Thermal Conductance of a Row of Cylinders Contacting Two Planes

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THERMAL CONDUCTANCE OF
A ROW OF CYLINDERS CONTACTING TWO PLANES

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Abstract

Analyses are presented for predicting the overall heat transfer coefficient for steady heat transfer between two smooth planes separated by a single row of a) uniformly spaced and b) nonuniformly spaced circular cylinders under vacuum conditions and negligible radiation heat transfer. The analyses incorporate the Hertzian theory as well as the thermal constriction theories of Veziroglu, valid for two-dimensional, rectangular heat channels, and of Yovanovich-Coutanceau, valid for circular cylinders, to determine the total thermal constriction resistance of a typical element. General expressions are developed relating the conductance to the thermal conductivities, modulii of elasticity, Poisson's ratios, cylinder diameter, spacing and apparent contact pressure. There is good agreement between the theory and some experimental data.

Nomenclature

a	=	half-width of contact area, Eq. (5)
Aa	=	apparent contact area
b	=	half-width of heat channel
D	=	cylinder diameter
e	=	modulus of elasticity
f	=	force per cylinder per unit length
F	=	total force on the joint
h_j	=	thermal conductance of the joint, Eq. (1)
k	=	physical property parameter, Eq. (6)
L	=	length of cylinders and apparent area
N	=	number of cylinders
P_a	=	apparent contact pressure

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P^* = dimensionless contact pressure, Eq. (15)
Q = total heat flow rate
R = thermal resistance
S = pitch of cylinders, $S = \alpha D$
T = temperature
ΔT_j = pseudo-temperature drop across joint
W = width of apparent area
α = dimensionless spacing, $\alpha = S/D$
λ = thermal conductivity
α^* = dimensionless spacing ratio, $\alpha^* = \alpha_1/\alpha_2$
θ = contact angle
ν = Poisson's ratio

Subscripts

1,2,3 = planes 1 and 2, and cylinder, respectively
a = apparent
j = joint

Introduction

Heat transfer across a joint consisting of a single row of long smooth circular cylinders in elastic contact with two smooth planes (Fig. 1) is currently of interest not only to aerospace engineers but also to cryogenic engineers. The aerospace engineer often is called upon to predict the total thermal resistance present in the available heat path from the heat source (usually some precision electromechanical instrument) to the heat sink. Quite often these precision instruments are mounted on gimbaled platforms, and, therefore, heat must be conducted across roller bearings which can be modeled as a row of solid cylinders contacting two planes. The design engineer must be able to calculate that fraction of the total resistance which is due to the bearings. Once he knows the parameters (geometric, physical, and thermal) which determine this bearing resistance, he can then make a proper engineering decision to alleviate it. On the other hand the cryogenic engineer is concerned about heat leakage into the cryogens. The path available for the heat leakage often includes stand-offs which can be modeled as Fig. 1.

A survey of the open literature shows that this problem has not been solved. It will be assumed that the cylinders have a length-to-diameter ratio of at least two, but preferably larger. Under vacuum conditions this will permit us to neglect end effects and so model the cylinders as being very long. The cylinder surface as well as the surfaces of the two planes are taken to be smooth so that the Hertzian theory can be used to predict the size of contact. This is not a severe

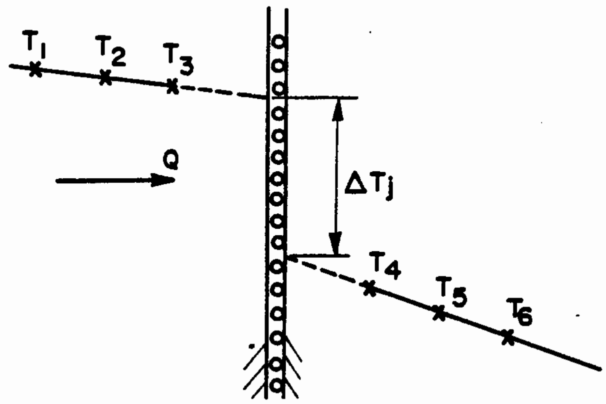

Fig. 1 Temperature field.

restriction, provided the actual surface conditions are relatively smooth and the load is relatively large.

The surfaces are assumed to be clean, free of oxides or lubricants. Neglecting oxides does not negate the results of this analysis. Provided the oxide layer does not alter the contact characteristics and its thermal properties are known, the effect of an oxide layer always can be added to the final results of this analysis. The presence of a lubricant changes completely the nature of the problem. Cylinders surrounded by a lubricant must be treated differently. This paper will be limited to the study of thermal constriction resistances only.

