effects of protective coatings or fouling deposits on the larger tube to be analyzed. The solution also considers curvature effects that are not present in other solutions available in the literature.[2]

The general solution will depend on several dimensionless geometric and thermal parameters. In general, the total resistance is given by

$$R_T = R_{1D} + R_s \tag{1}$$

where R_{1D} is the one-dimensional composite resistance of the system and R_s is the spreading resistance component. The total thermal resistance is defined as

$$R_T = (\bar{T}_s - T_\infty)/Q \tag{2}$$

where $\bar{T}_s$ is average source temperature.

The resistance for the configuration shown in Fig. 1 is a function of

$$R = f(a, b, c, h, k_1, k_2, \beta, \alpha, \mu) \tag{3}$$

or, in dimensionless form,

$$R^* = f(a/b, b/c, ha/k_1, k_2/k_1, \beta/\alpha, \mu) \tag{4}$$

Expressions will be presented for three flux distributions for the composite and isotropic systems. It will also be shown that the general solution for the compound annular sector simplifies to the equivalent solution for a two-dimensional compound strip[2] when the radius of curvature is large relative to the total thickness of the system.

Problem Statement

The governing equation for each layer in the system shown in Fig. 1 is Laplace's equation

$$\nabla^2 T_i(r,\psi) = \frac{\partial^2 T_i}{\partial r^2} + \frac{1}{r}\frac{\partial T_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T_i}{\partial \psi^2} = 0 \tag{5}$$

which is subject to a convective or mixed boundary condition on the interior surface

$$\left.\frac{\partial T_1}{\partial r}\right|_{r=a} = \frac{h}{k_1}[T_1(a,\psi) - T_\infty] \qquad 0 \le \psi \le \alpha \tag{6}$$

equality of the heat flux and temperature at the interface

$$k_1 \left.\frac{\partial T_1}{\partial r}\right|_{r=b} = k_2 \left.\frac{\partial T_2}{\partial r}\right|_{r=b} \qquad 0 \le \psi \le \alpha \tag{7}$$

$$T_1(b,\psi) = T_2(b,\psi) \qquad 0 \le \psi \le \alpha \tag{8}$$

and a specified flux distribution on the outer surface

$$\left.\frac{\partial T_2}{\partial r}\right|_{r=c} = \frac{q(\psi)}{k_2} \qquad \psi < \beta \tag{9}$$

$$\left.\frac{\partial T_2}{\partial r}\right|_{r=c} = 0 \qquad \beta < \psi \le \alpha \tag{10}$$

The following symmetry conditions are also required:

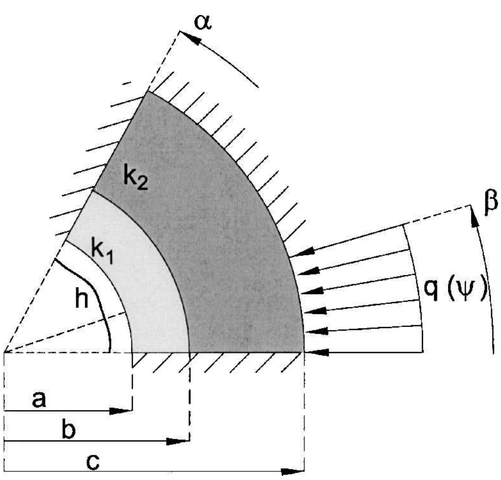

Fig. 1 Compound annular sector.

$$\left.\frac{\partial T_i}{\partial \psi}\right|_{\psi=0} = 0 \qquad a \le r \le c \tag{11}$$

$$\left.\frac{\partial T_i}{\partial \psi}\right|_{\psi=\alpha} = 0 \qquad a \le r \le c \tag{12}$$

where α varies, depending on the symmetry of the problem, that is, $\alpha = \pi/2$, for the case shown in Fig. 3. This allows for any number, $N \ge 1$, of equally spaced heat sources to be considered.

