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Resistance Approach for Annular Fins with Contact Conductance and End Cooling

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A simplified method, based on control volume, is developed to obtain the resistance of annular fins of constant thickness with base contact and end cooling. This approach is further extended to annular fins with variable thickness for which analytical solution is not possible. Such novel method is able to match the analytical solution for constant thickness fins with minimum number of control volumes and shows excellent agreement for various limiting cases of Biot numbers.

Nomenclature

a	tip thickness
A	cross-sectional area
b	base thickness
C_1	constant with modified Bessel functions
Bi	Biot number
h	heat transfer coefficient
I_1	modified Bessel function of first kind
k	thermal conductivity
K_1	modified Bessel function of second kind
L	fin length
m	non-dimensional fin parameter
n	number of control volume
Q	heat flow
r	radius
R	fin resistance
t	half-fin thickness
T	fin temperature
α	non-dimensional inner radius
β	non-dimensional outer radius
θ	semi-tapered angle
Θ	fin temperature excess of ambient
ψ	non-dimensional temperature
Ω	constant with modified Bessel functions
ρ	non-dimensional radial position

Subscripts

c	base contact
e	end cooling
f	convective surface
i	inner
j	index for control volume
o	outer
∞	ambient

Superscript

e	end conduction

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Introduction

ANNULAR fins are frequently used as extended surfaces to enhance the heat transfer rate in various applications such as air conditioning, heat exchangers and micro-electronic packages. When such fins are added to a surface in contact with the surrounding fluid, the following resistances play a key role in the entire heat transfer process: (a) the contact resistance due to the mechanical contact between the fin base and the previously exposed surface, (b) the conductive resistance to heat flow within the fin itself, and (c) the resistance to heat flow through the convective film of the surrounding fluid. Several analytical solutions for the steady-state heat conduction within an annular fin of constant thickness, perfect contact at the base, and insulated end (or some approximation for end cooling) already exist.[1,7] On the other hand, the one-dimensional steady-state analytical solution for annular fin with constant thickness along with base contact resistance and end cooling[3] involves modified Bessel functions that are difficult to compute and also computationally intensive, the details of which will be discussed later. However, in the case of a variable thickness annular fin, analytical solutions do not exist. Such problems are solved numerically, either through finite difference[4] or finite element[5] method, or by means of integral control volume approach.[6]

The main objective of the present work is to develop a simplified, yet accurate technique, by means of resistance method, for determining the fin resistance of constant thickness annular fin with base contact resistance and end cooling. It is also shown that such technique can be readily extended for an annular fin with variable thickness.

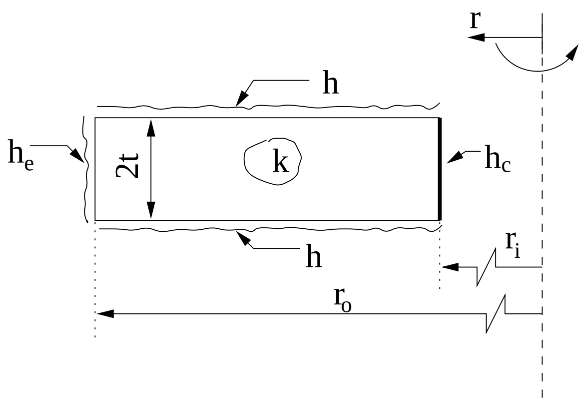

Fig. 1 Schematic diagram of a constant thickness annular fin.

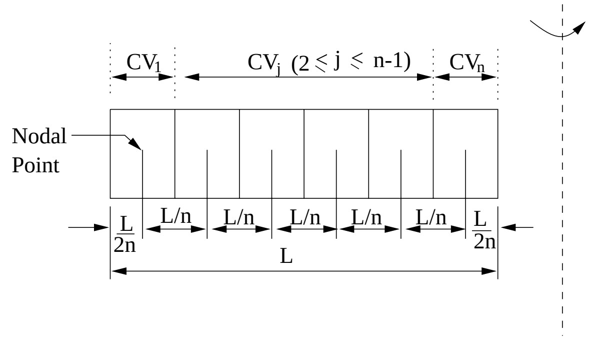

Fig. 2 Subdivision into n control volumes.

