ANALYTICAL MODELING OF NATURAL CONVECTION IN HORIZONTAL ANNULI

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Abstract

An analytical model is developed for natural convection in the two-dimensional region formed by an isothermal, heated horizontal cylinder concentrically located in a larger, cooled horizontal cylinder. The model is comprised of a combination of three asymptotic solutions, the diffusive limit, the laminar boundary layer limit, and the transition flow limit, and is applicable to a wide range of aspect ratios and inner and outer boundary shapes. Validation of the model is performed using numerical and experimental data from the literature for the circular annulus and a number of other geometries. The model and data are in good agreement, with an average RMS difference of 6% for the circular annulus and less than 9% for the other geometries.

Nomenclature

A	cross section area; (m^2)
d	diameter; (m)
$F(Pr)$	Prandtl number function
g	gravitational acceleration; (m/s^2)
$G_{\mathcal{L}}$	body gravity function
k	thermal conductivity; (W/mK)
k_{eff}	effective thermal conductivity; (W/mK)
$\mathcal{L}$	general characteristic length; (m)
m	mass; (kg)
n	combination parameter
$Nu_{\mathcal{L}}$	Nusselt number, $\equiv Q\mathcal{L}/(kP_i\Delta T)$
P	perimeter; (m)
Pr	Prandtl number, $\equiv \nu/\alpha$
Q	heat transfer rate per unit length; (W/m)
$Ra_{\mathcal{L}}$	Rayleigh number, $\equiv g\beta\Delta T\,\mathcal{L}^3/(\nu\alpha)$
$S^\star_{\mathcal{L}}$	dimensionless conduction shape factor
t	time; (s)
T	temperature; (^oC)
T_b	bulk fluid temperature; (^oC)
u	velocity; (m/s)

Greek Symbols

α	thermal diffusivity; (m^2/s)
β	thermal expansion coefficient; $(1/K)$
δ	gap spacing; (m)
ν	kinematic viscosity; (m^2/s)

Subscripts

bl	boundary layer
tr	transition
e	effective
i	inner boundary
o	outer boundary

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Introduction

Natural convection heat transfer in a horizontal two-dimensional annular region with uniform temperature boundary conditions on the inner and outer surfaces has been studied extensively by a number of researchers. These publications provide experimental and numerical CFD data for a wide range of different geometries, in particular the limiting case of the concentric circular annulus. The formulation of an analytical model to predict the total heat transfer rate through these annular geometries is of interest for a variety of industrial applications, including nuclear reactor design, energy conversion and storage and solar energy. The availability of easy-to-use, analytically based models that are applicable to a wide range of annulus geometries will provide the means to quickly and accurately predict operating temperature and heat transfer prior to more costly and time consuming CFD analysis or prototype testing.

The problem of interest in the current study involves natural convection in the two dimensional system consisting of an isothermal, heated, horizontal cylinder concentrically located in a larger isothermal, cooled, horizontal cylinder, for a variety of cylinder shapes. The limiting case of the horizontal circular annulus is shown in Fig. 1. Since the inner and outer boundaries are heated uniformly and do not intersect, heat transfer in the annulus occurs between the inner and outer boundaries only.

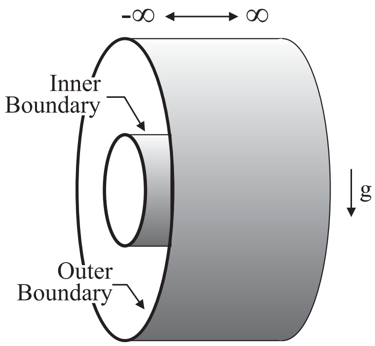


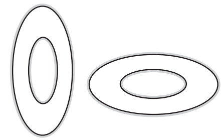
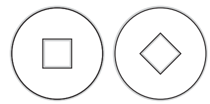
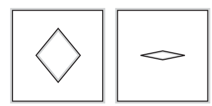
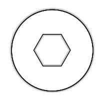
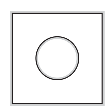
Fig. 1 Schematic of Horizontal Circular Annulus

The most widely studied horizontal annulus geometry in the literature is the concentric circular cylinders, with over 20 publications currently available that contain experimental and numerical data for the average heat transfer through the enclosed region. These include the experimental studies of Beckmann[1], Voigt and Krischer[2], Kraussold[3], Liu et al.[4], Grigull and Hauf[5], Lis[6], Koshmarov and Ivanov[7], Kuehn and Goldstein[8,9] and Collins et al.[10]. Numerical data for the concentric circular annulus are reported by Crawford and Lemlich[11], Abbott[12], Projahn et al.[13], Farouk and Guceri[14], Cho et al.[15], Prusa and Yao[16], Mahony et al.[17], Date[18], Rao et al.[19] and Yoo[20]. Teertstra et al.[21] present a comprehensive review of all experimental and numerical data, along with a comparison of available correlations and analytical models.

A number of studies for the horizontal annulus with different inner and outer boundary shapes are presented in the literature. Those that include average heat transfer, as summarized in Table 1, include the concentric elliptical cylinders of Lee and Lee[22], the square and hexagonal cylinders in a circular cylinder of Chang et al.[23] and Glakpe and Asfaw[24], the square and rhombic cylinders of Oosthuizen and Paul[25], Moukalled et al.[26] and Moukalled and Acharya[27], and the circle in a square cylinder of Moukalled and Acharya[27].