Finally, the heat flux $q(\psi)$ will take the following form:

$$q(\psi) = K[1 - (\psi/\beta)^2]^\mu \qquad 0 \le \psi < \beta \tag{13}$$

where $\mu > -1$ and

$$K = \frac{Q}{\beta c}\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\mu + \frac{3}{2}\right)}{\Gamma(\mu + 1)} \tag{14}$$

and $\Gamma(x)$ is the gamma function. The general solution will be obtained for three heat flux distribution parameter values, $\mu = -\frac{1}{2}$, 0, and $\frac{1}{2}$.

Solution

The solution may be obtained by means of separation of variables.[3,4] The solution is assumed to have the form $\theta(r,\psi) = R(r) \times \Psi(\psi)$, where $\theta(r,\psi) = T(r,\psi) - T_\infty$ is the temperature excess. Applying the method of separation of variables yields the following solution, which satisfies the thermal boundary conditions in the circumferential direction:

$$\theta_i(r,\psi) = A_i + B_i \ln(r) + \sum_{n=1}^{\infty}\left[C_i r^{\lambda_n} + D_i r^{-\lambda_n}\right]\cos(\lambda_n \psi) \tag{15}$$

where $\lambda_n = n\pi/\alpha$.

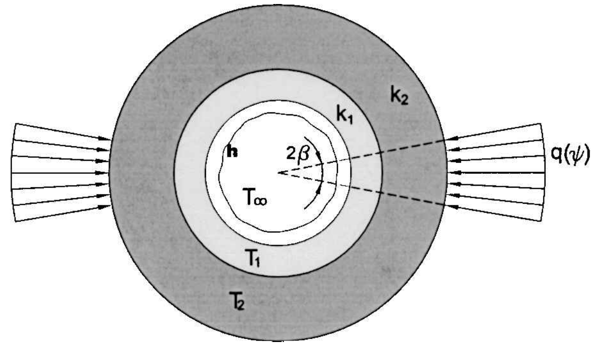

Fig. 3 Annulus with two heat sources.

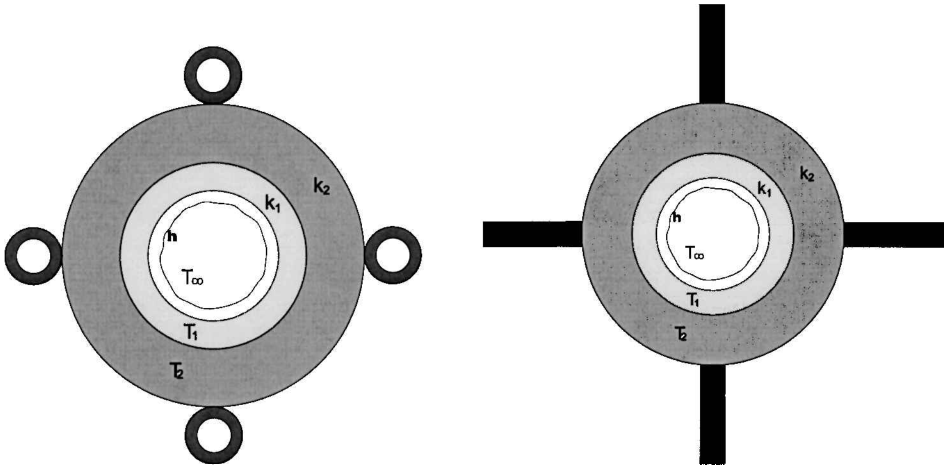

Fig. 2 Compound annulus with heat sources.

The solution contains two parts, a uniform flow solution and a spreading (or constriction) solution that vanishes when the heat flux is distributed over the entire element. Because the solution is a linear superposition of the two parts, they will be dealt with separately.

Uniform Flow Resistance

The Fourier solution method indicates that a uniform flow solution also satisfies the prescribed thermal boundary conditions. This part of the temperature field is always present and leads to a one-dimensional radial thermal resistance.

