
1 Introduction

Pin-fin heat sinks provide a large surface area for the dissipation of heat and effectively reduce the thermal resistance of the package at the cost of a higher pumping power. They often take less space and contribute less to the weight and cost of the product. For these reasons, they are widely used in applications where heat loads are substantial and/or where space is limited. They are also found to be useful in situations where the direction of the approaching flow is unknown or may change. They are usually mounted on circuit boards where significant clearances are available on the sides and at the top. Due to a higher resistance for the flow through the heat sink, the approaching cooling fluid takes a detour around the heat sink, which always results in a better hydraulic performance with lesser thermal performance.


Following Kern and Kraus [23], Sonn and Bar-Cohen [24] and Iyengar and Bar-Cohen [25,26] performed a least-material optimization of cylindrical pin-fin, plate-fin, and triangular-fin array geometries by extending the use of the least-material single fin analysis to multiple fin arrays. Bar-Cohen and Jelinek [27] developed guidelines and design equations for optimum plate-fin arrays.

It is obvious from the literature survey that all optimization studies, related to cylindrical pin-fin heat sinks, are limited to the optimization of fully shrouded heat sinks. The authors could not find any study related to the optimization of pin-fin heat sinks in the bypass flow. In this study, all relevant design parameters for pin-fin heat sinks, including clearance ratios, geometric parameters, and flow conditions, are optimized simultaneously by minimizing the dimensionless entropy generation rate \( N_s \) subject to manufacturing and design constraints.

2 Analysis

The front, side, and top views of an in-line pin-fin heat sink are shown in Fig. 1. The dimensions of the baseplate are \( W_2 \times L \times t_b \), where \( W_2 \) is the width of the heat sink, \( L \) is the length measured in the downstream direction, and \( t_b \) is the thickness of the baseplate. The dimensions of the duct are \( W \times H \), where \( W \) is the width and \( H \) is the height of the duct. The dimensions of the side bypass are \( W_1 \times H_1 \), whereas the dimensions of the top bypass are \( W \times H_2 \). The flow in the side and top bypass regions is assumed as an inviscid flow. The pin-fins can be arranged in an in-line or staggered manner.

Each pin-fins have a diameter \( D \) and a height \( H_{\text{fin}} \). The dimensionless longitudinal and transverse pitches are \( a = S_t / D \) and \( b = S_f / D \). The source of heat is applied to the bottom of the heat sink. The flow is assumed to be laminar, steady, and two dimensional. The duct velocity of the fluid is \( U_d \) and the ambient temperature is \( T_a \). There is no leakage of fluid from the top or sides. The wall temperature of the pin is \( T_w (> T_a) \) and the baseplate temperature is \( T_b \). The side and top clearance ratios are defined as

\[
CL_s = \frac{2W_1}{W_2}
\]

\[
CL_t = \frac{H_2}{H_1}
\]

Khan et al. [19] used the law of conservation of mass and energy and obtained the expressions for the average velocities in the side bypass, top bypass, and just in front of heat sink regions.

\[
U_1 = \frac{C_1U_d}{a_1C_2 + a_2C_1 + a_3C_1C_2}
\]
where the entropy generation rate in the heat sink can be obtained by following Khan [28] and Khan et al. [20–22] and applying the laws of conservation of mass and energy with the entropy balance and can be written as

$$\dot{S}_{\text{gen,hs}} = \dot{S}_{\text{gen,bp}} + \dot{S}_{\text{gen,hs}}$$  \hspace{1cm} (6)

Similarly, the entropy generation rate in the bypass region due to the fluid flow can be written as

$$\dot{S}_{\text{gen,bp}} = \frac{m\Delta P_{\text{bp}}}{\rho T_a}$$ \hspace{1cm} (7)

where $\Delta P_{\text{bp}}$ is the total pressure drop in the bypass regions and can be written as

$$\Delta P_{\text{bp}} = 2\Delta P_1 + \Delta P_1$$  \hspace{1cm} (9)

with

$$\Delta P_1 = \left(\frac{1}{2} \rho U_1^2\right) f_1 (L/D_h)$$ \hspace{1cm} (10)

$$\Delta P_2 = \left(\frac{1}{2} \rho U_2^2\right) f_2 (L/D_h)$$ \hspace{1cm} (11)