Statement of the Problem

Identical circular cylinders are placed between two large plane solids. The cylinders are parallel to each other, and the pitch is uniform. When the smooth plane solids make elastic contact with the cylinders, the contact areas are very narrow strips located diametrically opposite each other. When the planes have different elastic properties, the contact areas will differ in their widths. As the contact load increases, the contact areas increase in width, but the width will always remain much smaller than the diameter of the cylinders.

When the joint described above is placed in a vacuum and there is steady linear heat conduction Q in both solids (Fig.1) a pseudo-temperature drop ΔT_j will be observed across the joint. This temperature drop is a direct measure of the thermal resistance of the joint. This resistance is the result of the convergence and divergence of the heat flow lines in the neighborhood of the contact areas. For joints placed in a vacuum, and if there is negligible radiation heat transfer across the no contact regions, there is only one path available for heat transfer across the joint - that is, conduction through the contact areas.

The objective, then, is to develop a general expression for predicting the joint thermal resistance or its reciprocal, the thermal conductance, which is defined as

$$h_j = (Q/Aa)/\Delta T_j \qquad (1)$$

where (Q/Aa) is the heat flux based upon the apparent or nominal area of the joint.

The thermal problem will be treated as N typical heat channels (Fig. 2) thermally connected in parallel. Each heat channel will be subdivided into subelements: a cylinder and two rectangular heat channels (Fig. 3). Each rectangular heat channel will have a constriction resistance associated with it, and the cylinder will have two constrictive resistances because it has two different contact areas. These four constriction resistances will be thermally connected in series because all the heat entering a typical heat channel must flow through each subelement. The total resistance of a channel is, therefore, the sum of the four constriction resistances.

Total Resistance of a Typical Heat Channel

Figure 3 shows schematically the subelements used to determine the total resistance of a typical heat channel. Only one rectangular heat channel is shown because they are geometrically similar except for the contact width. Veziroglu[1] has shown that the symmetric constriction resistance of a rectangular channel of half-width b thermal conductivity λ_1, and half-width

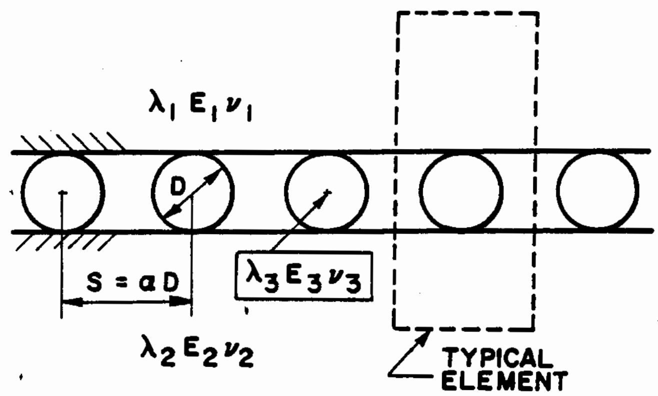

Fig. 2 Typical element for thermal analysis 1.

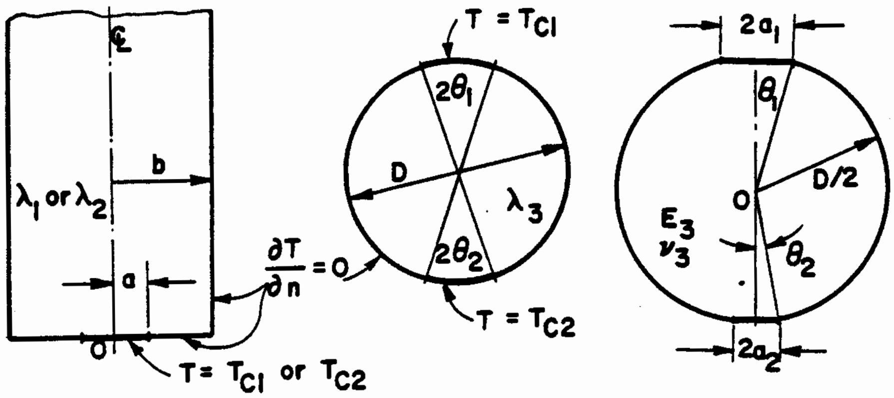

Fig. 3 Schematic of subelements.