Application of the boundary conditions yields the following result for the thermal resistance of the compound annulus on a per unit length basis:

$$R_{1D} = \frac{\ln(b/a)}{2\pi k_1} + \frac{\ln(c/b)}{2\pi k_2} + \frac{1}{2\pi h a} \tag{16}$$

or, in dimensionless form,

$$R_{1D}^* = \kappa \frac{\ln(1/\rho_1)}{2\pi} + \frac{\ln(1/\rho_2)}{2\pi} + \frac{\kappa}{2\pi Bi} \tag{17}$$

where $R^* = k_2 R L$, $0 < \rho_1 = a/b < 1$, $0 < \rho_2 = b/c < 1$, $\kappa = k_2/k_1$, and $Bi = ha/k_1$.

Spreading Resistance

The spreading resistance part requires a solution to the two-dimensional eigenvalue problem for each layer. Application of the thermal boundary conditions at the interior surface, Eq. (6), and, at the interface, Eqs. (7) and (8), gives the solution for C_1, C_2, and D_1 in terms of the unknown coefficient D_2. The solution to this complex set of equations was obtained with the computer algebra system Maple V (Ref. 5).

The final coefficient D_2 is obtained by taking a Fourier expansion of the exterior surface boundary condition Eqs. (9) and (10). This results in

$$D_2 = \frac{2}{\alpha G_n} \int_0^\beta \left(\frac{K}{k_2}\right) \left[1 - \left(\frac{\psi}{\beta}\right)^2\right]^\mu \cos(\lambda_n \psi)\, d\psi \tag{18}$$

where G_n is a constant that depends on the shell thicknesses, conductivities, and heat transfer coefficient. However, in practical applications the thermal resistance is of greater importance than the temperature field in each layer.

The thermal spreading resistance is defined as

$$R_s = \bar{\theta}_s / Q \tag{19}$$

where

$$\bar{\theta}_s = \frac{1}{\beta} \int_0^\beta \theta_2(c, \psi)\, d\psi \tag{20}$$

It is often convenient to define a dimensionless spreading resistance parameter

$$\psi_s = k_2 R_s L \tag{21}$$

where $L = 1$ will be assumed.

Combining the results for the final Fourier coefficient D_2 and the definition of the spreading resistance parameter results in the following expression for the dimensionless spreading resistance:

$$\psi_s = k_2 R_s = \frac{4}{\sqrt{\pi}} \frac{\Gamma\left(\mu + \frac{3}{2}\right)}{\Gamma(\mu + 1)} \cdot \sum_{n=1}^\infty \varphi_n \frac{\sin(n\pi\beta/\alpha)\alpha}{\beta^2 n^2 \pi^2} \times \int_0^\beta \left[1 - \left(\frac{\psi}{\beta}\right)^2\right]^\mu \cos(n\pi\psi/\alpha)\, d\psi \tag{22}$$

where the parameter φ_n determines the effect of shell thicknesses, layer conductivities, and heat transfer coefficient. It is defined as

$$\varphi_n = \left[\frac{(F_1 Bi + F_2 \lambda_n)\kappa + (F_3 Bi + F_4 \lambda_n)}{(F_4 Bi + F_3 \lambda_n)\kappa + (F_2 Bi + F_1 \lambda_n)}\right] \tag{23}$$

where

$$F_1 = 1 - (\rho_1)^{2\lambda_n} + (\rho_2)^{2\lambda_n} - (\rho_1\rho_2)^{2\lambda_n}$$

$$F_2 = 1 + (\rho_1)^{2\lambda_n} + (\rho_2)^{2\lambda_n} + (\rho_1\rho_2)^{2\lambda_n}$$

$$F_3 = 1 + (\rho_1)^{2\lambda_n} - (\rho_2)^{2\lambda_n} - (\rho_1\rho_2)^{2\lambda_n}$$

$$F_4 = 1 - (\rho_1)^{2\lambda_n} - (\rho_2)^{2\lambda_n} + (\rho_1\rho_2)^{2\lambda_n}$$

For an isotropic annulus, $\kappa = 1$, φ_n may be simplified to give

$$\varphi_n = \left[\frac{G_1 Bi + G_2 \lambda_n}{G_2 Bi + G_1 \lambda_n}\right] \tag{24}$$

where

$$G_1 = 1 - \rho^{2\lambda_n} \qquad G_2 = 1 + \rho^{2\lambda_n}$$

where $\rho = a/c$ and $k_2 = k_1 = k$.

