THERMAL CONSTRICTION RESISTANCE OF ARBITRARY PLANAR CONTACTS WITH CONSTANT FLUX

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Abstract

Steady-state thermal constriction resistances of singly-connected, planar contact areas of arbitrary shape with constant flux on insulated half-spaces are considered. An integral formulation and an extremely accurate, seminumerical integration procedure applicable to any shape are used to obtain numerical values. The customary definition of the constriction resistance (λaR) is compared with alternate definitions based upon the square root of the area and the average or centroidal temperature ($\lambda\sqrt{A}\,R$, $\lambda\sqrt{A}\,R_o$). The latter definition appears to be most useful because it varied by less than 5% for all shapes considered. Moreover, it is based upon easily computed quantities.

Nomenclature

A = contact area
A_o = contact area in the first quadrant
a = characteristic dimension of contat along x axis
B = beta function
b = characteristic dimension of contact along y axis
$G(\nu,v)$ = function defined in text
n = geometric parameter
Q = heat flow rate
q = heat flux.

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R = constriction resistance
$\bar{R}$ = constriction resistance based upon average temperature
R_o = constriction resistance based upon centroid temperature
T = temperature
$\bar{T}$ = average contact area temperature
T_o = centroidal temperature
u = dimensionless coordinate (= x/b)
v = dimensionless coordinate (= y/b)
x,y,z = Cartesian coordinates

Greek Symbols

α = aspect ratio (= b/a)
$\varepsilon_1, \varepsilon_2$ = incremental distances, dimensionless
Γ = gamma function
η = flux coordinate
θ = angular measure
λ = thermal conductivity
ξ = flux coordinate
μ = dimensionless coordinate (= ξ/b)
ν = dimensionless coordinate (= η/b)
ρ = radial vector in the xy plane
ρ_o = centroidal radial vector to boundary of contact area

Introduction

This paper concerns the thermal resistance of contact areas formed when either flat nominally, rough surfaces are in mechanical contact or smooth, curved surfaces are in elastic contact. In the first instance, each microcontact is assumed to be circular and usually is modeled as a contact on a half-space. The constriction resistance of a circular contact subjected to a uniform heat flux is known.[1] In the second instance, the general contact area is elliptical when, as an example, a sphere is in elastic contact with a race. It has been shown that this contact can be modeled as a contact on a half-space.[2]

The boundary-value problems just given require complex functions to describe the temperature field and the thermal resistance due to these contact areas. For example, in the circular contact solution, Bessel functions and exponentials[1] appear, whereas for elliptic contact, complete elliptic integrals of the first kind are required.[2] The square and rectangular contacts also have been studied,[3] and here trigonometric and exponential functions are required in the solution. It therefore appears that each family of shapes requires a

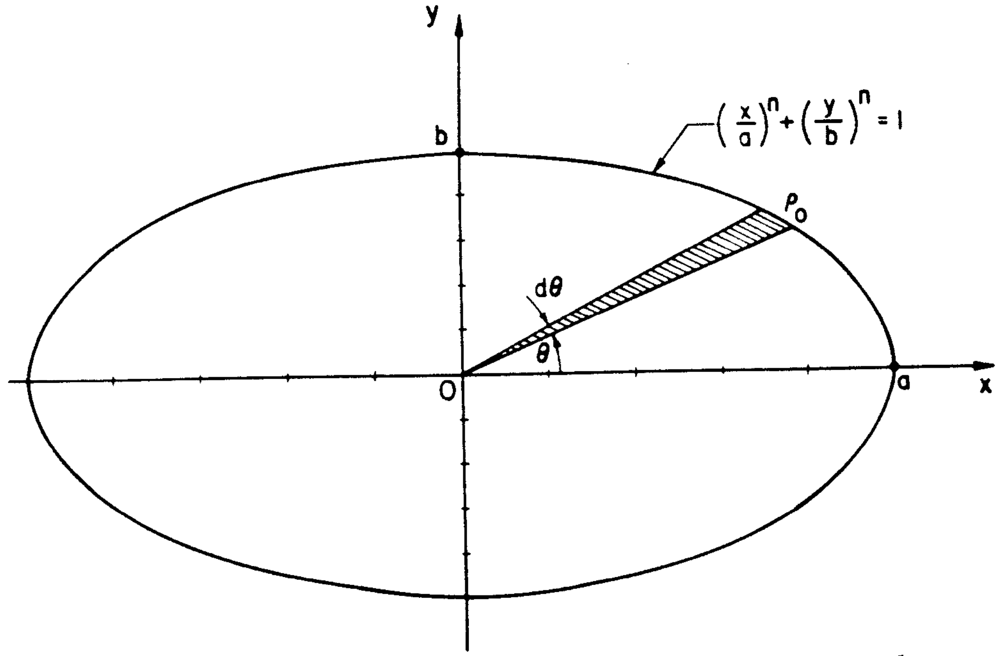

Fig. 1 Family of contact regions characterized by geometric parameter n.