of contact a_1, is given by

$$R_1 = (\pi\lambda_1)^{-1} \ln[1/\sin(\pi a_1/2b)] \quad (2)$$

per unit length of heat channel. A similar expression can be written for the resistance in the second rectangular channel where the half-width of contact is a_2 and the thermal conductivity is λ_2. Yovanovich and Coutanceau[2] recently determined the thermal resistance of a circular cylinder having identical sources and sinks placed diametrically opposite each other. The remainder of the boundary of the cylinder is perfectly insulated. The total resistance consists of two identical resistances: the resistance from the source to the midplane of the cylinder and the resistance from the midplane to the sink. The results of this analysis can be used to obtain the following expression:

$$R_3 = (\pi\lambda_3)^{-1} \ln[2(1 + \cos\theta_1)/\sin\theta_1] + (\pi\lambda_3)^{-1} \ln[2(1 + \cos\theta_2)/\sin\theta_2] \quad (3)$$

for the total constriction resistance across the cylinder when the heat enters through the contact area subtending an angle θ_1 at the center of the cylinder and leaves through a second contact area subtending an angle θ_2 (Fig. 3). The remainder of the cylinder boundary is impervious to heat flow.

The total constriction resistance of a typical heat channel is, therefore,

$$R = R_1 + R_2 + R_3 \quad (4)$$

Elastic Contact Between Cylinder and Plane

Elasticity theory[3] shows that the half-width of the contact area between the plane solid 1 and the cylinder is given by

$$a_1 = \sqrt{2fD(k_1 + k_3)} \qquad (5)$$

k_1 and k_3 are physical parameters given by

$$k_i = (1 - \nu_i^2)/\pi E_i \quad (i = 1,2,3) \qquad (6)$$

A similar expression can be written for the half-width of the contact area between the cylinder and plane solid 2.

Evaluation of Geometric Parameters

Consider the geometric parameter (a/b) that appears in Eq. (2). This parameter can be related to the pitch $S = \alpha D$ and the cylinder diameter in the following manner:

$$a_1/b = a_1/(S/2) - 2a_1/\alpha D \qquad (7)$$

By means of Eq. (6) the ratio in Eq. (7) becomes

$$a_1/b = (2/\alpha)\sqrt{2f(k_1 + k_3)/D} \qquad (8)$$

The geometric parameter (a/b) is always much less than unity - that is, the half-width of the contact areas are always much smaller than the half-width of the heat channels even when $\alpha = 1$. Therefore, for small values of (a/b) $\sin(\pi a/2b)$ can be approximated by $(\pi a/2b)$ with very little error. This result will be used in the next section.

Since the half-width of the contact areas are much smaller than the cylinder diameters (Fig. 3) the subtended angles are very small and therefore the geometric parameter appearing in Eq. (3) can be approximated in the following manner:

$$(1 + \cos\theta_1)/\sin\theta_1 \doteq (D/a_1)[1 - (a_1/D)^2/2] = D/a_1 \qquad (9)$$

Total Joint Resistance and Conductance

The results of the preceding section can be used to evaluate the total joint resistance in terms of the physical and geometric properties of the joint. The total constriction resistance of a typical element, Eq. (4), can now be written as

$$R = (\pi\lambda_1)^{-1}\ln[\alpha/\pi\sqrt{2f(k_1+k_3)/D}] + (\pi\lambda_2)^{-1}\ln[\alpha/\pi\sqrt{2f(k_2+k_3)/D}] +$$

$$(\pi\lambda_3)^{-1}\ln[2/\sqrt{2f(k_1+k_3)/D}]+(\pi\lambda_3)^{-1}\ln[2\sqrt{2f(k_2+k_3)/D}] \quad (10)$$

per unit length of cylinder. Since the typical heat channels are thermally connected in parallel, the total joint resistance is $R_j = R/LN$ and the number of cylinders forming the joint is $N = W/S = W/\alpha D$. It can be shown that the following relationship exists between the joint resistance and the typical heat channel resistance:

$$R_j = \alpha DR/Aa \quad (11)$$

Since, $Q = h_j\ Aa\ \Delta T_j = \Delta T_j/R_j$, therefore,

$$\alpha D\ h_j = L/R \quad (12)$$

where R is given by Eq. (10).

Equation (10) is not in its simplest form. It can be reduced to a more convenient form if we note that the force on each cylinder per unit length can be written as

$$f = P_a\ Aa/NL = \alpha D\ P_a \quad (13)$$

Therefore, we can replace $2f/D$ by $2\alpha\ P_a$ and write Eq. (10) as

$$R = (2\pi)^{-1}\ [1/\lambda_1 + 1/\lambda_2 - 2/\lambda_3][\ln(\alpha/2)] - \pi^{-1}\ [1/\lambda_1 + 1/\lambda_2]\ln\pi + (2\pi)^{-1}[1/\lambda_1 + 1/\lambda_3]\ln(1/P^*_{13}) + (2\pi)^{-1}\ [1/\lambda_2 + 1/\lambda_3]\ln(1/P^*_{23}) \quad (14)$$

where $P^*_{13} \equiv P_a(k_1 + k_3)$ and $P^*_{23} \equiv P_a(k_2 + k_3)$ (15)