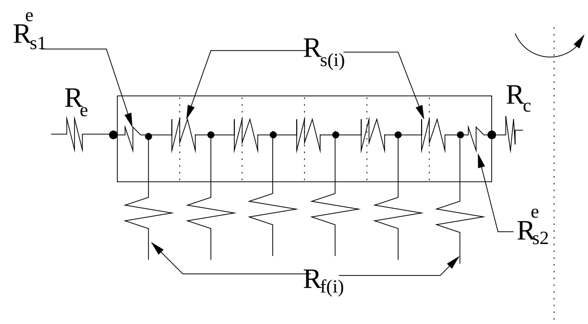

Fig. 3 Different resistances associated with each control volume.

Constant Thickness Fin

Analytical Method

Figure 1 shows a schematic diagram for an annular fin of constant thickness $2t$, thermal conductivity k, and inner and outer radii of r_i and r_o, respectively. The fin is cooled along the sides through a uniform film coefficient h and at the end through a uniform film coefficient h_e. The contact conductance, h_c, at the base is assumed to be uniform. The steady state one-dimensional governing heat transfer equation for the fin can be written as:[7]

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d\psi}{d\rho}\right] - m^2\psi = 0 \tag{1}$$

where $\psi = \Theta/\Theta_b$, with Θ defined as $\Theta = T(r) - T_\infty$ and $\Theta_b = T(r_i) - T_\infty$; $\rho = r/t$, and m^2 is the non-dimensional fin parameter. By introducing the following non-dimensional Biot numbers:

$$Bi = ht/k \quad (= m^2) \tag{2}$$

$$Bi_c = h_c t/k \tag{3}$$

$$Bi_e = h_e t/k \tag{4}$$

the base and end boundary conditions can be written as:

$$\rho = \alpha, \qquad \frac{d\psi}{d\rho} = -Bi_c[1-\psi] \tag{5}$$

$$\rho = \beta, \qquad \frac{d\psi}{d\rho} = -Bi_e\psi \tag{6}$$

where $\alpha = r_i/t$ and $\beta = r_o/t$. The solution of the governing equation subjected to the boundary conditions gives the fin resistance, R_{fin}, in the following form:

$$R_{fin} = [4\pi k t m \alpha C_1\{\Omega K_1(m\alpha) - I_1(m\alpha)\}]^{-1} \tag{7}$$

where I_1 and K_1 are the modified Bessel functions of the first and the second kind of order unity, respectively; C_1 and Ω are constants which involve modified Bessel functions, the details of which are provided by Yovanovich et al.[3] Therefore, it becomes obvious that the analytical solution for constant thickness annular fin requires evaluation of the Bessel functions, which at times may be difficult to obtain.

Resistance Method

As an alternative approach to the exact closed form solution, the resistance method is used to obtain the fin resistance accurately and easily with minimum computation effort. The entire fin length $L(= r_o - r_i)$ is divided into n control volumes, as shown in Fig. 2. The nodal points are located at the center of each control volume. Figure 3 shows the thermal resistances pertinent to each control volume due to the conduction within the fin material (R_s), the convective side (R_f) and end (R_e) coolings and the base contact (R_c). The conduction resistances, (R_s^e), for the two end control volumes (CV_1 & CV_n) are separately evaluated in order to simplify the thermal circuit. The general expressions for each resistance can be written as:

$$R_c = \frac{1}{h_c(4\pi r_i t)} \tag{8}$$

$$R_e = \frac{1}{h_e(4\pi r_o t)} \tag{9}$$

$$R_{f(j)} = \frac{1}{h\pi\left\{\left[r_o - \frac{(j-1)}{n}L\right]^2 - \left[r_o - \frac{j}{n}L\right]^2\right\}} \tag{10}$$

(for $j = 1, \ldots (n-1)$)

$$= \frac{1}{h\pi\left\{\left[r_o - \frac{(n-1)}{n}L\right]^2 - r_i^2\right\}} \tag{11}$$

(for $j = n$)

Table 1 Geometrical and physical parameters for the fin.