This assumes a fully-developed flow between parallel plates. Equation (11) shows that the entropy generation rate in the heat sink depends on the heat sink resistance and the pressure drop across the heat sink, provided that the heat load, mass flow rate, and ambient conditions are specified. The lumped heat sink resistance is given by

$$R_{\text{ls}} = R_m + R_{\text{fin}}$$  \hspace{1cm} (12)

where $R_m$ is the bulk material resistance given by

$$R_m = \frac{f_{bp}}{kA}$$ \hspace{1cm} (13)

and $R_{\text{fin}}$ is the overall resistance of the fins and the exposed baseplate, which can be written as

$$R_{\text{fin}} = \frac{1}{N} \frac{1}{R_c + R_{\text{fin}}} + \frac{1}{R_{\text{bp}}}$$ \hspace{1cm} (14)

where

$$R_c = \frac{1}{h_c A_c}$$ \hspace{1cm} (15)

$$R_{\text{fin}} = \frac{1}{h_{\text{fin}} A_{\text{fin}} \eta_{\text{fin}}}$$ \hspace{1cm} (16)

$$R_{\text{bp}} = \frac{1}{h_{\text{bp}} A_{\text{bp}}}$$

with

$$\eta_{\text{fin}} = \frac{\tanh(mH)}{mH}$$ \hspace{1cm} (16)

$$m = \sqrt{\frac{4 h_{\text{fin}}}{k D}}$$

Khan [28] developed the following analytical correlation for the dimensionless heat transfer coefficient for the cylindrical fin array:

3 Model Development

In the heat sink region, the entropy generation associated with heat transfer and frictional effects serves as a direct measure of the ability to transfer heat to the surrounding cooling medium. In the bypass regions, entropy generation is associated with the fluid flow only. A model that establishes a relationship between the total entropy generation rate and heat sink design parameters can be optimized in such a manner that all relevant design conditions combine to produce the best possible heat sink for the given constraints. The total entropy generation rate can be written as

$$U_2 = \frac{C_1 U_d}{a_1 C_2 + a_2 C_1 + a_3 C_1 C_2}$$ \hspace{1cm} (2)

$$U_{\text{app}} = \frac{C_1 C_2 U_d}{a_1 C_2 + a_2 C_1 + a_3 C_1 C_2}$$

where

$$C_1 = \sqrt{\frac{1 + K_1}{1 + \sigma_c K_1}} \quad \text{and} \quad C_2 = \sqrt{\frac{1 + K_2}{1 + \sigma_c K_2}}$$ \hspace{1cm} (3)

$$a_1 = \frac{A_1}{A_d}, \quad a_2 = \frac{A_2}{A_d}, \quad a_3 = \frac{A_3}{A_d}$$ \hspace{1cm} (4)

with

$$K_1 = f_1 \frac{L}{D h_1}, \quad K_2 = f_2 \frac{L}{D h_2}, \quad K_3 = f_3 (k_e + k_d + f_3 N_f)$$ \hspace{1cm} (5)
where \( C_3 \) is a constant, which depends on the longitudinal and transverse pitches, arrangement of the pins, and thermal boundary conditions. For isothermal boundary condition, it is given by

\[
C_3 = \begin{cases} 
0.2 + \exp(-0.55a) & \text{in-line arrangement} \\
0.61b(0.09a^{0.053}) & \text{staggered arrangement}
\end{cases}
\]

The heat transfer coefficient for the baseplate, \( h_{\text{base}} \), can be determined by considering it as a finite plate. Khan [28] developed the following analytical correlation for the dimensionless heat transfer coefficient for a finite plate:

\[
Nu_p = \frac{h_{\text{base}} L}{k_f} = 0.75Re^{1/2}Pr^{1/3}
\]

where \( L \) is the length of the baseplate in the streamwise direction. The mass flow rate through the pins is given by

\[
m = \rho U_{\text{app}} \nu b H_{\text{fin}} D
\]

The pressure drop associated with the flow across the pin-fins is given by

\[
\Delta P_{\text{hi}} = f_1 \rho U_{\text{max}}^2 N_S
\]

where the friction factor \( f \) depends on the Reynolds number and the array geometry, and it can be written as

\[
f_1 = \begin{cases} 
K_c[0.233 + 45.78/(b-1)1.1Re_D] & \text{in-line arrangement} \\
K_c[378.6/b^{0.131}]Re_D^{0.68/b^{0.129}} & \text{staggered arrangement}
\end{cases}
\]

where \( K_c \) is a correction factor depending on the flow geometry and the arrangement of the pins. It is given by

\[
K_c = \begin{cases} 
1.099b^{-1} & \text{in-line arrangement} \\
1.175(a/b)Re_D^{0.3124} + 0.5Re_D^{0.0807} & \text{staggered arrangement}
\end{cases}
\]