Evaluation of the the integral[6] in Eq. (22) gives the final general relation for the spreading resistance in the following form:

$$\psi_s = \frac{2}{\pi^2 \epsilon} \Gamma\left(\mu + \frac{3}{2}\right) \sum_{n=1}^\infty \left(\frac{2}{n\pi\epsilon}\right)^{\mu + \frac{1}{2}} \frac{\sin(n\pi\epsilon)}{n^2} J_{\mu + \frac{1}{2}}(n\pi\epsilon)\, \varphi_n \tag{25}$$

where $\epsilon = \beta/\alpha$.

The general solution is valid for any heat flux distribution defined by Eqs. (13) and (14) with $\mu > -1$. However, only three cases of practical interest will be presented. These are the uniform flux ($\mu = 0$), parabolic flux ($\mu = \frac{1}{2}$), and inverted parabolic flux ($\mu = -\frac{1}{2}$). The inverted parabolic flux distribution is representative of the isothermal boundary condition for values of $\epsilon = \beta/\alpha < 0.5^2$.

The general solution for the spreading resistance in an annular sector has the same form as that for a finite compound flux channel,[2] with the exception of the parameter φ_n. The parameter φ_n is a function of the radii ratio, conductivity ratio, and Biot number, whereas for the finite compound flux channel it is a function of the layer thicknesses, conductivity ratio, and Biot number. Further discussion on the similarities of the two solutions will be presented later as part of a parametric study.

Total Resistance

The total dimensionless resistance of the compound annulus may now be obtained by combining the uniform flow resistance and spreading resistance. For a compound annulus containing N equally spaced heat sources, the total thermal resistance is obtained from

$$R_T^* = \psi_s/(2N) + R_{1D}^* \tag{26}$$

or, for the basic element shown in Fig. 1, the total thermal resistance is

$$R_{T,e}^* = \psi_s + (2\pi/\alpha) R_{1D}^* \tag{27}$$

Results

Several special cases may be obtained from the general solution, Eq. (25). Each of these cases represents a particular heat flux distribution. Three cases of particular interest are inverted parabolic flux distribution $\mu = -\frac{1}{2}$, the uniform flux distribution $\mu = 0$, and the parabolic flux distribution $\mu = \frac{1}{2}$. In general, the exact flux distribution is not always known. These special cases provide a means to bound the results for the thermal spreading resistance.

Special Cases

The general solution for the compound annulus simplifies for three special cases of the the heat flux distribution. The results are given hereafter for $\mu = -\frac{1}{2}, 0, \frac{1}{2}$.

For $\mu = -\frac{1}{2}$,

$$\psi_s = \frac{2}{\pi^2 \epsilon} \sum_{n=1}^{\infty} \frac{J_0(n\pi\epsilon)\sin(n\pi\epsilon)}{n^2} \varphi_n \tag{28}$$

For $\mu = 0$,

$$\psi_s = \frac{2}{\pi^3 \epsilon^2} \sum_{n=1}^{\infty} \frac{\sin^2(n\pi\epsilon)}{n^3} \varphi_n \tag{29}$$

For $\mu = \frac{1}{2}$,

$$\psi_s = \frac{4}{\pi^3 \epsilon^2} \sum_{n=1}^{\infty} \frac{J_1(n\pi\epsilon)\sin(n\pi\epsilon)}{n^3} \varphi_n \tag{30}$$

The preceding cases have the following relationship:

$$\psi_s\left(\mu = -\tfrac{1}{2}\right) < \psi_s(\mu = 0) < \psi_s\left(\mu = \tfrac{1}{2}\right) \tag{31}$$

This allows the variation in the thermal resistance to be estimated when the precise heat flux distribution is not known.