Parameters	Values
r_i	$5\ mm$
r_o	$10\ mm$
t	$1\ mm$
k	$20\ W/mK$
h	$50\ W/m^2K$
h_e	$20\ W/m^2K$
h_c	$500\ W/m^2K$

$$R_{s(j)} = \ln\left[\frac{2nr_o - \{1+2(i-1)\}L}{2nr_o - \{3+2(i-1)\}L}\right]\frac{1}{4\pi kt} \quad (12)$$
$$(\text{for } j = 1, \ldots (n-1))$$

$$R^e_{s1} = \ln\left[\frac{r_o}{r_o - \frac{L}{2n}}\right]\frac{1}{4\pi kt} \quad (13)$$

$$R^e_{s2} = \ln\left[\frac{r_i + \frac{L}{2n}}{r_i}\right]\frac{1}{4\pi kt} \quad (14)$$

In order to obtain the total resistance, the resistances for each control volume are summed, starting from the tip and moving towards the base. Therefore, the equivalent resistance for each control volume can be written as:

$$\frac{1}{R_1} = \frac{2}{R_{f1}} + \frac{1}{(R^e_{s1}+R_e)} \quad (15)$$
$$\frac{1}{R_2} = \frac{2}{R_{f2}} + \frac{1}{(R_1+R_{s1})} \quad (16)$$
$$\frac{1}{R_3} = \frac{2}{R_{f3}} + \frac{1}{(R_2+R_{s2})} \quad (17)$$
$$\vdots \qquad \vdots$$
$$\frac{1}{R_n} = \frac{2}{R_{fn}} + \frac{1}{(R_{n-1}+R_{s(n-1)})} \quad (18)$$

where R_j $(j = 1, \ldots n)$ is the equivalent resistance for the j^{th} control volume. Hence, the overall fin resistance is expressed as

$$R = R_n + R_c + R^e_{s2} \quad (19)$$

Results and Discussions

The geometrical and the physical parameters used to calculate the fin resistance are provided in Table 1. With these typical values, the analytical solution for the fin resistance, given by Eq. (7), is $71.52K/W$. By using the resistance method, it is found that the analytical solution can be obtained with a minimum number of control volume, as shown in Fig. 4. It is observed that even with three control volumes ($n = 3$), the difference in the solution between the analytical and the resistance methods is only 0.04%. It is to be pointed out that previous attempts using a resistance approach[4] required a large number of iterations

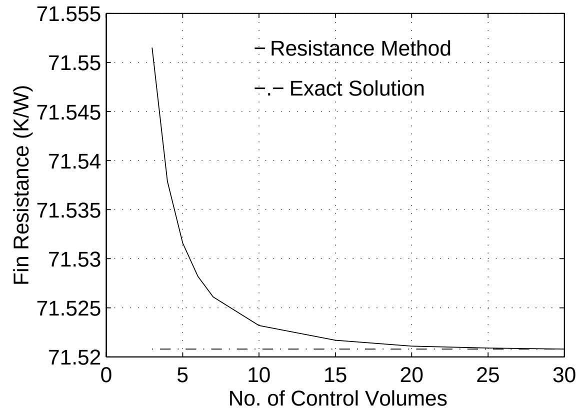

Fig. 4 Comparison of the result obtained by resistance method with exact solution.

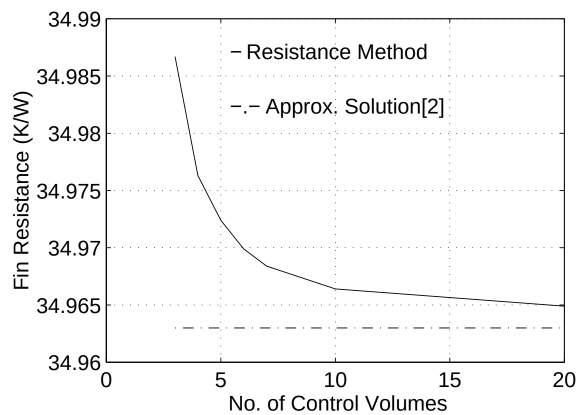

Fig. 5 Comparison of the solution obtained by resistance method with approximate method.[2]

(approximately 50 iterations) to obtain the analytical solution for the constant thickness fin. Therefore, it shows that the present approach is very efficient, yet accurate, in determining the fin resistance with minimum computational effort.