A variety of methods for predicting total heat transfer rate in horizontal annuli are currently available in the literature, ranging from correlations of experimental or numerical data to analytically-based modeling techniques. Many of the experimental and numerical studies include empirically-based correlations of the average heat transfer data; however, these correlations are limited to narrow application ranges and are developed for specific geometries, making them unsuitable for the general case. Raithby and Hollands[28] and Kuehn and Goldstein[29] present models applicable to the limiting case of the horizontal circular annulus only. Boyd[30] presents an empirical model for annuli with arbitrarily shaped boundaries; however, the complex formulation of this model requires that the boundary shape be clearly

Table 1 Review of horizontal annulus studies and data

Geometry	**Authors / Range of Independent Parameters**
	Lee & Lee[22] $P_o/P_i = 1.32$ $7 \times 10^2 \leq Ra_\delta \leq 10^4$
	Chang et al.[23] $1.96 \leq P_o/P_i \leq 3.93$ $4 \times 10^3 \leq Ra_{P_i} \leq 3.2 \times 10^6$
	Oosthuizen & Paul[25] $6 \leq P_o/P_i \leq 20$ $300 \leq Ra_{P_i} \leq 2.2 \times 10^4$
	Glakpe & Asfaw[24] $P_o/P_i = 2.09$ $100 < Ra_\delta < 10^5$
	Moukalled et al.[26] $1.25 \leq P_o/P_i \leq 1.875$ $2.4 \times 10^4 \leq Ra_{P_i} \leq 8.1 \times 10^8$ Moukalled & Acharya[27] $1.5 \leq P_o/P_i \leq 1.75$ $0 < Ra_{P_i} \leq 6.6 \times 10^9$
	Moukalled & Acharya[27] $2.12 \leq P_o/P_i \leq 6.37$ $0 < Ra_{P_i} \leq 6.7 \times 10^7$

defined, making it difficult to use for all but the simplest geometries. No models are currently available that predict average heat transfer rate in the annulus formed between differentially shaped isothermal cylinders for the full range of relative boundary size and Rayleigh number.

The objective of the current study is to develop an analytical modeling technique to predict the total heat transfer rate in the 2D annular region formed between isothermal convex inner and concave outer boundaries having similar or different shapes. The model will be comprised of a combination of three asymptotic solutions, the diffusive limit, the laminar boundary layer flow limit and the low Rayleigh number transition flow limit, and will be developed using physically based analysis techniques without empirically derived correlation coefficients. It will be applicable to the full range of Rayleigh number, from the conduction limit to the laminar boundary layer convection limit, and will be valid for a wide range of aspect ratios, including the limiting case of the external convection solution, $P_o/P_i \rightarrow \infty$.

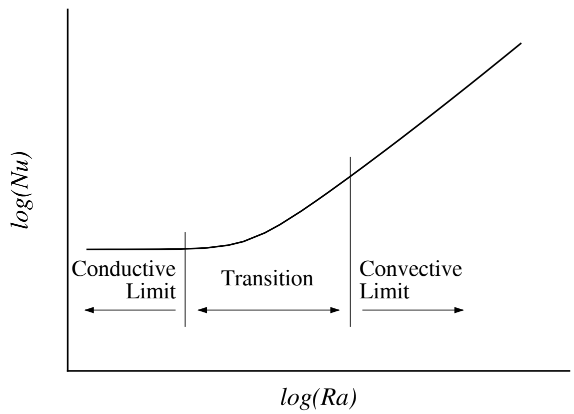


Fig. 2 Schematic of trends of total heat transfer rate

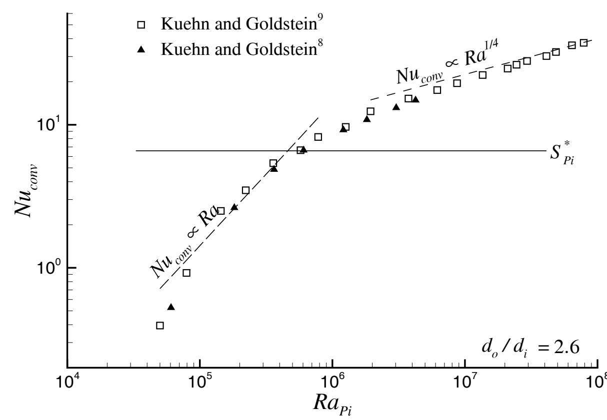


Fig. 3 Convection-only data[8,9]

Model Development

The parameter of interest in this research study is the total heat transfer rate (per unit length), Q, through the enclosure from the inner to outer boundaries. This total heat transfer rate is non-dimensionalized by the Nusselt number, defined using the overall temperature difference, $T_i - T_o$, and a arbitrary scale length, $\mathcal{L}$:

$$Nu_{\mathcal{L}} = \frac{Q\,\mathcal{L}}{k P_i\,(T_i - T_o)} \tag{1}$$

where P_i is the perimeter of the inner boundary. The Rayleigh number is defined using the same overall temperature difference and length scale:

$$Ra_{\mathcal{L}} = \frac{g\,\beta\,(T_i - T_o)\,\mathcal{L}^3}{\nu\,\alpha} \tag{2}$$

where all properties are evaluated at the bulk fluid temperature, T_b. For three-dimensional body shapes, Yovanovich[31] recommends that the square root of the active surface area be used as the characteristic length. Using $\sqrt{A}$ minimizes the differences in Nusselt number between bodies have similar shape and aspect ratio, which allows complex geometries to be approximated by simple, more easily characterized body shapes. The two-dimensional analog, the perimeter of the inner body, P_i, is selected as the characteristic length for the horizontal annulus problem.

The modeling procedure for natural convection heat transfer in the horizontal annulus is based on an analytical model developed by Teertstra[32] for three-dimensional enclosures. The basis of the model is described in Fig. 2, where the general behavior of the dimensionless heat transfer rate, Nu, as a function of Rayleigh number is shown. Three distinct regions can be identified based on critical values of the Rayleigh number: the diffusive limit, $Ra \ll Ra_{cr}$, where heat transfer is dominated by conduction, independent of Rayleigh number and equivalent to the dimensionless conduction shape factor, $Nu_{\mathcal{L}} = S^{\star}_{\mathcal{L}}$; the convective limit, $Ra \gg Ra_{cr}$, where heat transfer is dominated by convection at the inner and outer boundaries; and the transition region, typically spanning one to two decades of Ra, where values of Nu move smoothly between the limiting cases.