unique and different function for its closed-form solution; moreover, the resulting expression for the constriction resistance is different and seems to be unrelated. For contact areas different from these shapes, closed-form solutions are not available. It is not known whether other shapes will yield constriction resistances different from or similar to those shapes studied to date.

It is the objective of this paper to examine a special class of arbitrary, singly connected planar contact areas situated on half-spaces. This class of contacts will contain the circular, elliptic, square, and rectangle as special cases. The constant-flux condition will be assumed, and the integral method will be employed to determine the temperature field and the constriction resistance.

Problem Statement

Consider one, singly connected, planar contact area on an insulated half-space ($z \geq 0$) whose thermal conductivity is λ. The contact area is subjected to a uniform heat flux q. Its contour in the first quadrant is described by the following equation:

$$(x/a)^n + (y/b)^n = 1 \qquad (1)$$

where a and b are the characteristic dimensions along the x and y axes, respectively (Fig. 1).

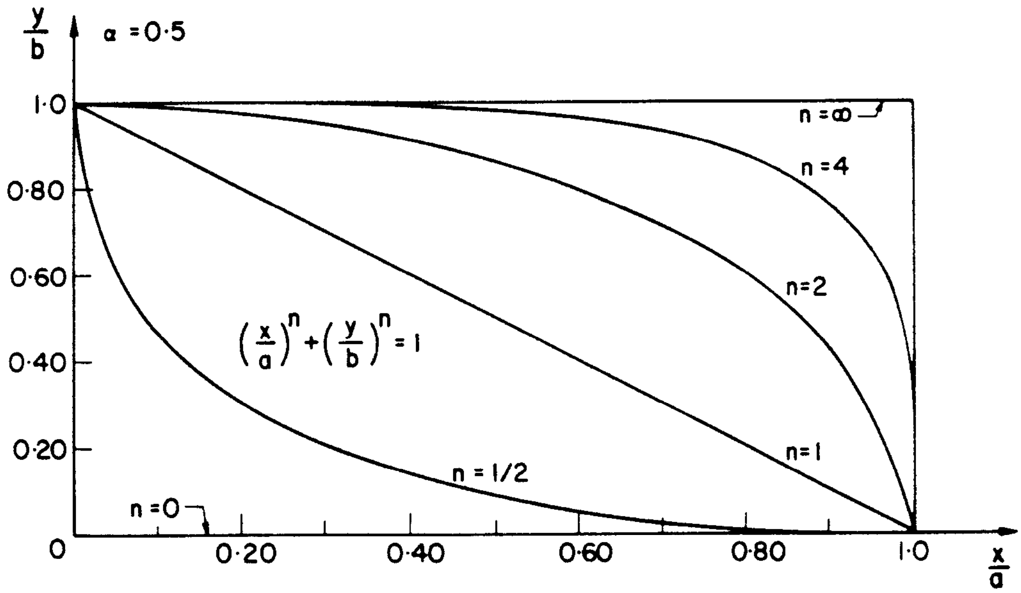

Fig. 2 First quadrant shape of contact regions for various n.

The effect of the geometric parameter n upon the shape of the contact area is shown in Fig. 2. When 1) $n = 2$, the contact is elliptical ($a > b$) or circular ($a = b$); 2) $n = 1$, the contact is diamond-shaped ($a > b$) or square ($a = b$); 3) $n > 2$, the contact is a rectangle ($a > b$) with rounded corners or a square ($a = b$) with rounded corners, and 4) $n < 1$, the contact is an astroid ($a = b$).