All the correlations for friction and correction factors are derived from graphs given by Žukauskas [29]. The velocity \( U_{\text{max}} \), in Eq. (21), represents the maximum average velocity seen by the array as the flow accelerates between pins and is given by

\[
U_{\text{max}} = \max \left\{ \frac{b}{b-1} U_{\text{app}}, \frac{b}{c-1} U_{\text{app}} \right\}
\]

where \( c = \sqrt{a^2 + (b/2)^2} \) is the dimensionless diagonal pitch. The dimensionless entropy generation rate can be written as

\[
N_e = \dot{\gamma} \text{gen}/(Q^2U_{\text{max}}/k_f)\nu T_c^4
\]

### 4 Optimization Procedure

The problem considered in this study is to minimize the dimensionless entropy generation rate, given by Eq. (7), for the optimal overall performance of the tube bank. If \( f(x) \) represents the dimensionless entropy generation rate that is to be minimized subject to equality constraints \( g_j(x_1, x_2, \ldots, x_n) = 0 \) and inequality constraints \( l_j(x_1, x_2, \ldots, x_n) \geq 0 \), then the complete mathematical formulation of the optimization problem may be written in the following form:

\[
\text{minimize } f(x) = N_e(x)
\]

subject to equality constraints

\[
g_j(x) = 0, \quad j = 1, 2, \ldots, m
\]

and inequality constraints

\[
l_j(x) \geq 0, \quad j = m + 1, \ldots, n
\]

The objective function can be redefined by using the Lagrangian function as follows:

\[
\mathcal{L}(x, \lambda, \chi) = f(x) + \sum_{j=1}^m \lambda_j g_j(x) - \sum_{j=m+1}^n \chi_j l_j(x)
\]

where \( \lambda_j \) and \( \chi_j \) are the Lagrange multipliers. The \( \lambda_j \) can be positive or negative but the \( \chi_j \) must be \( \geq 0 \). The necessary condition for \( x^* \) to be a local minimum of the problem, under consideration, is that the Hessian matrix of \( \mathcal{L} \) should be positive semidefinite, i.e.,

\[
v^T \nabla^2 \mathcal{L}(x, \lambda, \chi) v \geq 0
\]

For a local minimum to be a global minimum, all the eigenvalues of the Hessian matrix should be \( \geq 0 \).

A system of nonlinear equations is obtained, which can be solved using numerical methods such as a multivariable Newton–Raphson method. This method has been described in Ref. [30] and applied by Culham and Muzyczka [18] and Khan et al. [22] to study the optimization of plate or pin-fin heat sinks. In this study, the same approach is used to optimize the overall performance of a tube bank in such a manner that all relevant design conditions combine to produce the best possible tube bank for the given constraints. The optimized results are then compared for in-line and staggered arrangements.

A simple procedure was coded in MAPLE 10, a symbolic mathematics software, which solves the system of \( N \) nonlinear equations using the multivariable Newton–Raphson method. Given the Lagrangian \( L \), the solution vector \( [x] \), the initial guess \( [x_0] \), and the maximum number of iterations \( N_{\text{max}} \), the procedure systematically applies the Newton–Raphson method until the desired convergence criteria and/or the maximum number of iterations is achieved. The method is quite robust provided an adequate initial guess is made.

### 5 Results and Discussion

The parameters given in Table 1 are used as the default case to determine the thermal and hydraulic resistances and the entropy generation rate for both in-line and staggered pin-fin heat sinks in the bypass flow. The air properties are evaluated at the ambient temperature. The effect of the side and top clearance ratios on the thermal resistance and the total pressure drop is shown in Fig. 2. In each
case, the thermal resistance increases while the pressure drop decreases with the increase in the side and/or top clearance ratios, respectively. It is due to the fact that with the increase in the side or top clearance ratio, the pressure drop in the heat sink decreases, whereas thermal resistance increases since the effective resistance for the flow decreases. Note that for $CL_s=0$ and $CL_t=0$, the entire flow goes through the heat sink and the total thermal resistance of the heat sink is minimum. Thus, the thermal resistance is a direct measure of the deterioration of the thermal performance due to the presence of side and top bypass regions around the heat sink. However, with both side and top clearance ratios, the thermal performance increases further. In each case, the optimum point (where $R_{th}$ and $\Delta P$ intersect) increases with the increase in $CL_t$ but remains fixed for $CL_s$.

