AIAA 2004-2569

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37th AIAA Thermophysics Conference
28 June-1 July/Portland, OR

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ABSTRACT

A heat balance method (HBM) is presented for accurate and quick calculations of the overall thermal resistance and heat dissipation from spines, longitudinal and radial fins of arbitrary profile with contact conductance and end cooling. The proposed HBM is general and can be applied to all spines and fins, including many practical examples that are not included in text books and handbooks. A general outline of the HBM is presented and particular relationships are given for conduction and convection from discrete control volumes for spines, longitudinal and radial fins. It is shown that the HBM can be easily applied to radial fins of any profile (e.g., trapezoidal, rectangular, triangular, etc) that are mechanically attached to tubes. The HBM can be applied to radial fins with base contact resistance and end cooling. The HBM can be easily modified for variable heat transfer coefficients on the lateral surface; however, only uniform heat transfer coefficients are considered herein. The HBM is easily implemented in computer algebra systems, and its shown that the HBM gives very accurate results for 10 equal length control volumes. Its found by comparison of the numerical results with available exact results that the maximum error is less than 1%.

NOMENCLATURE

a	=	fin tip half-thickness or radius $[m]$
A	=	conduction area $[m^2]$
b	=	fin base half-thickness or radius $[m]$
Bi	=	fin Biot number $\equiv hb/k$
D	=	diameter of circular cylinder $[m]$
k	=	thermal conductivity $[W/mK]$
K	=	degree kelvin $\equiv 273.15 +^0 C$
h	=	heat transfer coefficient $[W/m^2K]$
h_c	=	fin base contact conductance $[W/m^2K]$
h_e	=	fin tip heat transfer coefficient $[W/m^2K]$
$I_0(\cdot), I_1(\cdot)$	=	modified Bessel functions of first kind of order 0 and 1
L	=	fin length $[m]$
L	=	radial fin length $L = r_o - r_i$ $[m]$
m	=	longitudinal fin parameter $\equiv \sqrt{h/kb}$ $[m]$
	=	spine parameter $\equiv \sqrt{2h/kb}$ $[m]$
N	=	number of control volumes
P	=	fin perimeter $[m]$
Q	=	heat flow rate $[W]$
Q_{fin}	=	heat flow rate from fin $[W]$
$Q_{f,e}$	=	heat flow rate from fin tip or end $[W]$
$Q_{f,j}$	=	heat flow rate from jth control volume $[W]$
r	=	radial coordinate $[m]$
R_{fin}	=	fin resistance $\equiv \theta_b/Q_{fin}$ $[K/W]$
r_i, r_o	=	inner and outer radii of radial fin $[m]$
$S^\star$	=	dimensionless convection surface $\equiv S/2wL$
T	=	fin temperature $[^\circ C]$
T_f	=	fluid temperature $[^\circ C]$
w	=	width of longitudinal fin $[m]$
x, y	=	Cartesian coordinate

Greek Symbols

β	=	dimensionless parameter $\equiv (b-a)/L$
ϵ	=	thickness or radii ratio $\equiv b/a$
η	=	fin efficiency
μ	=	fin profile parameter
θ	=	excess temperature $\equiv T(x) - T_f$ $[K]$
ξ	=	Cartesian coordinate and $\equiv x/L$

Subscripts

b	=	fin base
$base$	=	fin base
c	=	contact
e	=	fin end
end	=	fin end
fin	=	fin
i	=	inner
$ideal$	=	ideal fin heat flow rate
j	=	jth control volume
o	=	outer
$rect$	=	rectangular profile
$trap$	=	trapezoidal profile

INTRODUCTION

Convection heat transfer from a prime (bare) surface in natural and forced fluid flows can be significantly augmented or enhanced by the addition of *fins* or *extended* surfaces which are attached in some manner to the prime surface. The fins may be integral with the prime surface or they may be mechanically attached. If the fin base and the prime surface are integral, then the contact at the fin base is said to be perfect. If the fins are mechanically attached, then there is a contact resistance at the fin base which reduces the heat transfer from the fin.

The fin geometries are classified as (1) longitudinal or straight fins, (2) radial or annular fins, and (3) spines. The profiles are classified as (a) rectangular, (b) triangular or conical, (c) convex parabolic, and (d) concave parabolic. For a more detailed descriptions of fin geometries and fin profiles, one may consult Kern and Kraus[1], Kraus and Bar-Cohen[2], Kraus et al.[3], and Aziz[4].

Figures 1, 2 and 3 show the general profiles of the spine and longitudinal fins, and the radial fin, respectively. The half-thickness or radius at the fin tip is denoted as a, while the half-thickness or radius at the fin base is denoted as b, and $b \geq a$. The length of the fins is denoted as L, and for the radial fin its related to the inner and outer radii, r_i, r_o such that $L = r_o - r_i$. The local conduction area $A(x)$ and the local perimeter $P(x)$ are related to the local half-thickness or local radius denoted as $y(x)$ which is a function of a, b, and L through the fin profile parameter μ which can take on different values depending on the fin profile. The relationships are given in the figures. There are two cartesian coordinates: one located in the fin base called x, and the second located in the fin tip called ξ. They are related as $x + \xi = L$.

The fin thermal conductivity k is constant. The uniform heat transfer coefficient on the lateral surfaces is denoted as h. The uniform base contact conductance is h_c and the uniform tip heat transfer coefficient is h_e. In general, the fin resistance depends on several parameters, i.e., $R_{fin} = f(a, b, L, \mu, k, h, h_c, h_e)$.

Limiting Assumptions

The fin or extended surface equations and solutions are based on the so-called Murray-Gardner assumptions which are[1−4]:

1. The heat flow and temperature distribution in the fin are constant with time.
2. The fin material is homogeneous and isotropic.
3. The heat transfer coefficient is constant and uniform over the entire surface of the fin.
4. The temperature of the medium surrounding the fin is uniform.
5. The fin thickness relative to its length is sufficiently small that temperature gradients across the fin thickness may be neglected.
6. The temperature at the base of the fin is uniform and constant.
7. There is no contact resistance where the base of the fin joins is in contact with the prime surface.
8. There are no distributed heat sources within the fin.
9. The heat transfer through the fin tip or end is negligible compared to that leaving the fin through the lateral surface.
10. Heat transfer to or from the fin is proportional to the temperature excess between the fin and its surrounding medium, and the temperature excess is one-dimensional.