The total heat flow rate Q entering the contact area A is removed from the system through a boundary located at a distance that is large relative to some characteristic dimension of the contact area. The heat flow rate is related to the contact area as follows:

$$Q = 4q\,A_o \tag{2}$$

where A_o, the total area in the first quadrant, is given by

$$A_o = \int_0^b \int_0^{a[1-(y/b)^n]1/n} dx\,dy$$

$$= ab \int_0^1 [1 - v^n]^{1/n}\, du \tag{3}$$

with $u = x/b$ and $v = y/b$. The integral in Eq. (3) can be evaluated, and therefore we have[5]

$$A_o = \frac{ab}{n} B\left(\frac{n+1}{n}, \frac{1}{n}\right) \tag{4}$$

giving us

$$Q = \frac{4qab}{n} B\left(\frac{n+1}{n}, \frac{1}{n}\right) = qA \tag{5}$$

Equation (5) thus defines A in terms of the beta function B, which depends only upon the geometric parameter n. If for convenience we define the contact area aspect ratio $\alpha = b/a \leq 1$, then the product ab in Eq. (5) may be written as $a^2\alpha$.

Yovanovich[4] has demonstrated that this type of boundary-value problem can be solved using an integral method for the general case of arbitrary heat-flux distribution over an arbitrary contact area. In the special case of uniform flux, the constriction resistance, defined as the average temperature of the contact area divided by the total heat flow rate, is given by

$$\overline{R} = \frac{1}{qA^2} \iint_A T(x,y,0)dA \tag{6}$$

where T(x,y,0) is the local temperature within the contact due to the uniform flux q entering the contact at the point (ξ,η). The local temperature is[4]

$$T(x,y,0) = \frac{q}{2\pi\lambda} \iint_A \frac{d\xi d\eta}{\sqrt{(\xi - x)^2 + (\eta - y)^2}} \tag{7}$$

Substituting Eq. (7) into Eq. (6) and taking advantage of symmetry, we obtain the following expression for the constriction resistance:

$$\overline{R} = \frac{2}{\pi\lambda A^2} \iint_{A_o} \left[\iint_A \frac{d\xi d\eta}{\sqrt{(\xi - x)^2 + (\eta - y)^2}} \right] dxdy \tag{8}$$

where A_o and A have been defined in Eqs. (4) and (5), respectively.

It can be seen in Eq. (8) that the thermal problem has been reduced for the special case of uniform flux to a purely geometric problem, i.e., the analytic or numerical evaluation of double integral within a double integral over the contact area given by Eq. (1). The method employed to evaluate these multiple integrals will be considered in a subsequent section.

Dimensionless Constriction Resistance

It is customary in heat transfer to define a dimensionless resistance and to base this resistance upon the thermal conductivity of the half-space and some characteristic dimension of the system. If we choose a, the characteristic dimension along the x axis, the dimensionless resistance based upon the average contact area temperature will be $\lambda a \bar{R}$, which is the customary definition.[1] Another expression for the dimensionless constriction resistance may be obtained if we choose the square root of the total contact area as the characteristic dimension of the system. In this case, we have $\lambda \sqrt{A}\, \bar{R}$.

These two definitions of the characteristic dimension of the system yield the following two expressions:

$$\lambda a \bar{R} = \frac{1}{4\pi} \left[\frac{n}{\alpha B}\right]^{1/2} \frac{1}{A^{3/2}} \iint_A \left[\iint_{A_o} \frac{d\xi d\eta}{\sqrt{(\xi - x)^2 + (\eta - y)^2}} \right] dxdy \tag{9}$$

and

$$\lambda \sqrt{A}\, \bar{R} = \frac{1}{2\pi} \frac{1}{A^{3/2}} \iint_A \left[\iint_{A_o} \frac{d\xi d\eta}{\sqrt{(\xi - x)^2 + (\eta - y)^2}} \right] dxdy \tag{10}$$

Before proceeding to the evaluation and comparison of these integrals, we shall consider an alternate definition of the constriction resistance.

Centroidal Temperature and Constriction Resistance

When a singly connected, planar contact area having the contour given by Eq. (1) is subjected to a uniform flux, the maximum temperature within the contact area will occur at its centroid. Yovanovich[4] has shown that the centroidal temperature T_o can be obtained by means of the following expression:

$$T_o = \frac{2q}{\pi\lambda} \int_0^{\pi/2} \rho_o \, d\theta \tag{11}$$

where the radius vector ρ_o, from the centroid to any point on the contact area contour making an angle θ with the x axis as shown in Fig. 1, is given by

$$\rho_o = \frac{b}{[\sin^n\theta + \alpha^n \cos^n\theta]^{1/n}} \qquad \alpha \leq 1 \tag{12}$$