Thin Shell Limit

When the radius of curvature is large relative to the thickness of the shell, the results for the compound annulus may be modeled using the results for a rectangular flux channel (Fig. 4).

The solution for a compound flux channel was obtained by Yovanovich et al.[2] for the uniform flux distribution and for an isotropic flux channel for the heat flux distribution defined by Eqs. (13) and (14). The two solutions may be used to obtain an expression for the spreading resistance in a compound flux channel for any flux distribution parameter μ. This result for the configuration shown in Fig. 5 is

$$\psi_s = k_1 R_s = \frac{1}{\pi^2 \epsilon} \Gamma\left(\mu + \frac{3}{2}\right)$$

$$\times \sum_{m=1}^{\infty} \left(\frac{2}{m\pi\epsilon}\right)^{\mu + \frac{1}{2}} \cdot \frac{\sin(m\pi\epsilon)}{m^2} J_{\mu+\frac{1}{2}}(m\pi\epsilon)\, \phi_m \tag{32}$$

where the contributions of the layer thicknesses t_1 and t_2, the layer conductivities k_1 and k_2, and the uniform conductance h to the spreading resistance are determined by means of the parameter ϕ_m given by

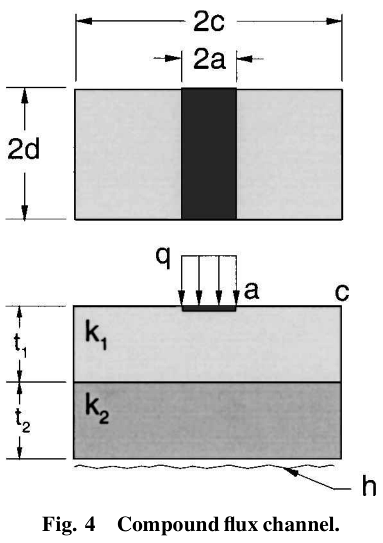

Fig. 4 Compound flux channel.

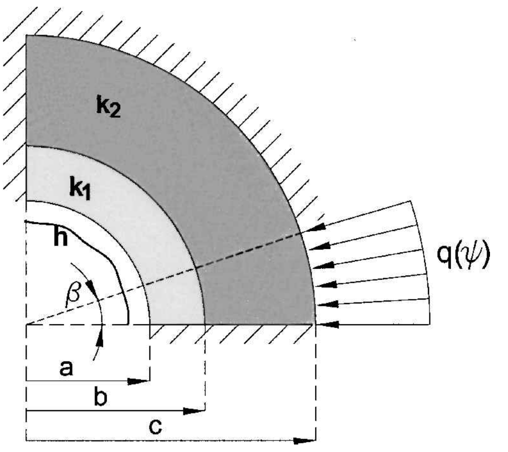

Fig. 5 Basic element for two heat sources.

$$\phi_m = \frac{(\alpha e^{4\delta t_1} + e^{2\delta t_1}) + \varphi\left(e^{2\delta(2t_1 + t_2)} + \alpha e^{2\delta(t_1 + t_2)}\right)}{(\alpha e^{4\delta t_1} - e^{2\delta t_1}) + \varphi\left(e^{2\delta(2t_1 + t_2)} - \alpha e^{2\delta(t_1 + t_2)}\right)} \tag{33}$$

where

$$\varphi = \frac{m\pi + Bi/\kappa}{m\pi - Bi/\kappa}$$

$$\alpha = (1 - \kappa)/(1 + \kappa)$$

with $\kappa = k_2/k_1$, $Bi = hc/k_1$, and $\delta = m\pi/c$.