For natural convection from an isothermal, convex body in an infinite, quiescent region, Yovanovich[31] recommends an analytic model based on the linear superposition of the diffusive limit, corresponding to pure conduction, with a convective asymptote. Teertstra[32] shows that this behavior also holds for natural convection in a three dimensional region formed between a heated body and a cooled enclosure. The validity of the linear superposition assumption for the annulus problem is demonstrated by examining the convection-only data of Kuehn and Goldstein[8,9] determined by:

$$Nu_{\text{conv}} = Nu_{P_i} - S^{\star}_{P_i} \tag{3}$$

These convection-only data are presented in Fig. 3, where it can be seen that two clearly defined asymptotic solutions exist for the convective portion of the heat transfer. For high values of Rayleigh number, the dominant mode of heat transfer is boundary layer convection, as indicated by the asymptotic relationship $Nu \propto Ra_{P_i}^{1/4}$ on Fig. 3. As the Rayleigh number decreases, the data approaches a second asymptote, where $Nu \propto Ra_{P_i}$. These two asymptotes for the convective portion of the heat transfer are combined using the composite solution technique of Churchill and Usagi[33]

$$Nu_{P_i} = S^{\star}_{P_i} + \left(Nu_{\text{tr}}^{-n} + Nu_{\text{bl}}^{-n}\right)^{-1/n} \tag{4}$$

where Nu_{tr} and Nu_{bl} are the asymptotic solutions corresponding to transition flow and laminar boundary layer flow, respectively. The following sections will describe the development of each of the three components of the model for the horizontal annulus.

Conduction Shape Factor

Correlations and models for the dimensionless conduction shape factor, $S^{\star}_{P_i}$, for a variety of annulus geometries are available in conduction texts[34] and handbooks[35]. The use of numerical simulations to provide conduction results is another alternative to determine the conduction shape factor. An alternate approach to approximate the conduction shape factor is based on a *two rule* method involving the equivalent concentric circular annulus. The dimensionless conduction shape factor for the concentric circles is[34]

$$S^{\star}_{P_i} = \frac{2\pi}{\ln\left(d_o/d_i\right)} \tag{5}$$

The diameter of the equivalent inner circle is determined by preserving the inner boundary perimeter, such that $d_i = P_i/\pi$, while the outer circle diameter is based on preserving the area between the boundaries, A

$$\begin{aligned} A &= \frac{\pi}{4}\left({d_o}^2 - {d_i}^2\right) \\ d_o &= \sqrt{\frac{4A}{\pi} + \frac{{P_i}^2}{\pi^2}} \end{aligned} \tag{6}$$

Substituting into Eq. (5) gives an approximate model for the dimensionless conduction shape factor for the general annulus as a function of the inner perimeter and cross sectional area

$$S^{\star}_{P_i} = \frac{2\pi}{\ln\sqrt{4\pi\left(A/P_i^2\right)+1}} \tag{7}$$

Laminar Boundary Layer Convection

For high Rayleigh number, the heat transfer in the annulus is dominated by laminar boundary layer convection, and a model to predict this asymptote is developed based on the following assumptions; that the boundary layers are sufficiently thin such that they do not intersect, and that the fluid in the core region is stationary and at a uniform, bulk temperature, T_b. From these assumptions the convection in the annulus can be treated as a series combination of film resistances at the inner and outer boundaries, R_i and R_o, respectively

$$R = R_i + R_o \tag{8}$$

Relating this total resistance to the Nusselt number gives

$$Nu_{\text{bl}} = \frac{1}{k\left(R_i+R_o\right)} = \frac{1}{kR_i\left(1+R_o/R_i\right)} \tag{9}$$

Based on the definition of film resistance, a new quantity, the dimensionless bulk temperature, ϕ, can be defined

$$R_i = \frac{T_i - T_b}{Q}\ ,\ \ R_o = \frac{T_b - T_o}{Q}\ ,\ \ \frac{R_i}{R_o} = \frac{T_i - T_b}{T_b - T_o} = \phi \tag{10}$$

Substituting ϕ and $Nu_i = 1/\left(kR_i\right)$ into Eq. (9) gives

$$Nu_{\text{bl}} = \frac{Nu_i}{1+1/\phi} \tag{11}$$

The natural convection at the inner and outer boundaries, Nu_i and Nu_o, are modeled based on the method presented by Yovanovich[31] and Jafarpur[36] for convex isothermal bodies. Converting the scale length of the expression presented by these authors in terms of the perimeter P gives

$$Nu_P = F\left(Pr\right) G_P {Ra_P}^{1/4} \tag{12}$$

where the Prandtl number function is defined as[37]

$$F\left(Pr\right) = \frac{0.67}{\left[1+\left(0.5/Pr\right)^{9/16}\right]^{4/9}} \tag{13}$$

The body gravity function G_P is determined based on the general expression of Lee et al.[38] modified for two dimensional geometries

$$G_P = \left[\frac{2^{4/3}}{P}\int_0^{P/2} \sin^{1/3}\phi\, dP\right]^{3/4} \tag{14}$$

where ϕ is the angle between the gravity vector and a vector normal to the surface and $\int_0^{P/2}$ represents integration over half of the perimeter from the lower to upper stagnation points (assumes symmetry about the vertical axis.) Applied to the inner boundary, the model yields

$$\begin{aligned} Nu_i &= F\left(Pr\right) G_{P_i} {Ra_i}^{1/4} \\ Ra_i &= \frac{g\beta\left(T_i - T_b\right){P_i}^3}{\nu\alpha} \\ &= Ra_{P_i}\cdot\left(\frac{T_i - T_b}{T_i - T_o}\right) = \frac{Ra_{P_i}}{\left(1+1/\phi\right)} \end{aligned} \tag{15}$$

Substituting this relationship for Nu_i into Eq. (9) gives

$$Nu_{\text{bl}} = \frac{F\left(Pr\right) G_{P_i} {Ra_{P_i}}^{1/4}}{\left(1+1/\phi\right)^{5/4}} \tag{16}$$

The dimensionless bulk temperature ϕ is determined based on a ratio of the film resistances at the inner and outer boundaries

$$\begin{aligned} \phi &= \frac{R_i}{R_o} = \frac{Nu_o}{Nu_i} = \left(\frac{G_{P_o}}{G_{P_i}}\right)\left(\frac{T_b - T_o}{T_i - T_b}\right)^{1/4}\left(\frac{P_o}{P_i}\right)^{3/4} \\ &= \left(\frac{G_{P_o}}{G_{P_i}}\right)\left(\frac{P_o}{P_i}\right)^{3/4}\frac{1}{\phi^{1/4}} \end{aligned} \tag{17}$$