Equation (14) clearly shows the relationship between the total constriction resistance of a typical heat channel and the physical and geometric properties of the joint. For the special case of a joint consisting of identical materials ($\nu_1 = \nu_2 = \nu_3 = \nu$; $E_1 = E_2 = E_3 = E$; $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$), Eq. (14) reduces to

$$R = (2/\pi\lambda)\ln[E/2\ P_a(1 - \nu^2)] \quad (16)$$

and the thermal conductance, according to Eq. (12), can be written in the following dimensionless form:

$$(\alpha\ D\ h_j/\lambda) = (\pi/2)/\ln[E/2\ P_a(1 - \nu^2)] \quad (17)$$

Effect of Eccentricity

An occasion may arise where the cylinders are not placed upon a uniform pitch (Fig. 4). The pitch between adjacent cylinders may differ, but the two pitches will be repeated alternately. For this arrangement a typical heat channel can be taken as shown in Fig. 4. It will be noted that this heat channel exists for every cylinder. Furthermore, it will be

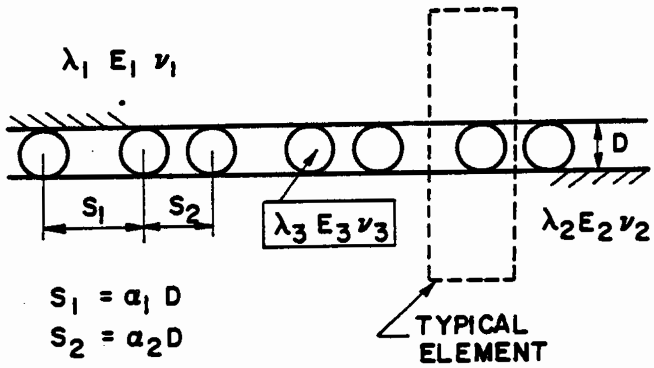

Fig. 4 Typical element for thermal analysis II.

noted that the center of the cylinder does not coincide with the centerline of the rectangular heat channel. The cylinder is displaced a distance e from the centerline of the heat channel (Fig. 5). The eccentricity will augment the constriction resistance within the rectangular heat channels but will not influence the constriction resistance within the cylinders. This effect of eccentricity has been considered by Veziroglu[1]. He found that the expression for the symmetric constriction, Eq. (2), should be altered to the following form to take into consideration the displacement from the symmetric position:

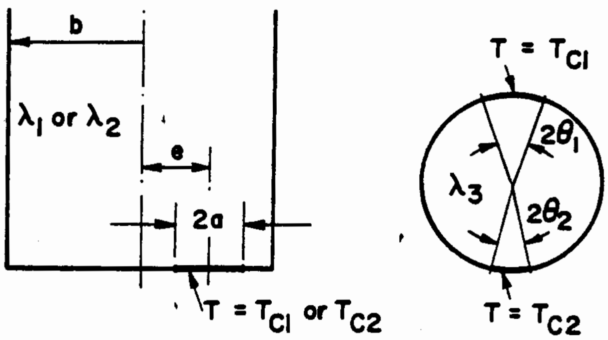

Fig. 5 Schematic of subelements.

$$R_1 = (\pi\lambda_1)^{-1} \ln[1/\cos(\pi e/2b) \sin(\pi a_1/2b)] \tag{18}$$

It can be easily proven by means of Fig. 4 that

$$(\pi e/2b) = (\pi/2)(\alpha_1 - \alpha_2)/(\alpha_1 + \alpha_2) \tag{19}$$

where $\alpha_1 > a_2$.

Without going into all the details of this analysis, which follows closely the work done in the previous sections, it can be stated that, for the case of eccentricity, the total constriction resistance of a typical heat channel can be expressed as.

$$\begin{aligned}\pi R = & (1/\lambda_1 + 1/\lambda_2)\{\ln(\sec[(\pi/2(\alpha_1 - \alpha_2)/(\alpha_1 + \alpha_2)]/\pi)\} + \\ & 2^{-1}(1/\lambda_1 + 1/\lambda_2 - 2/\lambda_3)\ln[(\alpha_1 + \alpha_2)/4)] + \\ & 2^{-1}(1/\lambda_1 + 1/\lambda_3)\ln(1/P^*_{13}) + 2^{-1}(1/\lambda_2 + 1/\lambda_3)\ln(1/P^*_{23})\end{aligned} \tag{20}$$

Equation (20) is now the general equation for the total resistance of a typical heat channel. It can be shown that it reduces to Eq. (14) when there is no eccentricity ($\alpha_1 = \alpha_2 = \alpha$).