The result given by Eq. (32) may be compared with the result for the compound annulus Eq. (25) provided that

$$\psi_{s,\text{annulus}} = 2\psi_{s,\text{strip}} \tag{34}$$

because the solution for the strip is for two elements in parallel.

Parametric Studies

The general result given by Eq. (25) depends on seven parameters. The large number of parameters makes it difficult to address the effect that each parameter has on the solution. A number of simplifications will be made such that the important characteristics may be examined. First, the effect of the heat flux distribution parameter μ will not be considered because Eq. (31) describes this effect. In all cases presented in this section, $\mu = 0$. Furthermore, because the effect of conductivity ratio κ will increase or decrease the thermal spreading parameter accordingly, only the isotropic case $\kappa = 1$ will be considered. Finally, the solution is also dependent on the size of the sector α through the definition of the eigenvalues. The sector having $\alpha = \pi/2$, as shown in Fig. 5, is chosen. These simplifications result in

$$\psi = f(\epsilon, Bi, \rho) \tag{35}$$

This functional dependence will illustrate the effects of relative contact size ϵ, film coefficient Bi, and relative shell thickness ρ.

The solution for the spreading parameter ψ are presented in Figs. 6–8 for three values of epsilon: $\epsilon = 0.25$, 0.5, and 0.75, as a function of the radii ratio ρ for several values of the Biot number Bi. All of the results were computed using MATLAB® (Ref. 7) mathematics software. In all cases, at some critical value of ρ, the results approach a constant value regardless of the value of the Biot number. The asymptotic result is equal to the value of the the spreading parameter for a semi-infinite flux channel.[2] This is a result of the parameter $\varphi \to 1$ in Eq. (25) for all terms in the summation. This reduces the general solution to that reported by Yovanovich et al.[2] for the semi-infinite flux channel.

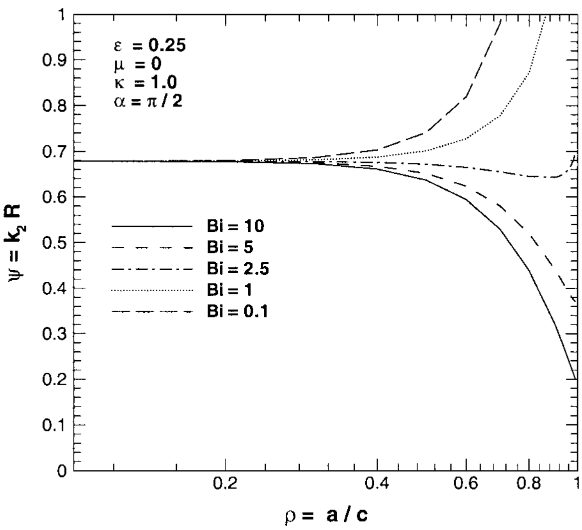

Fig. 6 Spreading parameter for $\epsilon = 0.25$.

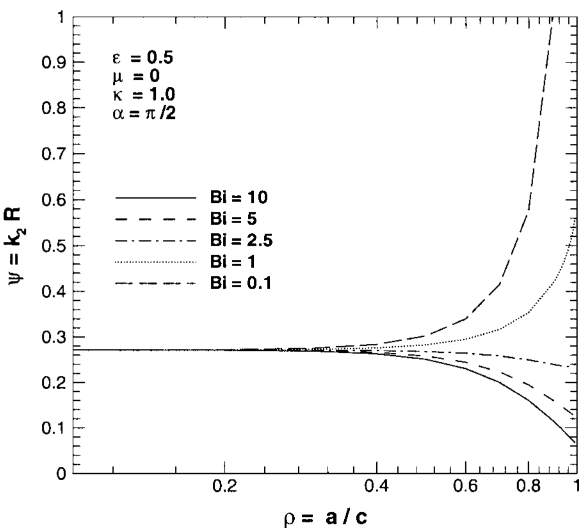

Fig. 7 Spreading parameter for $\epsilon = 0.5$.