Many analytical solutions are available[1−4] for longitudinal, radial fins, and spines which are based on the limiting assumptions of Murray-Gardner. The results are often presented as *fin efficiency* or *fin effectiveness* in graphical form or as analytical relationships[1−4].

Assumptions 6, 7 and 9 are too restrictive because they preclude many important practical applications. To overcome these restrictions Yovanovich[5] developed a non-iterative control volume approach to systems with one-dimensional conduction with convection heat losses. This novel approach permitted the relaxation of assumptions 6, 7 and 9, and it was applied to longitudinal fins, radial fins, and spines of any profile with uniform and variable heat transfer coefficient along the lateral surface. It was demonstrated that only a few (5 to 10) control volumes yielded very accurate numerical results for the fin resistance and fin efficiency.

Yovanovich[6] formulated the fin equation in orthogonal curvilinear coordinates. He demonstrated that the general fin equation with general boundary conditions, contact resistance at the fin base and convective cooling at the fin tip, reduced to many special cases[1−4]. There are many other cases which arise from the general fin equation, e.g., solutions which are valid for conduction through cylindrical and spherical shells, to name only two special cases.

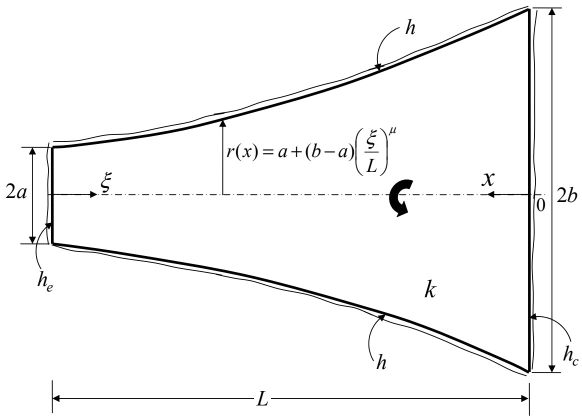

Fig. 1 General Spine Profile

Mitra and Yovanovich[7] recently presented a resistance network method applied to radial fins of arbitrary profile with contact resistance at the fin base and convective cooling at the fin end. It was shown that only a few internal resistors (5 to 10) give very accurate numerical results for the fin resistance and fin efficiency when compared against available analytical results. It was shown that the fin resistance converges rapidly to a constant value as the number of internal resistors increases.

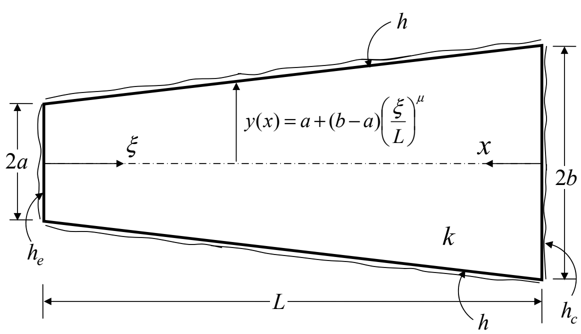

Fig. 2 General Longitudinal Profile

Assumptions 6, 7 and 9 will be relaxed in the development of the heat balance method; therefore the relationships will be general and applicable to all fins which have contact resistance at the fin base and significant cooling at the fin tip. The proposed HBM will be applied to longitudinal fins, radial fins, and spines having arbitrary profile with uniform heat transfer coefficient over the entire lateral surface, with contact conductance at the fin base, and convective cooling of the fin tip. The HBM will yield accurate numerical values for discrete excess temperatures, fin resistance, and fin heat transfer rate.

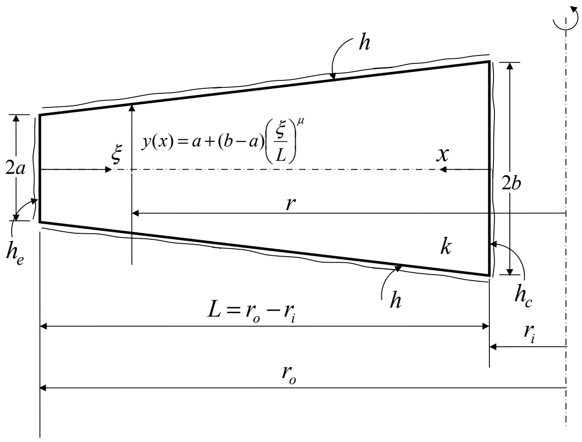

Fig. 3 General Radial Fin Profile

HEAT BALANCE METHOD FOR SPINES, LONGITUDINAL AND RADIAL FINS

A simple and direct numerical method based on heat balances on discrete control volumes will be outlined and implemented to demonstrate the accuracy and efficacy of this approach to one-dimensional conduction with convection problems. The system (spines, longitudinal and radial fins) is divided into N equal length control volumes of length L/N where $N \geq 3$ as shown in Fig. 4. For radial (annular) systems, the length is defined as $L = r_o - r_i$ where r_i and r_o represent the inner and outer radii, respectively. The system consists of two *boundary* control volumes (one at the system base and one at the system tip) and $(N-2)$ *interior* control volumes. The origin of the cartesian coordinate x is located in the base of the system and $0 \leq x \leq L$.