A new expression for the constriction resistance may be derived by defining the resistance as the quotient of the centroidal temperature and the total heat flow rate. Therefore,

$$R_o = \frac{T_o}{Q} = \frac{2b}{\pi\lambda A}\int_0^{\pi/2} \frac{d\theta}{[\sin^n\theta + \alpha^n \cos^n\theta]^{1/2}} \tag{13}$$

As before, the centroidal constriction can be normalized. Using the various characteristic dimensions defined earlier, we obtain the following expressions:

$$\lambda a R_o = \frac{n}{2\pi B}\int_0^{\pi/2} \frac{d\theta}{[\sin^n\theta + \alpha^n \cos^n\theta]^{1/n}} \tag{14}$$

$$\lambda\sqrt{A}\, R_o = \frac{1}{\pi}\left[\frac{\alpha n}{B}\right]^{1/2}\int_0^{\pi/2} \frac{d\theta}{[\sin^n\theta + \alpha^n \cos^n\theta]^{1/n}} \tag{15}$$

Equations (14) and (15) are clearly functions of the geometric parameters n and α, because B depends upon n only.

Integral Evaluation

This section will consider the evaluation of the integrals appearing in Eqs. (9) and (10). We define the following dimensionless variables:

$$u = x/b, \qquad v = v/b \tag{16a}$$

$$\mu = \xi/b, \qquad \nu = \eta/b \tag{16b}$$

and rewrite the integrals in the following manner with the aid of Fig. 3:

$$I = b^3 \int_0^{+1}\int_0^{+u_1(v)}\int_{-1}^{+1}\int_{-\mu(\nu)}^{\mu_1(\nu)} \frac{d\mu\, d\nu}{\sqrt{(\mu-u)^2 + (\nu-v)^2}}\, du dv \tag{17}$$

with

$$u_1 = (1/\alpha)[1-v^n]^{1/n}; \quad \mu_1 = (1/\alpha)[1-\nu^n]^{1/n} \tag{18}$$

Integration with respect to μ yields

$$I = b^3 \int_0^{+1} \int_0^{u_1(v)} \int_{-1}^{+1} \Big[\ell n\Big((\mu_1 - u) + \sqrt{(\mu_1 - u)^2 + (\nu - v)^2} \Big)$$

$$- \ell n\Big(-(\mu_1 + u) + \sqrt{(\mu_1 + u)^2 + (\nu - v)^2} \Big) d\nu \Big] du dv \qquad (19)$$

The integrand may be interpreted as the temperature at point (u,v) due to a uniform flux q acting on the strip of infinitesimal width $d\nu$ between $-\mu_1(\nu)$ and $+\mu_1(\nu)$ ($-\xi_1$ and $+\xi_1$ in dimensional coordinates shown in Fig. 3).

Except for contact areas having linear or piecewise linear boundaries, μ_1 is not a simple function of ν. Consequently, it becomes difficult or even impossible to carry out the integration formally with respect to ν. Fortunately, however, u is independent of ν, so that the order of integration with respect to these two variables may be interchanged. Integration of the first term in Eq. (19) with respect to u gives

$$\int_0^{u_1} \ell n \left[(\mu_1 - u) + \sqrt{(\mu_1 - u)^2 + (\nu - v)^2} \right] d\mu$$

$$= \mu_1 \, \ell n \left[\mu_1 + \sqrt{\mu_1^2 + (\nu - v)^2} - \sqrt{\mu_1^2 + (\nu - v)^2} \right]$$

$$- (\mu_1 - u_1) \ell n \left[(\mu_1 - u_1) + \sqrt{(\mu_1 - u_1)^2 + (\nu - v)^2} \right]$$

$$+ \sqrt{(\mu_1 - u_1)^2 + (\nu - v)^2} \qquad (20)$$

Simply substituting $-\mu_1$ for μ_1 in Eq. (20) gives the result for the integration of the second term of Eq. (19) with respect to u.