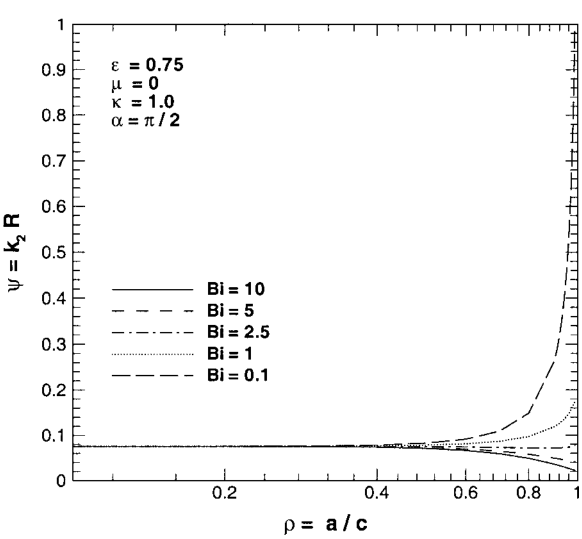

Fig. 8 Spreading parameter for $\epsilon = 0.75$.

Equivalence of Annular Sector and Flux Channel Solutions

Finally, it is desirable to compare the solutions for an annular sector and finite flux channel to examine the effect that curvature has on the spreading parameter. In the limit of $\rho \to 0$, the solution exhibits characteristics of the semi-infinite flux channel. In the limit of $\rho \to 1$, the solution should exhibit characteristics of the finite flux channel as curvature effects become negligible. To compare both Eqs. (25) and (32) on the same plot, a new dependent parameter will be defined. This parameter is chosen to be the ratio of the shell thickness to the size of the flux channel. To convert the annular sector to a flux channel, as shown in Fig. 9, the following parameters will be defined:

$$t_e = c - a \tag{36}$$

$$r_e = (a + c)/2 \tag{37}$$

such that

$$\tau_e = t_e/\alpha r_e = (2/\alpha)[(1 - \rho)/(1 + \rho)] \tag{38}$$

$$Bi_e = \alpha Bi \tag{39}$$

$$\epsilon_e = \epsilon \tag{40}$$

where the subscript e denotes the the equivalent rectangular flux channel parameter in terms of the annular sector parameters. A comparison between Eqs. (25) and (32) is made for a range of Biot number, ϵ, and ρ using Eqs. (38–40). This allows both limits of thick and thin shells to be compared with the flux channel solution as the overall thickness increases or decreases. The results of this comparison are presented in Figs. 10–12.

It is seen in Figs. 10–12 that, for all τ, the solutions are equivalent. In Figs. 10–12, the maximum deviation is approximately 1.3%. This

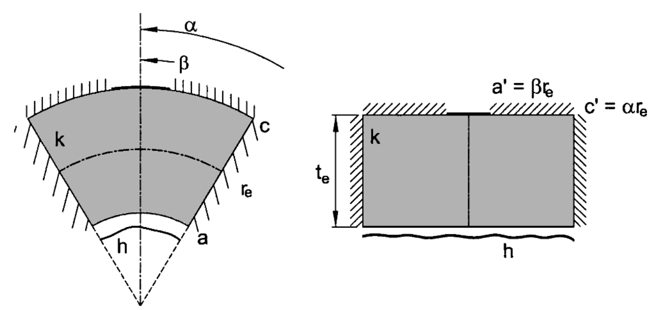

Fig. 9 Conversion of annular sector to rectangular flux channel.