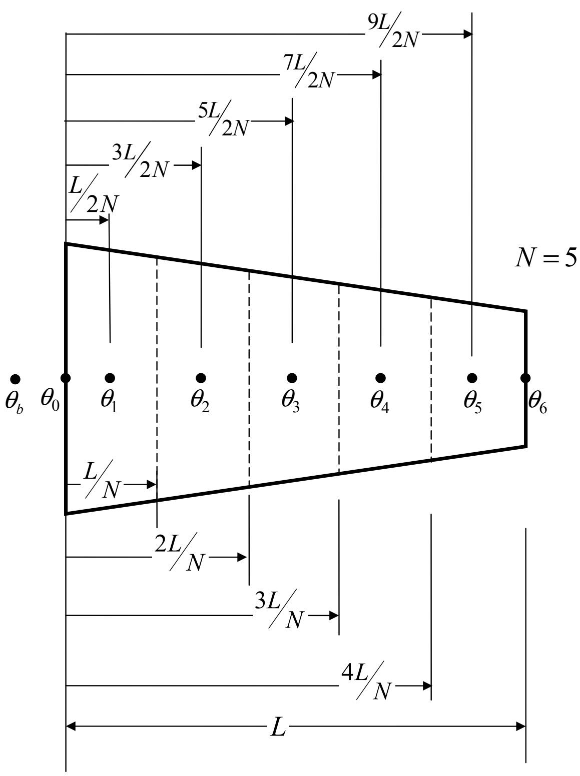

Fig. 4 Locations of Temperature Nodes

Excess temperature nodes $\theta_j = T_j - T_f$ where $1 \leq j \leq N$ are located in the center of the control volumes such that $x_j = (j-1/2)L/N$. One node, labelled θ_0, is located in the base surface and another node, labelled θ_{N+1}, is located in the system tip.

If there is contact resistance at the system base, then a node, labelled θ_b, is located in the base $x = 0^-$. The nodes $\theta_b = T_b - T_f$ and θ_0 are connected by the contact resistance $R_c = 1/h_c A_b$ where $h_c > 0$ is the base contact conductance. If there is perfect contact at the base, i.e., $h_c \to \infty$, then $\theta_0 = \theta_b$. Approximate values of the system heat transfer rate, Q_{fin}, and the system resistance, R_{fin}, will be obtained from the approximate values of the excess temperatures which will depend on the number of control volumes.

In order to make heat balances on the control volumes it is necessary to define the heat conduction and convection relationships at the system base and tip, and on the control volume boundaries. The heat conduction and convection relationships will depend on the conduction areas and the convection surface areas which may be defined accurately for the spine, and longitudinal, and radial systems.

Conduction Relationships on Control Volume Boundaries

In the general case there are N control volumes and $N + 2$ relationships for the conduction rates into and out of the control volumes. The conduction rates into and out of the control volumes, shown in Fig. 5, are

$$\left.\begin{aligned}
Q_{base} &= h_c A_0 \left(\theta_b - \theta_0\right) \\
Q_0 &= \frac{kA(L/4N)}{L/2N} \left(\theta_0 - \theta_1\right) \\
Q_j &= \frac{kA_j}{L/N} \left(\theta_j - \theta_{j+1}\right) \\
Q_N &= \frac{kA(L - L/4N)}{L/2N} \left(\theta_N - \theta_{N+1}\right) \\
Q_{end} &= h_e A_N \theta_{N+1}
\end{aligned}\right\} \tag{1}$$

where $1 \leq j \leq N - 1$. The distance between the excess temperature nodes θ_0 and θ_1 at the system base, and θ_N and θ_{N+1} at the system tip, is $L/2N$.

Convection Relationships at the Control Volume Boundaries

In the general case there are $N + 1$ relationships for the convection losses from the lateral boundaries and the system tip. The convection loses from the conrol volumes are shown in Fig. 5, and they are given by the following relationships:

$$\left.\begin{aligned}
Q_{f,j} &= h_j S_j \theta_j \\
Q_{f,e} &= h_e A_N \theta_{N+1}
\end{aligned}\right\} \tag{2}$$

where $1 \leq j \leq N$ and h_j represents the average value of the heat transfer coefficient on the *jth* control volume surface. If the heat transfer coefficient is uniform, then $h_j = h$ for $1 \leq j \leq N$. The heat transfer coefficient at the system tip (end) which is denoted as h_e is assumed to be uniform over the system tip.

Heat Balance Relationships

There are $N + 2$ heat balance relationships:

$$\left.\begin{aligned}
\text{HB}_{base} &= Q_{base} - Q_0 = 0 \\
\text{HB}_1 &= Q_0 - Q_1 - Q_{f,1} = 0 \\
\text{HB}_j &= Q_{j-1} - Q_j - Q_{f,j} = 0 \qquad 2 \leq j \leq N - 1 \\
\text{HB}_N &= Q_{N-1} - Q_N - Q_{f,N} = 0 \\
\text{HB}_{end} &= Q_N - Q_{f,e} = 0
\end{aligned}\right\} \tag{3}$$

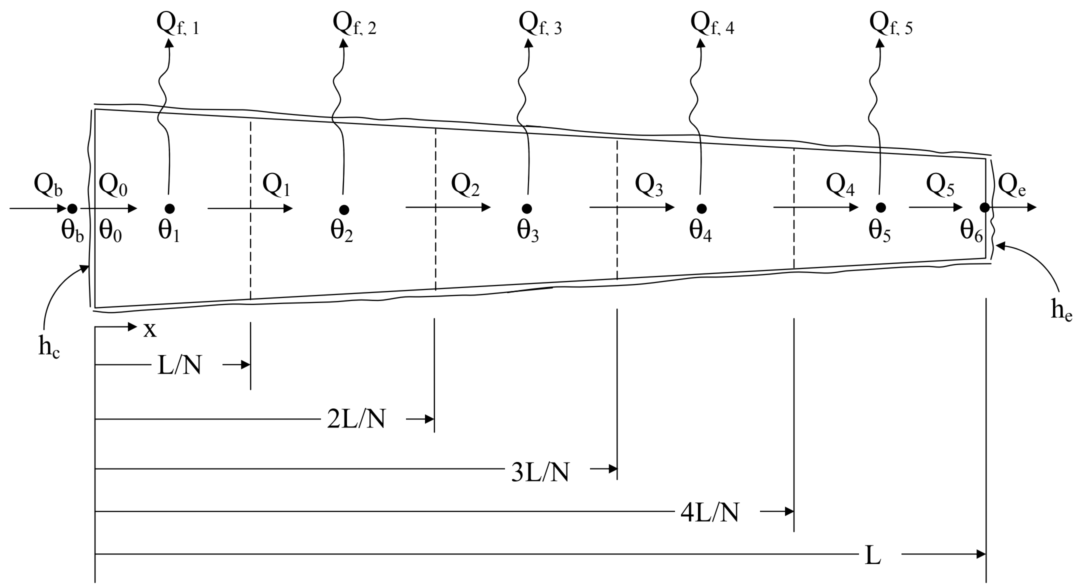

Fig. 5 Heat Balances on Control Volumes

The heat balance relationships give $N+2$ equations for the excess temperature nodes θ_j where $0 \leq j \leq N+1$.