Solving for ϕ and substituting into Eq. (16) gives the final expression for the laminar boundary layer convection

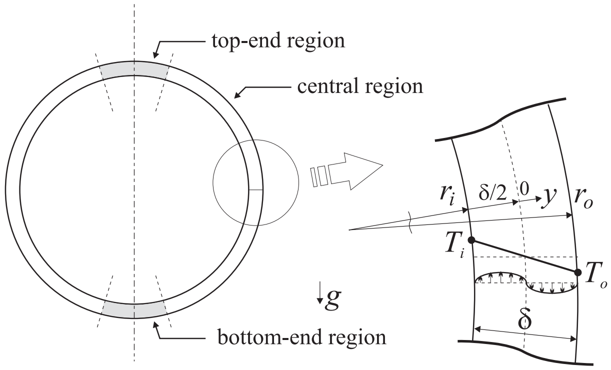


Fig. 4 Transition flow in horizontal circular annulus

asymptote for the horizontal annulus

$$Nu_{\text{bl}} = \frac{F(Pr)\, G_{P_i}\, Ra_{P_i}^{1/4}}{\left[1+\left(\frac{G_{P_i}}{G_{P_o}}\right)^{4/5}\left(\frac{P_i}{P_o}\right)^{3/5}\right]^{5/4}} \qquad (18)$$

For the limiting case of the circular annulus where $G_P = 1.028$, the ratio of the body gravity functions reduces to zero and the model expression simplifies to

$$Nu_{\text{bl}} = \frac{(1.028)F(Pr)\, Ra_{P_i}^{1/4}}{\left[1+\left(\frac{d_i}{d_o}\right)^{3/5}\right]^{5/4}} \qquad (19)$$

Transition Flow Convection

The third asymptotic component in the horizontal annulus model occurs in the transition between the conduction and laminar boundary layer convection limits. As the Rayleigh number decreases, the boundary layers on the inner and outer surfaces of the annulus grow and eventually merge. For values of *Ra* near or less than this critical value, three distinct regions form within the annulus, as shown in Fig. 4 for the example of the circular annulus; the top and bottom-end regions and the central region. The heat transfer in the central region is dominated by conduction in the radial direction, and the energy equation expressed in polar coordinates[39] for this geometry reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \qquad (20)$$

The radial temperature distribution tends to induce a small, buoyancy-driven flow in the annulus, upwards with respect to the gravity vector in the inner half of the central region and downwards in the outer half. Assuming steady state, constant properties, and no radial velocity component, the momentum equation[39] for flow tangential to the boundary in the circular annulus simplifies to the following

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r\,u)\right) = g\,\beta\,(T - T_b) \qquad (21)$$

At the limit of narrow gap spacing with respect to the inner and outer boundaries, $\delta << d_i,\ d_o$, the annulus is modeled based on the equivalent problem of flow between vertical, parallel plates, as shown in Fig. 5. The energy and momentum equations are transformed as follows

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) \rightarrow \frac{d^2T}{dy^2} = 0 \qquad (22)$$

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r\,u)\right) \rightarrow \frac{d^2u}{dy^2} = g_e\,\beta\,(T - T_b) \qquad (23)$$

Solving these coupled equations as presented by Rohsenow and Choi[40] gives an expression for the velocity distribution across in the central region

$$u = \frac{g_e\beta}{12\nu}(T_i - T_o)\left(\frac{\delta_e}{2}\right)^2\left[\left(\frac{y}{\delta_e/2}\right)^3 - \left(\frac{y}{\delta_e/2}\right)\right] \qquad (24)$$

where g_e and δ_e represent effective values from the equivalent circular annulus. As a result of the buoyancy-induced flow in the central region, natural convection heat transfer occurs in the top and bottom-end regions

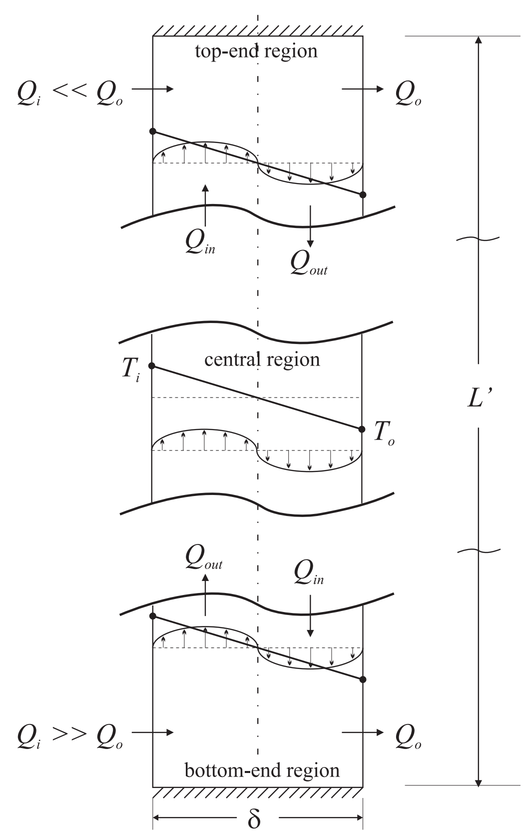


Fig. 5 Schematic of vertical cavity problem

of the vertical cavity, as described by Batchelor[41] and Eckert and Carlson[42]. As shown in Fig. 5, an enthalpy balance can be performed on each end region, relating the conduction heat transfer, Q_i and Q_o, with the heat transferred by advection to and from the central region, Q_{out} and Q_{in}. The net enthalpy flux into and out of the central region is determined based on an integration of the temperature and velocity profiles: for the inner half of the cavity:

$$Q = \int_{\delta/2}^{0} \rho \, d_p \, u \, (T - T_o) \; dy \tag{25}$$

where the temperature rise is defined with respect to the outer wall temperature, T_o. Applying this methodology to the control volume formed at the bottom of the cavity, the enthalpy flux into and out of bottom-end region can be determined

$$Q_{out} = \frac{23}{120} \frac{\rho \, c_p \, g_e \, \beta \, (T_i - T_o)^2 \, \delta_e^3}{96 \nu} \tag{26}$$

$$Q_{in} = \frac{7}{120} \frac{\rho \, c_p \, g_e \, \beta \, (T_i - T_o)^2 \, \delta_e^3}{96 \nu} \tag{27}$$