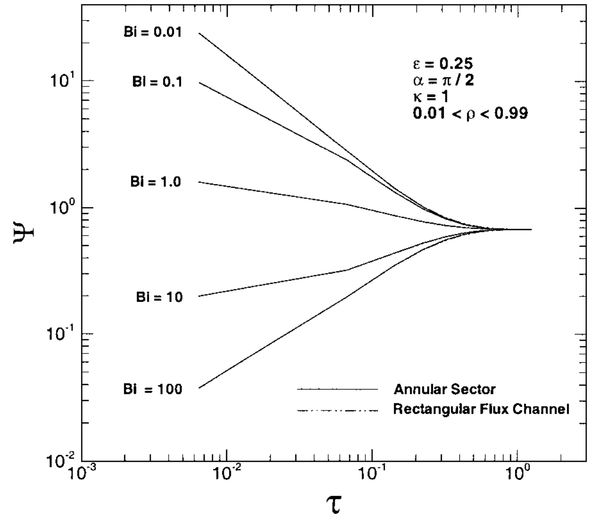

Fig. 10 Comparison of flux channel and annular sector results for $\epsilon = 0.25$.

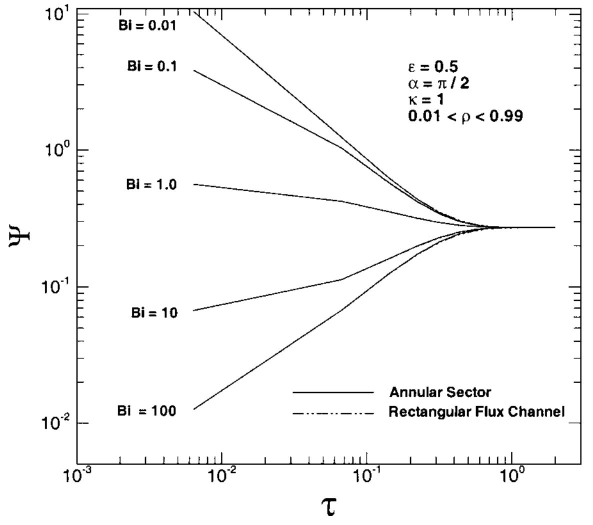

Fig. 11 Comparison of flux channel and annular sector results for $\epsilon = 0.5$.

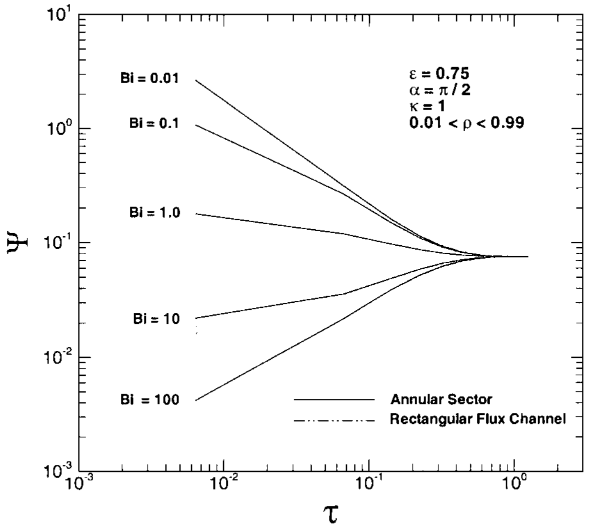

Fig. 12 Comparison of flux channel and annular sector results for $\epsilon = 0.75$.

suggests that the effect of curvature on the spreading parameter is small. In most practical applications, the strip solution would provide adequate results. However, the total resistance, which includes the one-dimensional flow solution, will be affected by the curvature of the system.

Concluding Remarks

A general solution for the thermal spreading resistance in a compound annular sector flux channel was obtained. This solution may be used to model the spreading resistance in a compound annulus for any number of equally spaced heat sources (or sinks). Results were obtained for three heat flux distributions: the uniform flux, parabolic flux, and inverse parabolic flux profiles. It was shown that the general solution for the annular sector is similar to that of the finite rectangular flux channel. For the case of a thin shell, it was shown that the results approach that of the finite rectangular flux channel, and, in the case of a thick shell, the results also approach that of a semi-infinite rectangular flux channel. Finally, it was shown that the effect of curvature on the thermal spreading parameter is small.

Acknowledgment

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