The effect of side and top clearances on the dimensionless entropy generation rate is shown in Fig. 3 for fixed pin diameter and volume flow rate. The optimum entropy generation rate remains constant with the increase in side clearance and the decrease in top clearance. It is due to the complex behavior of the entropy generation rate due to the increase/decrease in side/top clearance ratios.

Figure 4 shows the effects of the pin diameter and side clearance ratios on the dimensionless entropy generation rate for an in-line arrangement. It is obvious from the figure that for a fixed top clearance ratio, the dimensionless entropy generation rate decreases with the increase in side clearance ratio and the decrease in pin diameter. In the in-line arrangement, with the decrease in
pin diameter, the heat transfer surface area decreases, which allows the increase in pressure drop keeping the longitudinal or transverse pitch ratios fixed.

The effect of the pin height on the dimensionless entropy generation rate is shown in Fig. 5 for an in-line arrangement. For each pin height, the optimum entropy generation rate and the optimum Reynolds number exist, which decrease with the decrease in pin height and the increase in Reynolds number based on the pin diameter and the maximum velocity $U_{\text{max}}$ within the pin-fins. It was observed that $U_{\text{max}}$ is maximum when the top clearance ratio is zero and decreases with an increase in $C_{L_s}$.

The effect of the side clearance ratio on the thermal resistance and the total pressure drop in both in-line and staggered arrangements is shown in Fig. 6 for a fixed top clearance ratio. As expected, in both arrangements, the thermal resistance increases and the pressure drop decreases with an increase in the side clearance ratio. For a fully shrouded heat sink (i.e., $C_{L_s}=0$ and $C_{L_t}=0$), the pressure drop is maximum, whereas the thermal resistance is minimum. The pressure drop in the heat sink decreases with an increase in the side clearance ratio since the effective resistance for the flow decreases. The staggered arrangement shows a higher pressure drop in the heat sink and a lower thermal resistance for any $C_{L_s}$.

The effects of the pin height on the performance of cylindrical pin-fin heat sinks in the side bypass flow are shown in Fig. 7 for both arrangements. The optimum dimensionless entropy generation rate decreases with the decrease in pin heights for both arrangements. The in-line arrangement performs much better for all three cases. The optimum Reynolds number increases with a decrease in pin height for both arrangements. Figure 7 also shows the effects of the Reynolds number on the performance of heat sinks in the side bypass flow for both arrangements. For different pin heights, in-line arrangements give better performance for higher Reynolds numbers and smaller pin heights.

6 Conclusions

An optimal design of cylindrical pin-fin heat sinks in side and top bypass flows is obtained for both the in-line and staggered arrangements. The effects of side and top clearance ratios, pin diameter, pin height, and Reynolds numbers are examined with respect to their roles in influencing optimum design conditions and the overall performance of the pin-fin heat sink. It is demonstrated that
1. The thermal resistance increases, whereas the pressure drop decreases with the increase in the side and/or top clearance ratios.
2. The dimensionless entropy generation rate decreases with the increase in the side/top clearance ratio and a decrease in the pin diameter.
3. The optimum dimensionless entropy generation rate and Reynolds numbers decrease with the decrease in pin height.
4. The staggered arrangement shows a higher pressure drop and a lower thermal resistance for any clearance ratio.
5. The in-line arrangement shows better performance for any pin height and side/top clearance ratio.

Acknowledgment

The authors gratefully acknowledge the financial support of Natural Sciences and Engineering Research Council of Canada and the Center for Microelectronics Assembly and Packaging.