For the bottom end region, it is assumed that the boundary layer on the inner, heated surface is much thinner than that on the outer, cooled surface, such that $Q_i >> Q_o$. Therefore, applying an energy balance gives the net heat transfer into the bottom-end region

$$Q_i = Q_{out} - Q_{in} = \frac{\rho \, c_p \, g_e \, \beta \, (T_i - T_o)^2 \, \delta_e^3}{720 \nu} \tag{28}$$

Repeating the analysis for the top-end region yields the same result, leading to the conclusion that the total heat transfer rate due to convection for the transition flow asymptote is equivalent to Q_i from Eq. (28). Non-dimensionalizing Q_i using a Nusselt number based on the general scale length $\mathcal{L}$ gives:

$$Nu_{\mathcal{L}} = \frac{Q \, \mathcal{L}}{k L' \, (T_i - T_o)} = \frac{1}{720} \frac{\mathcal{L}}{L'} \frac{g_e}{g} Ra_{\delta_e} \tag{29}$$

The effective length of the vertical cavity, L', is modeled based on the arithmetic average of the inner and outer perimeters, and the effective gravitational coefficient, g_e, is calculated from an area-weighted integration over a cylindrical surface at the midpoint of the enclosed region

$$g_e = \frac{1}{A} \iint_A g \sin\theta \, dA = \frac{\pi}{4} g \tag{30}$$

Substituting these relationships and $\mathcal{L} = P_i$ into Eq. (29) and simplifying yields the final expression for the transition flow asymptote for the horizontal annulus

$$Nu_{\rm tr} = \frac{1}{90\pi} \frac{(\delta_e / P_i)^3}{(1 + P_o/P_i)} Ra_{P_i} \tag{31}$$

The effective gap spacing, δ_e, is modeled using the same equivalent circular annulus used in the conduction analysis. From Eq. (6) the effective gap spacing for any annular region can be approximated as a function of the perimeter of the inner boundary, P_i, and the enclosed area, A

$$\frac{\delta_e}{P_i} = \frac{1}{2\pi} \left[\sqrt{\frac{4\pi A}{{P_i}^2} + 1} - 1 \right] \tag{32}$$

In the case of the circular annulus, where the gap spacing reduces to $\delta = (d_o - d_i)/2$, the transition flow asymptote simplifies to the following

$$Nu_{\rm tr} = \frac{1}{720 \pi^4} \frac{(d_o/d_i - 1)^3}{(1 + d_o/d_i)} Ra_{P_i} \tag{33}$$

Model Validation

The model for natural convection in the horizontal annulus is validated using experimental and numerical average heat transfer data for ten different geometries from the literature, having both similar and different inner and outer boundary shapes. In Figs. 6 - 8, the simplified model for the circular annulus is compared with data from a number of previous studies, including Rao et al.[19], Yoo[20], Kuehn and Goldstein[8,9], Farouk and Guceri[14], Prusa and Yao[16] and Mahony et al.[17]. Substituting the simplified expressions for the diffusive, laminar boundary layer and transition flow limits into the general expression, Eq. (4), and simplifying yields

$$Nu_{P_i} = \frac{2\pi}{\ln\left(\frac{d_o}{d_i}\right)} + \left\{ \left[\frac{(1.028)\, F(Pr)\, Ra_{P_i}^{1/4}}{\left(1 + \left(\frac{d_i}{d_o}\right)^{3/5}\right)^{5/4}} \right]^{-2} + \left[\frac{1}{720\pi^4} \frac{\left(\frac{d_o}{d_i} - 1\right)^3}{\left(1 + \frac{d_o}{d_i}\right)} Ra_{P_i} \right]^{-2} \right\}^{-1/2} \tag{34}$$

where the value of the combination parameter $n = 2$ is selected that provides the best fit of the available data. The model is compared with data for four different diameter ratios: $d_o/d_i = 1.175$ and 1.2 in Fig. 6, $d_o/d_i = 2.6$ in Fig. 7 and $d_o/d_i = 5$ in Fig. 8. In each case the model is in good agreement with the data, with an average RMS difference of 6%, and all trends in the data are clearly reflected by the model, such as the critical Rayleigh number and the onset of transition and boundary layer convection.