System Heat Transfer Rate

The heat transfer rate through the system is given by the convection losses and the base conduction relationships:

$$\left.\begin{array}{lrcl} \text{convection losses} & Q_{fin} & = & \displaystyle\sum_{j=1}^{N} Q_{f,j} + h_e A_N \theta_{N+1} \\ \text{base conduction} & Q_{fin} & = & \dfrac{kA(L/4N)}{L/2N}\,(\theta_0 - \theta_1) \end{array}\right\} \tag{4}$$

The calculated values of Q_{fin} based on conduction from the system base into the first control volume and the convection losses are equal.

System Resistance

The system (fin) resistance is

$$R_{fin} = \frac{\theta_b}{Q_{fin}} \tag{5}$$

which includes the effect of the base contact resistance.

Fin Efficiency

The fin efficiency is defined for perfect base contact; its given by

$$\eta = \frac{Q_{fin}}{Q_{ideal}} \tag{6}$$

where the ideal (maximum) heat transfer rate from the system surface is

$$Q_{ideal} = [hS + h_e\, A_N]\, \theta_0 \tag{7}$$

where h is the uniform heat transfer coefficient on the total lateral convection surface area S. This relationship excludes the effect of base contact resistance. If we let $h_c \to \infty$, then $\theta_0 \to \theta_b$.

CONDUCTION AND CONVECTION SURFACE AREAS

The conduction areas and convection surfaces areas must be defined for the spine, longitudinal and radial systems.

Spines of Arbitrary Profile

For spines of arbitrary profile the conduction area and the convection surface areas are related to the local radius:

$$r = a + (b-a)\left(1 - \frac{x}{L}\right)^{\mu} \qquad 0 \le x \le L \tag{8}$$

where a and b are the radii of the axisymmetric system at the system tip and base respectively. The profile parameter is $\mu \ge 0$. If the system profile is rectangular, then $\mu = 0$ and $a = b$, and the system is called a circular fin or a pin fin. If the system profile is conical, then $\mu = 1$, and if $a < b$, the profile is called trapezoidal. If $\mu = 1/2$ and $a = 0$, the profile is called a convex parabolic profile, and if $\mu = 2$ and $a = 0$, the profile is called a concave parabolic profile.

The local conduction area for the spines is

$$A = \pi r^2 = \pi \left[a + (b-a)\left(1 - \frac{x}{L}\right)^{\mu}\right]^2 \qquad 0 \le x \le L \tag{9}$$

The base and tip conduction areas are $A_b = A(0) = \pi b^2$ and $A_e = A(L) = \pi a^2$ for all values of μ. The relationship for the conduction area at discrete locations $x_j = jL/N$ is

$$A_j = A(x_j) = \pi \left[a + (b-a)\left(1 - \frac{j}{N}\right)^{\mu}\right]^2 \qquad 0 \le j \le N \tag{10}$$

The differential of the convection surface area is given by

$$dS = 2\pi r ds = 2\pi r \sqrt{1 + \left(\frac{dr}{dx}\right)^2}\, dx \tag{11}$$

Numerical integration is required for large values of dr/dx. If the system is slender, i.e., $dr/dx \le 0.1$, then the relationship can be integrated for arbitrary values of μ. The general relationship for the total convection area is

$$S = 2\pi b L \left(\frac{1 + \mu\epsilon}{1 + \mu}\right) \qquad \text{where} \quad 0 \le \epsilon = a/b \le 1 \tag{12}$$

The convection surface area for the jth control volume may be obtained from the following integral:

$$S_j = 2\pi b L \int_{(j-1)/N}^{j/N} \left[\epsilon + (1-\epsilon)(1-\xi)^{\mu}\right] d\xi \tag{13}$$

Evaluation of the integral yields the general relationship:

$$\frac{S_j}{2\pi b L} = \frac{\epsilon}{N} - \left(\frac{1-\epsilon}{1+\mu}\right)\left[\left(\frac{N-j}{N}\right)^{1+\mu} - \left(\frac{N-j+1}{N}\right)^{1+\mu}\right] \tag{14}$$

with $1 \le j \le N$.

Longitudinal Fins of Arbitrary Profile

For longitudinal fins of arbitrary profile the conduction and the convection surface areas are related to the local half thickness:

$$y = a + (b-a)\left(1 - \frac{x}{L}\right)^{\mu} \qquad 0 \le x \le L \tag{15}$$

where a and b are the half thicknesses at the system tip and base, respectively. The profile parameter is $\mu \ge 0$. For the rectangular profile $a = b$ and $\mu = 0$. For the trapezoidal profile $a < b$ and $\mu = 1$. For the concave parabolic profile $a < b$ and $\mu = 2$, and for the convex parabolic profile $a < b$ and $\mu = 1/2$.

The local conduction area for longitudinal fins of width w is

$$A = 2wy = 2w\left[a + (b-a)\left(1 - \frac{x}{L}\right)^{\mu}\right] \qquad 0 \le x \le L \tag{16}$$

The base and tip conduction areas are $A_b = A(0) = 2wb$ and $A_e = A(L) = 2wa$, respectively, for all values of μ. The relationship for the conduction area at discrete locations $x_j = jL/N$ is

$$A_j = A(x_j) = 2w\left[a + (b-a)\left(1 - \frac{j}{N}\right)^{\mu}\right] \qquad 0 \le j \le N \tag{17}$$