The thermal conductance of the joint when eccentricity is present is related to the total resistance in the following manner:

$$(\alpha_1 + \alpha_2)D\ h_j = 2/R \tag{21}$$

Equation (21) reduces to Eq. (12) when there is no eccentricity.

Another interesting special case of Eq. (20) results when the materials are identical:

$$R = (2/\pi\lambda)\ln\{\sec[(\pi/2)(\alpha_1 - \alpha_2)/(\alpha_1 + \alpha_2)]/\pi P^*\} \tag{22}$$

where, for this special case;

$$\pi P^* = 2P_a(1 - \nu^2)/E \tag{23}$$

For identical materials the joint conductance with eccentricity, $h_j(\alpha_1 > \alpha_2)$, relative to the joint conductance without eccentricity, $h_j(\alpha_1 = \alpha_2)$, can be related to the total heat channel resistances, with and without eccentricity, in the following manner:

$$h_j(\alpha_1 > a_2)/h_j(\alpha_1 = \alpha_2) = [R(\alpha_1 > \alpha_2)/R(\alpha_1 = \alpha_2)]^{-1} \tag{24}$$

because $2\alpha = \alpha_1 + \alpha_2$. The ratio of the resistance can be written as

$$R(\alpha_1 > \alpha_2)/R(\alpha_1 = \alpha_2)$$
$$= 1 + \ln\{\sec[(\pi/2)(\alpha^* - 1)/(\alpha^* + 1)]\}/\ln(1/\pi P^*) \qquad (25)$$

where $\alpha^* = \alpha_1/\alpha_2$. The maximum effect of the eccentricity for a particular contact pressure will occur when

$$(\alpha^*_{max} - 1)/(\alpha^*_{max} + 1) = (\alpha_1 + \alpha_2 - 2)/(\alpha_1 + \alpha_2) \qquad (26)$$

The effect of uneven spacing is a minimum at very light contact pressures and is a maximum at very high contact pressures where the contact area width is the largest.

Test equipment and procedure

Tests were carried out in a conventional thermal contact resistance system.[4] The tests were done in a vacuum 10^{-4} mm Hg. The two large plane solids were 1 in. x 1 in. x 1 in. stainless steel (303) having the test surface optically flat. Four copper-constantan thermocouples were inserted to the centerline into both solids. The thermocouple nearest the test surface was located 1/8 in. from the surface, and the other three were located on 1/4 in. centers. The cylinders were 1/32 in. diameter of stainless steel (303) 1 in. long, relatively smooth and clean. Heat was supplied by a nichrome resistance wire cemented into machined grooves in the heater block. The heater operated on a.c. current with variable resistance controllers. The heat input was about 10 watts, and the heat flux based upon the apparent area was about 5000 Btu/ hr ft^2 °F. The apparent contact pressure ranged from 100 to 4000 psi. The pseudo-temperature drop across the joint ranged from a low value of about 8.5°F to a high of about 50°F. The low values corresponded to the highest loads and the smallest pitch ($\alpha = 1$), while the high values corresponded to the lowest loads and the largest pitch ($\alpha = 4$). Tests were conducted at pitches corresponding to $\alpha = 1,2,$ and 4. The error in the copper-constantan thermocouples used in the tests did not exceed 3/4% with an average joint temperature ranging from 150 to 280°F. The thermocouple millivolt output was monitored through a digital recorder having negligible error.

Results and discussions

The tests results agree well with the theory, Eq. (17). The maximum discrepancies between data and theory occur at the lightest contact pressure. At this load the data for $\alpha = 1$

fall 5% below the theoretical values. As α is increased, this error decreases to about 1-2% for $\alpha = 4$. The difference between data and theory decreases with increasing contact pressure for all values of α. At the highest pressures all the data agree to within 1-2% of the theoretical values. The best agreement over the entire load range occurs when the spacing is a maximum.

It is believed that oxides or perhaps the surface roughness of the cylinders may be responsible for the observed lower values at the low loads when the spacing is a minimum.

Future work will need to compare the theory with data obtained for different materials in contact and to test for the effect of eccentricity. The theory should be extended to include the effects of oxides and surface roughness, especially at light contact pressures. An analysis incorporating the ffects of lubricants would be desirable, although very difficult to achieve.

Conclusion

A theory has been presented that yields two expressions for the thermal conductance of uniformly spaced and for nonuniformly spaced cylinders. The theory agrees with a limited amount of test data, in spite of the fact that the cylinders were neither absolutely smooth nor absolutely clean.

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