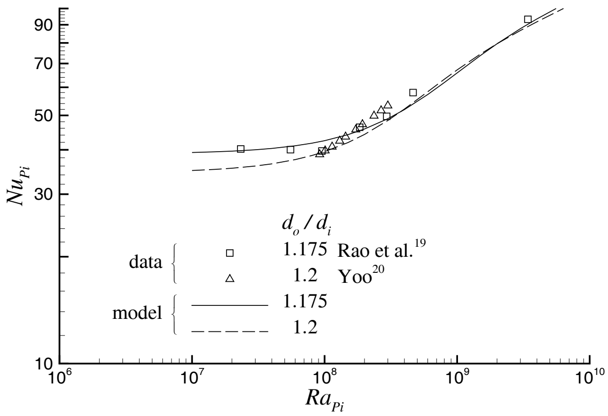


Fig. 6 Model validation: circular annulus, $d_o/d_i = 1.175, 1.2$

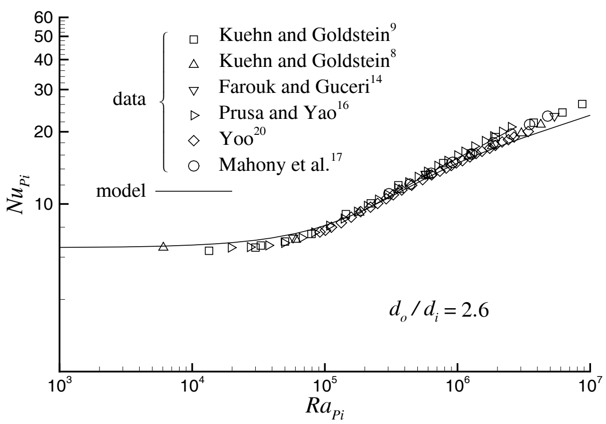


Fig. 7 Model validation: circular annulus, $d_o/d_i = 2.6$

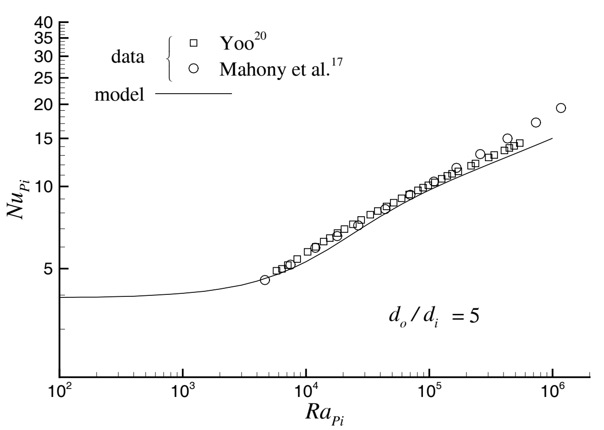


Fig. 8 Model validation: circular annulus, $d_o/d_i = 5$

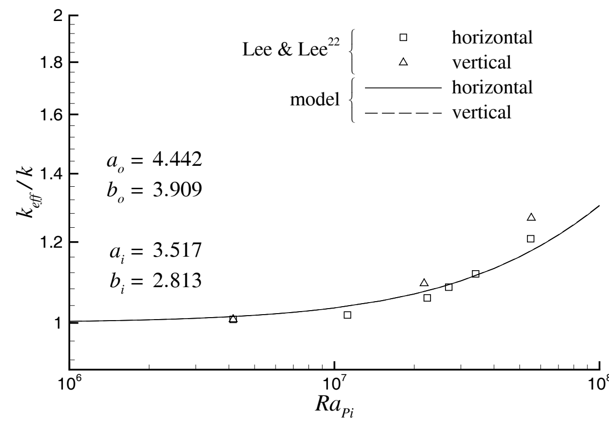


Fig. 9 Model validation: concentric elliptic cylinders

Figure 9 compares the model with numerical data of Lee and Lee[22] for the horizontally and vertically-oriented concentric elliptic cylinders, as shown in the schematic in Table 1. Average heat transfer rate is reported by these authors in terms of an effective conductivity, k_{eff}, defined as the apparent value of the thermal conductivity required for pure conduction to equal natural convection. The effective conductivity, normalized using the thermal conductivity, k, is related to the Nusselt number and conduction shape factor by:

$$\frac{k_{eff}}{k} = \frac{Nu_{P_i}}{S^{\star}_{P_i}} \tag{35}$$

When dimensionless effective conductivity is used to quantify the average heat transfer rate, the results for all geometries approach a common asymptote, $k_{eff}/k \to 1$ at the diffusive limit.

The results of the comparison of the model with numerical data of Lee and Lee[22], expressed in terms of k_{eff}/k, as well as the semi-major and semi-minor axes dimensions, a and b, are presented in Fig. 9. The model predictions are determined based on the enclosed area, A, the inner and outer perimeters, P_i and P_o, and body gravity functions for the inner and outer boundaries, G_i and G_o, and the equations derived in the previous sections. A combination parameter value of $n = 1$ was selected in this case, which provided a much better fit of the data than $n = 2$. Figure 9 shows the good agreement between the model and the data, and also demonstrates that there are no significant differences in the results due to orientation, as expected for the conduction-dominated conditions.

The model is also compared with data for the annulus formed between a square inner and circular outer cylinder of Chang et al.[23] in Fig. 10, where results for two different sizes of inner cylinder, corresponding to $P_o/P_i = 3.93$ and 1.96, and two different orientations, denoted square and diamond as shown in Table 1, are

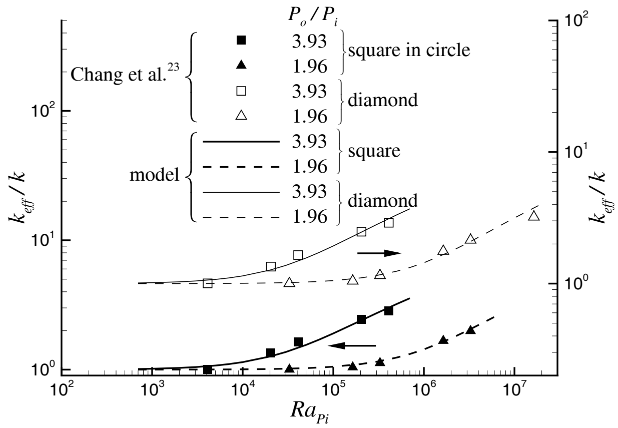


Fig. 10 Model validation: square and diamond in circle

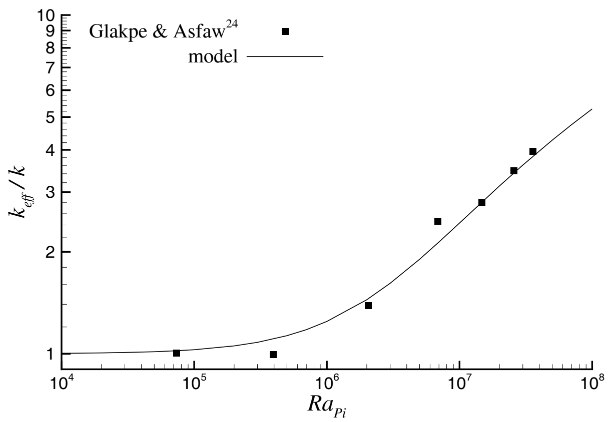


Fig. 11 Model validation: hexagon in circle

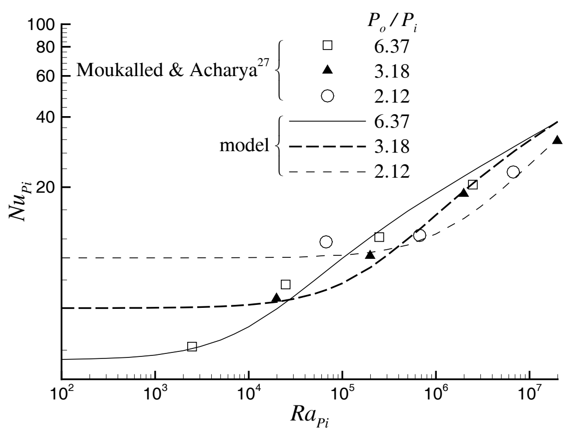


Fig. 12 Model validation: circle in square

reported. Once again, a combination parameter value of $n = 1$ was used in Eq. (4) and the model and data are in good agreement, with an RMS difference of less than 6%.