The differential of the convection surface area is given by

$$dS = 2wds = 2w\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \tag{18}$$

The total convection surface area is given by

$$S = 2wL\int_0^1 \sqrt{1 + \mu^2\beta^2(1-\xi)^{2\mu-2}}\,d\xi \tag{19}$$

where $\beta = (b-a)/L$ and $\xi = x/L$. The dimensionless convection surface area depends on μ and β:

$$S^{\star} = \frac{S}{2wL} = f(\mu, \beta) \tag{20}$$

For the rectangular profile, $\mu = 0$, $S^{\star} = 1$ and for the trapezoidal profile, $\mu = 1$, $S^{\star} = \sqrt{1+\beta^2}$. The dimensionless convection surface area for the concave parabolic profile, $\mu = 1/2$, is

$$S^{\star} = \frac{1}{2}\sqrt{4+\beta^2} + \frac{\beta^2}{4}\sinh^{-1}\left(\frac{2}{\beta}\right) \tag{21}$$

and for the convex parabolic profile, $\mu = 2$, it is

$$S^{\star} = \frac{1}{2}\sqrt{1+4\beta^2} + \frac{1}{4\beta}\sinh^{-1}(2\beta) \tag{22}$$

If the longitudinal fins are slender, i.e., the slenderness parameter is

$$\mu\beta \le 0.1 \tag{23}$$

then the total convection surface area is approximately

$$S = 2wL \tag{24}$$

The convection surface area for the control volumes is given by the following integral:

$$S_j = 2wL\int_{(j-1)/N}^{j/N}\left[1 + \mu^2\left(\frac{b-a}{L}\right)^2\xi^{2\mu-2}\right]^{1/2} d\xi \qquad 1 \le j \le N \tag{25}$$

For the general case, numerical integration is required to calculate values of S_j. For the trapezoidal profile, $\mu = 1$ and $a < b$, the integral gives

$$S_j = \frac{2wL}{N}\sqrt{1 + \left(\frac{b-a}{L}\right)^2} \qquad 1 \le j \le N \tag{26}$$

For slender longitudinal fins, $(b-a)/L \le 0.1$, the convection surface area of all control volumes is

$$S_j = \frac{2wL}{N} \tag{27}$$

Radial Fins of Arbitrary Profile

The radial fin of arbitrary profile has inner and outer radii r_i and r_o, respectively. The fin half thickness at the fin tip is a and the half thickness at the fin base is b. The local half thickness is given by the following general relationship:

$$y = a + (b-a)\left(\frac{\xi}{L}\right)^{\mu} \qquad 0 \le \xi \le L \tag{28}$$

where the fin length is defined as $L = r_o - r_i$ and the fin profile parameter is μ. The origin of the local cartesian coordinate x is located in the fin base and its related to the circular coordinate as $r + x = r_o$. If the fin profile is rectangular, then $\mu = 0$ and $a = b$. If the fin profile is trapezoidal, $\mu = 1$ and $a < b$. The triangular profile has $\mu = 1$ and $a = 0$. Many different fin profiles may be modeled by appropriate selection of the values of a, b and μ.

The local half thickness may be expressed in circular coordinates as

$$y = a + (b - a)\left(\frac{r_o - r}{r_o - r_i}\right)^{\mu} \qquad r_i \le r \le r_o \tag{29}$$

The local conduction area is given by

$$A = 4\pi r\left[a + (b - a)\left(\frac{r_o - r}{r_o - r_i}\right)^{\mu}\right] \qquad r_i \le r \le r_o \tag{30}$$

The conduction areas at the fin base and tip are $A_b = A(r_i) = 4\pi r_i b$ and $A_e = A(r_o) = 4\pi r_o a$, respectively. The conduction areas which are located at $r = r_i + jL/N$ are given by

$$A_j = 4\pi\left(r_i + j\frac{L}{N}\right)\left[a + (b - a)\left(\frac{r_o - r_i - jL/N}{r_o - r_i}\right)^{\mu}\right] \tag{31}$$

where $0 \le j \le N$. The base and tip conduction areas are $A_0 = 4\pi r_i b$ and $A_N = 4\pi r_o a$, respectively.

The differential of the convection surface area is

$$dS = 4\pi(r_o - x)\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx \tag{32}$$

Thus, the general relationship for the total surface area may be expressed as

$$S = 4\pi L\int_0^1 (r_o - L\xi)\left[1 + \mu^2\beta^2(1 - \xi)^{2\mu - 2}\right]^{1/2} d\xi \tag{33}$$

with $\xi = x/L$ and $\beta = (b - a)/L$. The integral yields closed form relationships for the rectangular profile, $\mu = 0$ and $a = b$, the trapezoidal profile, $\mu = 1$ and $a < b$, which reduces to the triangular profile, $\mu = 1$, and $a = 0$. For the concave profile, $\mu = 1/2$, and the convex profile, $\mu = 2$, the relationships are very complex, and therefore, numerical integration is recommended. The relationship for the rectangular profile is

$$S_{rect} = 2\pi\left(r_o^2 - r_i^2\right) \tag{34}$$

The relationship for the trapezoidal profile is

$$S_{trap} = 2\pi\left(r_o^2 - r_i^2\right)\left[1 + \left(\frac{b - a}{r_o - r_i}\right)^2\right]^{1/2} \tag{35}$$

which shows its relationship to the rectangular profile.

For slender radial fins where $|dy/dx| < 0.1$, the total convection surface area for the trapezoidal, concave parabolic and convex parabolic profiles may be approximated by the total surface area for the rectangular profile. In general, the slenderness parameter is

$$\mu\beta \le 0.1 \tag{36}$$

The general relationship for the surface area of the *jth* control volume is

$$S_j = 4\pi L\int_{(j-1)/N}^{j/N} (r_o - L\xi)\left[1 + \mu^2\beta^2(1 - \xi)^{2\mu - 2}\right]^{1/2} d\xi \tag{37}$$

For the rectangular profile we have

$$S_{rect,j,} = 4\pi\,(r_o - r_i)\left\{\frac{r_0}{N} - \frac{(r_0 - r_i)}{2}\left[\left(\frac{j}{N}\right)^2 - \left(\frac{j-1}{N}\right)^2\right]\right\} \tag{38}$$

For the trapezoidal profile we have

$$S_{trap,j} = 4\pi\,(r_o - r_i)\sqrt{1+\beta^2}\left\{\frac{r_0}{N} - \frac{(r_0 - r_i)}{2}\left[\left(\frac{j}{N}\right)^2 - \left(\frac{j-1}{N}\right)^2\right]\right\} \tag{39}$$

with $\beta = (b-a)/L$.