The present resistance method can be applied to obtain various approximate solutions which involve correction factor for tip convection and zero base resistance.[2] For such cases, a good agreement can be obtained with the approximate solution by treating the end and the side convection coefficients to be equal, as shown in Fig. 5 with $h_e = h = 50\ W/m^2K$. The fin resistances are also calculated for different limiting cases of Biot numbers and they are compared with the analytical expressions given below:

$$R = \frac{\ln(r_o/r_i)}{4\pi kt}\left\{\begin{array}{l} Bi_c \to \infty \\ Bi_e \to \infty \quad Bi \to 0 \end{array}\right. \quad (20)$$

$$R = \frac{\ln(r_o/r_i)}{4\pi kt} + \frac{1}{h_e 4\pi r_o t} \quad (21)$$
$$\left\{\begin{array}{l} Bi_c \to \infty \\ 0 < Bi_e < \infty \quad Bi \to 0 \end{array}\right.$$

Table 2 Comparison of fin resistance at different limiting cases. $(n = 3)$

Biot Number	Exact Solution (K/W)	Resistance Method (K/W)
$Bi \to 0$, $Bi_e \to \infty$, $Bi_c \to \infty$	2.7579	2.7579
$Bi \to 0$, $0 < Bi_e < \infty$, $Bi_c \to \infty$	400.645	400.638
$Bi \to 0$, $0 < Bi_e < \infty$, $0 < Bi_c < \infty$	432.476	432.469

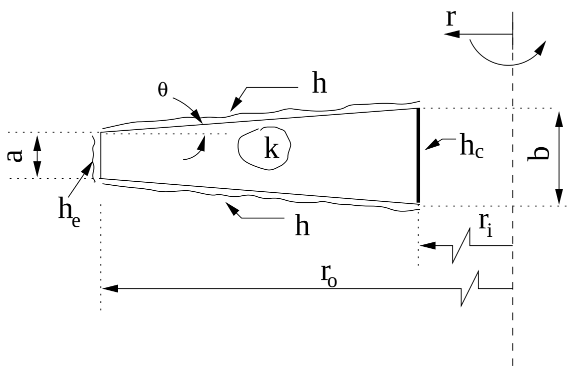

Fig. 6 Schematic diagram of a variable thickness annular fin.

$$R = \frac{\ln(r_o/r_i)}{4\pi kt} + \frac{1}{h_e 4\pi r_o t} + \frac{1}{h_c 4\pi r_o t} \qquad (22)$$
$$\left\{ \begin{array}{l} 0 < Bi_c < \infty \\ 0 < Bi_e < \infty \quad Bi \to 0 \end{array} \right.$$

Table 2 summarizes the results and it is found that by using only three control volumes, the resistance method yields solutions same as those obtained analytically.

Variable Thickness Fin

Resistance Method

So far, it is observed that the present resistance method evaluates the fin resistance accurately for a constant thickness annular fin. As a next step, this approach is extended to estimate the resistance of a variable thickness fin for which analytical solutions are not available in literature. Figure 6 shows a schematic diagram for a variable thickness annular fin whose tip and base thicknesses are a and b respectively and θ is semi-tapered angle of the fin. By applying the resistance method to this geometry, the placement of the control volumes and the associated thermal resistances remain the same as before.

The main difficulty with a variable thickness fin is to obtain expressions for resistances due to conduction within the fin, which are different from those stated in Eqs. (12)-(14). To overcome this difficulty, Fourier's Law of heat conduction is used, which is as follows:

$$\Delta Q = kA\frac{\Delta T}{\Delta r}$$

where ΔQ is the heat flow through a cross-section area A and over a thickness of Δr with a temperature difference of ΔT. By using the definition of resistance, it can be shown that

$$\Delta R = \frac{\Delta T}{\Delta Q} = \frac{\Delta r}{kA} \qquad (23)$$

where ΔR is the conduction resistance for the thickness Δr. This is a good approximation, as with the increase in the number of control volumes, Δr becomes smaller and therefore in the limiting case, the conduction resistance approaches the exact value. Hence, expressions for the resistances due to conduction within the fin, base conductance and tip convection take the following form:

$$R_{s(j)} = \frac{L/n}{4\pi k\left(r_o - j\frac{L}{n}\right)\left[\frac{a}{2} + j\frac{L}{n}\tan\theta\right]} \qquad (24)$$
$$(\text{for } j = 1, \ldots (n-1))$$

$$R^e_{s1} = \frac{L/(2n)}{4\pi k\left(r_o - \frac{L}{4n}\right)\left[\frac{a}{2} + \frac{L}{4n}\tan\theta\right]} \qquad (25)$$

$$R^e_{s2} = \frac{L/(2n)}{4\pi k\left(r_i + \frac{L}{4n}\right)\left[\frac{a}{2} + \left(r - \frac{L}{4n}\right)\tan\theta\right]} \qquad (26)$$

$$R_e = \frac{1}{2\pi h_e r_o a} \qquad (27)$$

$$R_c = \frac{1}{2\pi h_c r_i b} \qquad (28)$$

It is to be noted that for resistances due to convection, the effective heat transfer area is taken as the projection of the corresponding area for a constant thickness annular fin. Therefore, the convective resistances are obtained by multiplying the corresponding resistances of the constant thickness fin, given in Eqs. (10)-(11), with the cosine of the semi-tapered angle of the fin. The final expression for the fin resistance can be obtained in a similar manner as shown in Eqs. (15)-(19).

Results and Discussions

In case of a variable thickness fin, two additional geometrical parameters, a and b, are required along with those listed in Table 1. The typical values used for a and b are $2mm$ and $4mm$, respectively. Figure 7 shows the variation of the fin resistance with the number of control volumes, starting with $n = 3$. It is observed that the fin resistance asymptotically approaches to a constant value within 20 control volumes. It is also found that the difference between the asymptotic value and that obtained with three control volumes is only

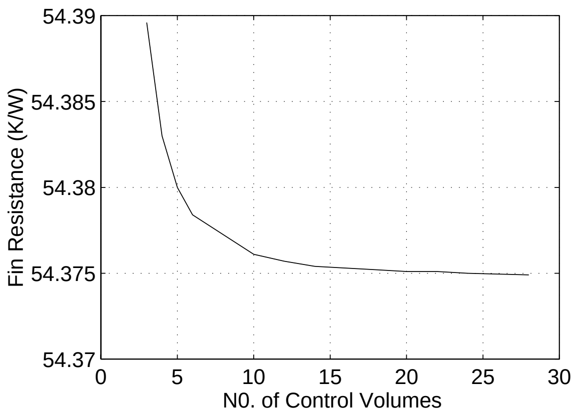

Fig. 7 Fin resistance for a variable thickness annular fin.

Table 3 Comparison of the fin resistance for $a = 2mm, b = 2mm, 2t = 2mm$ and $\theta = 0$ $(n = 30)$.

L (mm)	Constant Thickness (K/W)	Variable Thickness (K/W)
5	71.5208	71.5207
50	38.4551	38.4437

0.02%. Since, no closed form analytical solution is possible for this case, therefore the accuracy of the solution could not be claimed with certainty. However, the solution for the variable thickness fin can be checked with the limiting case, when the tapered fin approaches a constant thickness fin. Therefore, the variable thickness fin solution should approach the constant thickness result by setting $\theta = 0$ in Eqs. (24) - (26). Table 2 shows excellent agreement between the constant thickness fin and the variable thickness fin for $\theta = 0$. Hence, this approach presents a simple way of finding fin resistances for different complex geometries, for which only numerical solutions are possible.

Conclusion

A resistance approach, involving control volumes, is developed for calculating the fin resistances of annular fins with contact conductance and end cooling. The method is first applied to a constant thickness fin and it is found that an accurate and a quick solution can be obtained with only a few number of control volumes. Various limiting cases of Biot numbers are also calculated and excellent agreements are obtained with analytical results. This approach is further extended to variable thickness fins for which no closed form solution is available. It is observed that the resistance method can also be applied successfully for such variable thickness annular fins.

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