Figure 11 compares data of Glakpe and Asfaw[24] for a hexagonal shaped cylinder concentrically located in a circular cylinder, as shown Table 1, with the model using $n = 1$ for Eq. (4). Using k_{eff}/k as the independent variable, the model and the data are seen to be in close agreement.

Finally, numerical data for circular cylinders inside a square cylinder of Moukalled and Acharya[27] are compared with the model. Three different inner cylinder diameters are examined, resulting in perimeter ratio values of $P_o/P_i = 6.37$, 3.18 and 2.12, and the RMS difference between the data and the model is 9%.

Summary

An analytical model has been developed that predicts total heat transfer rate due to natural convection in the 2D, annular region formed between isothermal inner and outer boundaries having similar or different shapes. The model is based on a combination of asymptotic expressions for three limiting cases, the diffusive limit, the laminar boundary layer limit and the transition flow limit. The model is applicable to a wide range of annulus geometries, and a simplified expression is presented for the limiting case of the concentric circular annulus. Agreement between the model and existing numerical and experimental data from the literature is quite good, with an average RMS difference of approximately 6 - 9% for all cases examined in this work.

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References

[1]Beckmann, W., "Die Wärmeiübertragung in Zylindrischen Gasschichten bei Natürlicher Konvektion," *Forschung auf dem Gebiete des Ingenieurwesens,* Vol. 2, 1931, pp. 165 - 178.

[2]Voigt, H. and Krischer, D., "Die Wärmeübertragung in Zylindrischen Luftschichten bei Natürlicher Konvektion," *Forschung auf dem Gebiete des Ingenieurwesens,* Vol. 2, 1932, pp. 303 - 306.

[3]Kraussold, H., "Wärmeabgabe von Zylindrischen Flüssigkeitschichten bei Natürlicher Konvektion," *Forschung auf dem Gebiete des Ingenieurwesens,* Vol. 5, 1934, pp. 186 - 191.

[4]Liu, C., Mueller, W.K. and Landis, F., "Natural Convection Heat Transfer in Long Horizontal Cylindrical Annuli," *ASME International Developments in Heat Transfer,* Paper #117, 1961, pp. 976 - 984.

[5]Grigull, U. and Hauf, W., "Natural Convection in Horizontal Cylindrical Annuli," *Proceedings of the 3rd International Heat Transfer Conference,* Vol. 2, 1966, pp. 182 - 195.

[6]Lis, J., "Experimental Investigation of Natural Convection Heat Transfer in Simple and Obstructed Horizontal Annuli," *Proceedings of the 3rd International Heat Transfer Conference,* Vol. 2, 1966, pp. 196 - 204.

[7]Koshmarov, Y.A. and Ivanov, Y.E., "Experimental Study of Heat Transfer Through a Rarefied Gas Between Coaxial Cylinders," *Heat Transfer - Soviet Research,* Vol. 5, No. 1, 1973, pp. 29 - 36.

[8]Kuehn, T.H. and Goldstein, R.J., "An Experimental and Theoretical Study of Natural Convection in the Annulus Between Horizontal Concentric Cylinders," *Journal of Fluid Mechanics,* Vol. 74, No. 4, 1976, pp. 695 - 719.

[9]Kuehn, T.H. and Goldstein, R.J., "An Experimental Study of Natural Convection Heat Transfer in Concentric and Eccentric Horizontal Cylindrical Annuli," *Journal of Heat Transfer,* Vol. 100, 1978, pp. 635 - 640.

[10]Collins, M.W., Kaczynski, J. and Stasiek, J., "A Combined Numerical and Experimental Investigation of Natural Convection in a Horizontal Concentric Annulus," *Advanced Computational Methods in Heat Transfer,* Editors L.C. Wrobel, C.A. Brebbia and A.J. Nowak, Springer - Verlag, Berlin, Vol. 2, 1990, pp. 51 - 62.

[11]Crawford, L. and Lemlich, R., "Natural Convection in Horizontal Concentric Cylindrical Annuli," *Industrial and Engineering Chemistry Fundamentals,* Vol. 1, No. 1, 1962, pp. 260 - 264.

[12]Abbott, M.R., "A Numerical Method for Solving the Equations of Natural Convection in a Narrow Concentric Cylindrical Annulus with a Horizontal Axis," *Quarterly Journal of Mechanics and Applied Mathematics,* Vol. 17, No. 4, 1964, pp. 471 - 481.

[13]Projahn, U., Rieger, H. and Beer, H., "Numerical Analysis of Laminar Natural Convection between Concentric and Eccentric Cylinders," *Numerical Heat Transfer,* Vol. 4, 1981, pp. 131 - 146.

[14]Farouk, B. and Guceri, S.I., "Laminar and Turbulent Natural Convection in the Annulus between Horizontal Concentric Cylinders," *Journal of Heat Transfer,* Vol. 104, 1982, pp. 631 - 636.

[15]Cho, C.H., Chang, K.S. and Park, K.H., 1982, "Numerical Simulation of Natural Convection in Concentric and Eccentric Horizontal Circular Annuli," *Journal of Heat Transfer,* Vol. 104, pp. 624 - 630.

[16]Prusa, J. and Yao, L.S., "Natural Convection Heat Transfer between Eccentric Horizontal Cylinders," *Journal of Heat Transfer,* Vol. 105, 1983, pp. 108 - 116.

[17]Mahony, D.N., Kumar, R. and Bishop, E.H., "Numerical Investigation of Variable Property Effects on Laminar Natural Convection of Gases between Two Horizontal Isothermal Concentric Cylinders," *Journal of Heat Transfer,* Vol. 108, 1986, pp. 783 - 789.