IMPLEMENTATION OF HEAT BALANCE METHOD

The heat balance method will be used to calculate the heat dissipation and the fin efficiency of spines, longitudinal and radial fins[1,3]. In all cases the base contact is perfect and the fin tip is adiabatic. Calculations will be done with $N = 3, 5, 10, 20$ control volumes to show the rapid convergence to the exact values.

Spines

Spines of rectangular, $\mu = 0$, conical, $\mu = 1$, concave parabolic, $\mu = 2$, and convex parabolic, $\mu = 1/2$, profiles are examined. The geometric and thermophysical parameters for the spines are given in Table 1.

The fin Biot number is

$$Bi = \frac{hb}{k} = 0.001840$$

which is much smaller than the critical value of 0.1. The fin parameter values for the analytical solutions are

$$m = \sqrt{\frac{2h}{kb}} = 13.1876\,m^{-1} \quad \text{and} \quad mL = 1.31876$$

Table 1 Values of Parameters for Spines[3]

$b = 4.60\ mm$	$h = 40\ W/m^2 \cdot K$
$L = 100\ mm$	$T_b = 100°C$
$k = 100\ W/m \cdot K$	$T_f = 25°C$

Kraus et al[3]. reported theoretical values for the four spines which are listed in Table 2.

Table 2 Fin Efficiencies and Heat Transfer Rates[3]

Profile of Spine	η	$Q_{fin}(W)$
Rectangular ($\mu = 0$)	0.657	5.70
Conical ($\mu = 1$)	0.796	3.45
Concave parabolic ($\mu = 2$)	0.858	2.48
Convex parabolic ($\mu = 1/2$)	0.744	4.30

To approximate the adiabatic tip and the perfect base contact, the following values were used: $h_e = 10^{-20}\,W/m^2 \cdot K$ and $h_c = 10^{20}\,W/m^2 \cdot K$. Computer algebra systems were used to implement the heat balance method for the four types of spines. The calculated values are listed in Table 3. The numerical values of η and Q_{fin} converge rapidly to the exact values for $N \geq 5$.

Table 3 Convergence of HBM

$\mu = 0$		$Q_{ideal} = 8.671\ W$
N	η	Q_{fin}
3	0.6398	5.547
5	0.6507	5.642
10	0.6554	5.683
20	0.6566	5.694
exact	0.657	5.70

$\mu = 1$		$Q_{ideal} = 4.335\ W$
N	η	Q_{fin}
3	0.7677	3.328
5	0.7862	3.408
10	0.7939	3.442
20	0.7958	3.450
exact	0.796	3.45

$\mu = 1/2$		$Q_{ideal} = 5.781\ W$
N	η	Q_{fin}
3	0.7200	4.162
5	0.7351	4.249
10	0.7415	4.286
20	0.7431	4.296
exact	0.744	4.30

$\mu = 2$		$Q_{ideal} = 2.890\ W$
N	η	Q_{fin}
3	0.8212	2.373
5	0.8452	2.443
10	0.8548	2.471
20	0.8571	2.477
exact	0.858	2.48

If $N = 10$, the differences between the exact and approximate values for η for all spines fall in the range -0.25% to -0.37%. The heat balance method is easy to implement, and it is rapid and accurate.

As another example of the implementation of the HBM we will obtain the set of general heat balance equations for a circular spine where $\mu = 0$ and $a = b$ if there is base contact resistance characterized by $h_c \neq \infty$, with tip cooling $h_e \neq 0$, and variable convection coefficient $h(x)$. The average value of $h(x)$ for the jth control volume is denoted as h_j. We choose the number of control volumes to be $N = 5$. For $N = 5$, the heat balances on the finite length control volumes yield the following general equation set for the excess temperature nodes.

$$\left\{ \begin{array}{rcll} h_c A_0 \left(\theta_b - \theta_0\right) - \dfrac{10kA_0}{L}\left(\theta_0 - \theta_1\right) & = & 0 & (1) \\ \dfrac{10kA_0}{L}\left(\theta_0 - \theta_1\right) - \dfrac{5kA_1}{L}\left(\theta_1 - \theta_2\right) - h_1 S_1 \theta_1 & = & 0 & (2) \\ \dfrac{5kA_1}{L}\left(\theta_1 - \theta_2\right) - \dfrac{5kA_2}{L}\left(\theta_2 - \theta_3\right) - h_2 S_2 \theta_2 & = & 0 & (3) \\ \dfrac{5kA_2}{L}\left(\theta_2 - \theta_3\right) - \dfrac{5kA_3}{L}\left(\theta_3 - \theta_4\right) - h_3 S_3 \theta_3 & = & 0 & (4) \\ \dfrac{5kA_3}{L}\left(\theta_3 - \theta_4\right) - \dfrac{5kA_4}{L}\left(\theta_4 - \theta_5\right) - h_4 S_4 \theta_4 & = & 0 & (5) \\ \dfrac{5kA_4}{L}\left(\theta_4 - \theta_5\right) - \dfrac{10kA_5}{L}\left(\theta_5 - \theta_6\right) - h_5 S_5 \theta_5 & = & 0 & (6) \\ \dfrac{10kA_5}{L}\left(\theta_5 - \theta_6\right) - h_e A_5 \theta_6 & = & 0 & (7) \end{array} \right\} \quad (40)$$

For the fin parameter values given in Table 4 obtain the heat balance equations, and calculate the excess temperatures θ_j, calculate the fin heat transfer rate, and calculate the fin resistance.

Table 4 Values of Parameters of Circular Fin

$a = 3\ mm$	$h_c = 50000\ W/m^2 \cdot K$
$L = 40\ mm$	$h_e = 75\ W/m^2 \cdot K$
$k = 180\ W/m \cdot K$	$\theta_b = 100\ K$
$h = 55\ W/m^2 \cdot K$	

For a circular fin of constant cross section, $A_j = \pi a^2$ for $0 \le j \le N$, and $S_j = 2\pi a L/N$ for $1 \le j \le N$.