Nomenclature

- \( A_b \) = area of the baseplate = \( L \times W_2 \), m²
- \( A_f \) = frontal face area of the heat sink, m²
- \( a \) = dimensionless longitudinal pitch = \( S_L / D \)
- \( b \) = dimensionless transverse pitch = \( S_T / D \)
- \( c \) = dimensionless diagonal pitch = \( S_D / D \)
- \( C_1, C_2 \) = constants defined in Eq. (3)
- \( C_{L_s} \) = side clearance ratio = \( 2W_1 / W_2 \)
- \( C_{L_T} \) = top clearance ratio = \( H_2 / H_1 \)
- \( D \) = pin diameter, m
- \( D_h \) = hydraulic diameter, m
- \( f \) = friction factor
- \( g, I \) = equality and inequality constraints
- \( G \) = volume flow rate, m³/s
- \( H \) = height of the duct, m
- \( h \) = average heat transfer coefficient, W/m² K
- \( j \) = number of imposed constraints
- \( K_c \) = correction factor defined in Eq. (23)
- \( K_1, K_2, K_3 \) = constants defined in Eq. (5)
- \( k \) = thermal conductivity, W/m K
- \( k_e, k_r \) = contraction and expansion coefficients in the heat sink region
- \( L \) = Lagrangian function
- \( L \) = length of heat sink in the flow direction, m
- \( m \) = mass flow rate, kg/s
- \( m \) = fin performance parameter defined in Eq. (16), m⁻¹
- \( N \) = total number of pins in the heat sink = \( N_f N_L \)
- \( N_L \) = number of pins in the longitudinal direction
- \( N_T \) = number of pins in the transverse direction
- \( N_u, u \) = Nusselt number based on the pin diameter
- \( \alpha \) = dimensionless entropy generation rate = \( S_{gen} / (Q U_{max} k_f V_T^2) \)
- \( P \) = pressure, Pa
- \( Pr \) = Prandtl number = \( v / \alpha \)
- \( Q \) = total heat transfer rate, W
- \( R_c \) = contact resistance between fins and the baseplate, K/W
- \( R_{film} \) = thermal resistance of exposed (unfinned) surface of the baseplate, K/W
- \( R_{fin} \) = resistance of a fin, K/W
- \( R_m \) = material resistance of the baseplate, K/W
- \( R_{D_f} \) = Reynolds number based on the pin diameter = \( D_f^2 / \nu \)
- \( R_{D_h} \) = Reynolds number based on the hydraulic diameter = \( D_h U / \nu \)
- \( S_D \) = diagonal pitch, m
- \( S_L \) = longitudinal distance between two consecutive pins, m
- \( S_T \) = transverse distance between two consecutive pins, m
- \( T \) = temperature, K
- \( t_b \) = thickness of the baseplate, m
- \( U \) = velocity, m/s
- \( U_{app} \) = approach velocity, m/s
- \( U_{max} \) = maximum velocity in the minimum flow area, m/s
- \( W \) = width of the duct, m
- \( W_2 \) = width of the heat sink, m

Greek Symbols

- \( \Delta P \) = pressure drop, Pa
- \( \eta_{fin} \) = fin efficiency = tanh\((mH_{fin})/(mH_{app})\)
- \( \gamma \) = aspect ratio = \( H_{fin} / D \)
- \( \mu \) = absolute viscosity of fluid, kg/m s
- \( \nu \) = kinematic viscosity of fluid, m²/s
- \( \rho \) = fluid density, kg/m³

Subscripts

- \( 1 \) = side bypass
- \( 2 \) = top bypass
- \( a \) = ambient
- \( b \) = baseplate or unfinned surface of baseplate
- \( d \) = duct
- \( f \) = fluid
- \( fin \) = single fin
- \( \text{fins} \) = all fins with exposed baseplate area
- \( hs \) = heat sink
- \( m \) = bulk material
- \( T \) = thermal
- \( w \) = wall

References

H20851 11  |  H20852 11

H20851 12  |  H20852 12

H20851 13  |  H20852 13

H20851 14  |  H20852 14

H20851 15  |  H20852 15

H20851 16  |  H20852 16

H20851 17  |  H20852 17

H20851 18  |  H20852 18

H20851 19  |  H20852 19

H20851 20  |  H20852 20

H20851 21  |  H20852 21

H20851 22  |  H20852 22

H20851 23  |  H20852 23

H20851 24  |  H20852 24

H20851 25  |  H20852 25

H20851 26  |  H20852 26

H20851 27  |  H20852 27

H20851 28  |  H20852 28

H20851 29  |  H20852 29

H20851 30  |  H20852 30