[18]Date, A.W., 1986, "Numerical Predictions of Natural Convection Heat Transfer in Horizontal Annulus," *International Journal of Heat and Mass Transfer,* Vol. 29, pp. 1457 - 1464.

[19]Rao, Y-F, Miki, Y., Fukuda, K., Takata, Y. and Hasegawa, S., "Flow Patterns of Natural Convection in Horizontal Cylindrical Annuli," *International Journal of Heat and Mass Transfer,* Vol. 28, 1985, pp. 705 - 714.

[20]Yoo, J-S, "Dual Steady Solutions in Natural Convection between Horizontal Concentric Cylinders," *International Journal of Heat and Fluid Flow,* Vol. 17, 1996, pp. 587 - 593.

[21]Teertstra, P. and Yovanovich, M. M., "Comprehensive Review of Natural Convection in Horizontal Circular Annuli," 7th AIAA/ASME Joint Thermophysics and Heat Transfer Conference, Albuquerque, NM, 1998, June 15 - 18.

[22]Lee, J.H. and Lee, T.S., "Natural Convection in the Annuli between Horizontal Confocal Elliptic Cylinders," *International Journal of Heat and Mass Transfer,* Vol. 24, 1981, pp. 1739 - 1742.

[23]Chang, K.S., Won, Y.H. and Cho, C.H., "Patterns of Natural Convection around a Square Cylinder Placed Concentrically in a Horizontal Circular Cylinder," *Journal of Heat Transfer,* Vol. 105, 1983, pp. 273 - 280.

[24]Glakpe, E.K. and Asfaw, A., "Prediction of Two-Dimensional Natural Convection in Enclosures with Inner Bodies of Arbitrary Shape," *Numerical Heat Transfer,* Vol. 20, 1987, pp. 279 - 296.

[25]Oosthuizen, P.H. and Paul, J.T., "Finite Element Study of Natural Convective Heat Transfer from a Prismatic Cylinder in an Enclosure," *Numerical Methods in Heat Transfer,* ASME HTD-Vol. 62, 1986, pp. 13 - 21.

[26]Moukalled, F., Diab, H and Acharya, S., "Laminar Natural Convection in a Horizontal Rhombic Annulus," *Numerical Heat Transfer, Part A,* Vol. 24, 1993, pp. 89 - 107.

[27]Moukalled, F. and Acharya, S., "Natural Convection in the Annulus between Concentric Horizontal Circular and Square Cylinders," *Journal of Thermophysics and Heat Transfer,* Vol. 10, 1996, pp. 524 - 531.

[28]Raithby, G.D. and Hollands, K.G.T., "A General Method of Obtaining Approximate Solutions to Laminar and Turbulent Free Convection Problems," *Advances in Heat Transfer,* Editors T.F. Irvine Jr. and J.P. Hartnett, Academic Press, New York, Vol. 11, 1975, pp. 265 - 315.

[29]Kuehn, T.H. and Goldstein, R.J., "Correlating Equations for Natural Convection Heat Transfer Between Horizontal Circular Cylinders," *International Journal of Heat and Mass Transfer,* Vol. 19, 1976, pp. 1127 - 1134.

[30]Boyd, R.D., “A Unified Theory for Correlating Steady Laminar Natural Convective Heat Transfer Data for Horizontal Annuli,” *International Journal of Heat and Mass Transfer,* Vol. 24, 1981, pp. 1545 - 1548.

[31]Yovanovich, M.M., “New Nusselt and Sherwood Numbers for Arbitrary Isopotential Geometries at Near Zero Peclet and Rayleigh Numbers,” *AIAA 22nd Thermophysics Conference,* AIAA-87-1643, 1987.

[32]Teertstra, P., “Models and Experiments for Laminar Natural Convection from Heated Bodies in Enclosures,” Department of Mechanical Engineering, University of Waterloo, Waterloo, Ontario, Canada, 2003.

[33]Churchill, S.W. and Usagi, R., “A General Expression for the Correlation of Rates of Transfer and Other Phenomenon,” *American Institute of Chemical Engineers Journal,* Vol. 18, 1972, pp. 1121 - 1128.

[34]Incropera, F. P. and DeWitt, D. P., *Fundamentals of Heat and Mass Transfer,* 4th ed., Wiley, New York, 1996. pp. 96 - 99.

[35]Yovanovich, M.M., “Conduction and Thermal Contact Resistances (Conductances)”, *Handbook of Heat Transfer,* 3rd. ed., McGraw Hill, New York, Chapter 3, 1998, pp. 3.1 - 3.73.

[36] Jafarpur, K., “Analytical and Experimental Study of Laminar Free Convective Heat Transfer from Isothermal Convex Bodies of Arbitrary Shape,” Ph.D. Thesis, Department of Mechanical Engineering, University of Waterloo, 1992.

[37]Churchill, S.W. and Churchill, R.U., “A Comprehensive Correlating Equation for Heat and Component Heat Transfer by Free Convection,” *American Institute of Chemical Engineers Journal,* Vol. 21, 1975, pp. 604 - 606.

[38]Lee, S., Yovanovich, M. M. and Jafarpur, K., “Effects of Geometry and Orientation on Laminar Natural Convection Heat Transfer from Isothermal Bodies,” *Journal of Thermophysics and Heat Transfer,* Vol. 5, 1991, pp. 208 - 216.

[39]Arpaci, V.S. and Larsen, P.S., “Convective Heat Transfer,” Prentice Hall, NJ, 1984, pp. 499 - 500.

[40]Rohsenow, W.M. and Choi, H.Y., *Heat, Mass and Momentum Transfer,* Prentice-Hall, NJ, 1961, pp. 143 - 146.

[41]Batchelor, G.K., “Heat Transfer by Free Convection Across a Closed Cavity Between Vertical Boundaries at Different Temperatures,” *Quarterly of Applied Mathematics,* Vol. 12, No. 3, 1954, pp. 209 - 223.

[42]Eckert, E.R.G. and Carlson, W.O., “Natural Convection in an Air Layer Enclosed Between Two Vertical Plates with Different Temperatures” *International Journal of Heat and Mass Transfer,* Vol. 2, 1961, pp. 106 - 120.