For the circular fin parameters listed in Table 4, the equation set for the excess temperature nodes $\theta_0 \ldots \theta_6$ for constant conduction area A and perimeter P, with uniform heat transfer coefficients h, h_e, and contact conductance h_c are

$$\left\{\begin{array}{lcl}
\frac{9}{20}\pi\,(100-\theta_0) - \frac{81}{200}\pi\,(\theta_0-\theta_1) & = & 0 \\
\frac{81}{200}\pi\,(\theta_0-\theta_1) - \frac{81}{400}\pi\,(\theta_1-\theta_2) - \frac{33}{12500}\pi\,\theta_1 & = & 0 \\
\frac{81}{400}\pi\,(\theta_1-\theta_2) - \frac{81}{400}\pi\,(\theta_2-\theta_3) - \frac{33}{12500}\pi\,\theta_2 & = & 0 \\
\frac{81}{400}\pi\,(\theta_2-\theta_3) - \frac{81}{400}\pi\,(\theta_3-\theta_4) - \frac{33}{12500}\pi\,\theta_3 & = & 0 \\
\frac{81}{400}\pi\,(\theta_3-\theta_4) - \frac{81}{400}\pi\,(\theta_4-\theta_5) - \frac{33}{12500}\pi\,\theta_4 & = & 0 \\
\frac{81}{400}\pi\,(\theta_4-\theta_5) - \frac{81}{200}\pi\,(\theta_5-\theta_6) - \frac{33}{12500}\pi\,\theta_5 & = & 0 \\
\frac{81}{200}\pi\,(\theta_5-\theta_6) - \frac{27}{40000}\pi\,\theta_6 & = & 0
\end{array}\right\} \tag{41}$$

The factors that appear in the equation set are exact values. The calculated excess temperatures are listed in Table 5.

Table 5 Numerical (Theoretical) Values of Excess Temperatures K

$\theta_b = 100.000\ (100.000)$	$\theta_3 = 86.182\ (86.396)$
$\theta_0 = 97.314\ (97.371)$	$\theta_4 = 83.740\ (84.082)$
$\theta_1 = 94.487\ (94.448)$	$\theta_5 = 82.391\ (82.864)$
$\theta_2 = 89.749\ (89.837)$	$\theta_6 = 82.117\ (82.726)$

The numerical and analytical values of the fin heat transfer rate and fin resistance are listed in Table 6 for 5 control volumes.

Table 6 Theoretical and Numerical Values of Fin Dissipation and Resistance for $N = 5$

	Theoretical	Numerical
$Q_{fin},\ W$	3.797	3.717
$R_{fin},\ K/W$	26.338	26.901

The numerical values for the fin heat transfer rate and the fin resistance are in very good agreement with the theoretical values.

Longitudinal Fin of Triangular Profile

In order to illustrate the efficacy of the heat balance method it will be used to find the approximate relationships for the triangular fin with the parameter values listed in Table 7. The contact at the base of the fin is perfect.

Table 7 Values of Triangular Fin Parameters

$b = 16\,mm$	$h = 100\,W/m^2 \cdot K$
$L = 80\,mm$	$T_0 = 115°\text{C}$
$w = 1\,m$	$T_f = 15°\text{C}$
$k = 25\,W/m \cdot K$	

The fin parameters $Q_{ideal}, Q_{fin}, R_{fin}$, and η will be calculated for $N = 3, 5, 10$ and 20 to show the rapid convergence to the theoretical values. The fin parameter values are

$$\begin{aligned} Bi &= \frac{hb}{k} = 0.0640 \\ m &= \sqrt{\frac{h}{kb}} = 15.81139\,m^{-1} \\ mL &= 1.264911 \end{aligned}$$

The theoretical values are

$$\left.\begin{aligned} Q_{ideal} &= 1631.69\,W \\ \eta &= \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)} = 0.6073 \\ Q_{fin} &= \eta\, Q_{ideal} = 990.951\,W \\ R_{fin} &= \frac{\theta_0}{Q_{fin}} = 0.1009\,K/W \end{aligned}\right\} \tag{42}$$

The approximate values calculated by means of the heat balance method are listed in Table 8. The convergence of the numerical values are rapid; the differences between the values of $N = 5$ and $N = 20$ are less than 0.4%.

Table 8 Numerical Values for Triangular Fin

N	Q_{fin}	R_{fin}	η
3	974.01	0.1027	0.5969
5	980.77	0.1020	0.6011
10	983.66	0.1017	0.6028
20	984.39	0.1016	0.6033

If we choose $N = 10$, the 11 equations for the excess temperature nodes θ_j for $0 \le j \le 10$ are given in the following equation set:

$$\left\{\begin{aligned} \theta_0 &= 100 \\ 200\,\theta_0 - 291.631686\,\theta_1 + 90\,\theta_2 &= 0 \\ 90\,\theta_1 - 171.631686\,\theta_2 + 80\,\theta_3 &= 0 \\ 80\,\theta_2 - 151.631686\,\theta_3 + 70\,\theta_4 &= 0 \\ 70\,\theta_3 - 131.631686\,\theta_4 + 60\,\theta_5 &= 0 \\ 60\,\theta_4 - 111.631686\,\theta_5 + 50\,\theta_6 &= 0 \\ 50\,\theta_5 - 91.631686\,\theta_6 + 40\,\theta_7 &= 0 \\ 40\,\theta_6 - 71.631686\,\theta_7 + 30\,\theta_8 &= 0 \\ 30\,\theta_7 - 51.631686\,\theta_8 + 20\,\theta_9 &= 0 \\ 20\,\theta_8 - 31.631686\,\theta_9 + 10\,\theta_{10} &= 0 \\ 10\,\theta_9 - 11.631686\,\theta_{10} &= 0 \end{aligned}\right\} \tag{43}$$

Table 9 Numerical and Theoretical Values of Excess Temperatures

Node	Numerical	Theoretical
θ_0	115.00	115.00
θ_1	110.08	110.22
θ_2	100.88	101.11
θ_3	92.27	92.60
θ_4	84.24	84.64
θ_5	76.75	77.20
θ_6	69.78	70.27
θ_7	63.30	63.83
θ_8	57.28	57.84
θ_9	51.71	52.28
θ_{10}	46.56	47.13

The agreement between the numerical and exact values of θ_j is very good for all excess temperature nodes as seen in Table 9.

Radial Fin of Rectangular Profile

The HBM will be used to calculate excess temperatures, fin heat transfer rate, fin resistance, and fin efficiency of a radial fin of rectangular profile where $\mu = 0$. There is perfect thermal contact at the fin base and the fin tip is adiabatic.

The fin parameter values are listed in Table 10.

Table 10 Values of Parameters of Radial Fin

$r_i = 10\, mm$	$h = 120\, W/m^2 \cdot K$
$r_o = 40\, mm$	$h_c = 10^{12}\, W/m^2 \cdot K$
$2b = 2\, mm$	$h_e = 0\, W/m^2 \cdot K$
$k = 380\, W/m \cdot K$	$\theta_b = T_b - T_f = 80\, K$

Heat balances on the $N = 5$ control volumes yield the following equation set for the excess temperatures θ_j.

$$\left\{\begin{array}{lcl} 4 \times 10^{15}\, \pi\, (80 - \theta_0) - \dfrac{437\pi}{15}\, (\theta_0 - \theta_1) & = & 0 \\ \dfrac{437\, \pi}{75}\, (\theta_0 - \theta_1) - \dfrac{304\, \pi}{75}\, (\theta_1 - \theta_2) - \dfrac{117\, \pi}{3125} \theta_1 & = & 0 \\ \dfrac{304\, \pi}{75}\, (\theta_1 - \theta_2) - \dfrac{418\, \pi}{75}\, (\theta_2 - \theta_3) - \dfrac{171\, \pi}{3125} \theta_2 & = & 0 \\ \dfrac{418\, \pi}{75}\, (\theta_2 - \theta_3) - \dfrac{532\, \pi}{75}\, (\theta_3 - \theta_4) - \dfrac{9\, \pi}{125} \theta_3 & = & 0 \\ \dfrac{532\, \pi}{75}\, (\theta_3 - \theta_4) - \dfrac{646\, \pi}{75}\, (\theta_4 - \theta_5) - \dfrac{279\, \pi}{3125} \theta_4 & = & 0 \\ \dfrac{646\, \pi}{75}\, (\theta_4 - \theta_5) - \dfrac{1463\, \pi}{75}\, (\theta_5 - \theta_6) - \dfrac{333\, \pi}{3125} \theta_5 & = & 0 \\ \dfrac{1463\, \pi}{75}\, (\theta_5 - \theta_6) - \dfrac{\pi}{6.25 \times 10^{23}} \theta_6 & = & 0 \end{array}\right\} \tag{44}$$

The set of 7 equations for θ_j with $0 \leq j \leq N + 1 = 6$ are applicable to radial fins with *perfect* contact at the fin base and with an adiabatic fin tip. The contact conductance value $h_c = 10^{12}\, W/m^2 \cdot K$ was assumed to approximate perfect contact at the fin base.

Computer algebra systems were used to solve the set of equations for the values of θ_j. The numerical and analytical values of excess temperatures are listed in Table 11.

The numerical values for the excess temperature at the control volume nodes lie below the analytical values. The percent differences are below 0.7%, and the differences decrease with increasing j. The differences between the numerical and analytical values for $T_j = \theta_j + T_f$ are even smaller. The agreement is excellent.

Table 11 Numerical and Analytical Values of Excess Temperatures

θ_j	Numerical	Analytical
θ_1	75.83	75.91
θ_2	70.53	70.53
θ_3	67.37	67.35
θ_4	65.57	65.53
θ_5	64.77	64.72
θ_6	64.77	64.63

The numerical values for the excess temperatures at the control volume nodes lie below the analytical values. The percent differences are below 0.7%, and the differences decrease with increasing j. The differences between the numerical and analytical values for $T_j = \theta_j + T_f$ are even smaller. The agreement is excellent.

The ideal heat transfer rate for this radial fin is

$$Q_{ideal} = h2\pi \left(r_o^2 - r_i^2\right) \theta_0 \tag{45}$$

The fin resistance is

$$R_{fin} = \frac{\theta_0}{Q_{fin}} \tag{46}$$

and the fin efficiency is given by

$$\eta = \frac{Q_{fin}}{Q_{ideal}} \tag{47}$$

The fin heat transfer rate is given by

$$Q_{fin} = \sum_{j=1}^{N} hS_j\theta_j = \frac{kA(r_i + L/4N)}{L/2N} (\theta_0 - \theta_1) \tag{48}$$

where $L = r_o - r_i$, and S_j is given by Eq. (38).

The results of the numerical calculations are listed in Table 12.

Table 12 Comparisons of Numerical and Analytical Values

Fin Parameters	Numerical	Analytical
Q_{fin} (W)	76.360	76.338
R_{fin} (K/W)	1.0477	1.0480
η	0.8440	0.8437

Five control volumes give very accurate numerical values.

SUMMARY AND DISCUSSION

A relatively simple and direct method based on heat balances on discrete control volumes has been presented for calculating excess temperatures, heat flow rates, and total resistances of spines, and longitudinal and radial fins of arbitrary profile with contact conductance and end cooling. The approach, called the heat balance method (HBM), was outlined and general relationships were presented for the conduction areas and the convection surface areas for discrete control volumes for the spines, and longitudinal and radial fins of arbitrary profile. The implementation of the HBM was demonstrated by means of several examples which included spines, and longitudinal and radial fins. It was shown that the HBM yields a set of equations of the excess temperature nodes which are assigned to the center of each control volume and the base and tip of the fin. The set of equations is solved quickly and accurately by means of computer algebra systems such as Maple, Matlab, Mathcad, Mathematica, and spreadsheets. It was shown that very accurate values are obtained with 5 or more equal length control volumes. The HBM may be applied to fins with variable heat transfer coefficient along the lateral surfaces, and to fin profiles for which there are no known analytical solutions.

ACKNOWLEDGEMENTS

The author thanks the Natural Sciences and Engineering Research Council (NSERC) of Canada for its continual financial support. Also the assistance of Dr. Waqar Khan in the preparation of the figures and the paper is greatly appreciated.

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