Equation (17) now can be written as

$$I = b^3 \int_{v=0}^{1} \int_{\nu=-1}^{1} G(\nu, v) \, d\nu dv \qquad (21)$$

where the integrand in Eq. (21) is given by

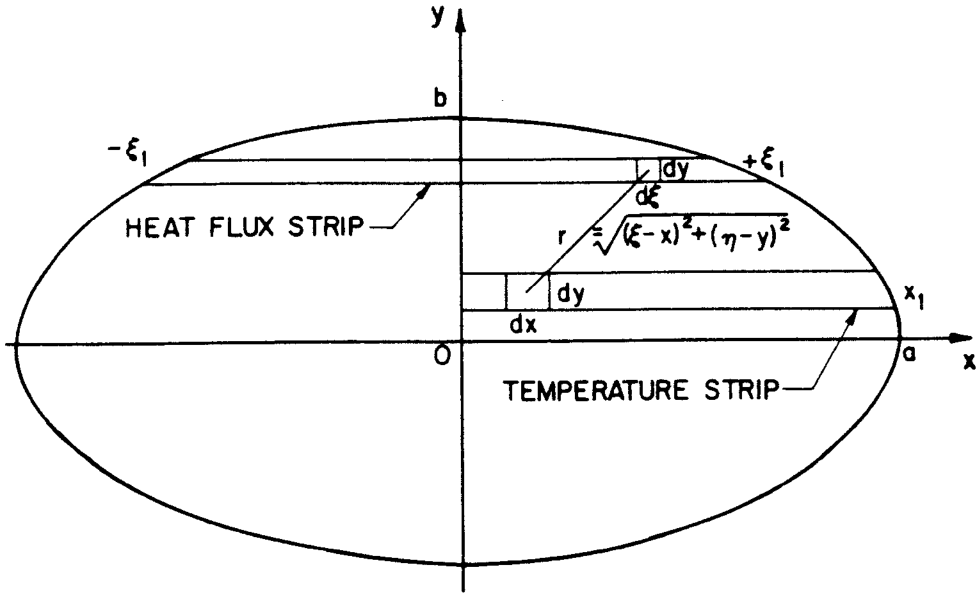

Fig. 3 Temperature and heat flux strips.

$$G(\nu,v) = 2\mu_1 \ln|\nu-v| - (\mu_1 - u_1)\ln\left[(\mu_1-u_1) + \sqrt{(\mu_1-u_1)^2+(\nu-v)^2}\right]$$

$$- (\mu_1+u_1)\ln\left[-(\mu_1+u_1) + \sqrt{(\mu_1+u_1)^2 + (\nu-v)^2}\right]$$

$$+ \sqrt{(\mu_1-u_1)^2 + (\nu-v)^2} - \sqrt{(\mu_1+u_1)^2 + (\nu-v)^2} \qquad (22)$$

Disregarding scalar multiplicative constants, we may interpret $G(\nu,v)$ as the integral of the temperature along the strip of width dv between o and $+u_1$ due to the uniform flux on the strip of width $d\nu$ between $-\mu_1$ and $+\mu_1$.

The presence of the logarithmic singularities in the first two terms on the right-hand side of Eq. (22) when $\nu = v$ and $\mu_1(\nu) = u_1(v)$ would appear to make it impossible to evaluate I accurately by a numerical technique. However, it is not at all difficult to prove that the error introduced by simply neglecting the interval $v - \varepsilon_1 < \nu < v + \varepsilon_2$ when the quadrature with respect to ν is carried out tends to zero as ε_1 and ε_2 tend to zero. Consequently, the authors adopted a quadrature procedure, which involved dividing the intervals of integration of v and ν into M and N strips of equal width, respectively. Then the dimensionless resistances given by Eqs. (9) and (10) could be computed from the expressions

$$\lambda a\overline{R} = \frac{\alpha}{8\pi}\left[\frac{n}{B}\right]^2 \Delta\nu\Delta v \sum_{i=1}^{M}\sum_{j=1}^{N} G(\nu_j, v_i) \qquad (23)$$

and

$$\lambda\sqrt{A}\ \bar{R} = \frac{1}{4\pi}\left[\frac{\alpha n}{B}\right]^{3/2} \Delta\nu\Delta v \sum_{i=1}^{M}\sum_{j=1}^{N} G(\nu_j, v_i) \qquad (24)$$

In these equations, ν_j and v_i denote midpoints of the jth and ith strips. The obvious requirement that no $\nu_j - v_i$ term be so small as to cause numerical difficulties was avoided by maintaining a ratio of M:N = 5:8, by limiting N to 80, and by doing all calculations in double precision.

The beta function appearing in Eqs. (23) and (24) was replaced by gamma functions:

$$B\left(\frac{n+1}{n}, \frac{1}{n}\right) = \frac{\Gamma(\frac{n+1}{n})\ \Gamma(1/n)}{\Gamma\left[(n+2)/n\right]} \qquad (25)$$

Numerical Results and Discussion

Computer programs were written for the numerical evaluation of Eqs. (14, 15, 23, and 24) with Eq. (25). The integrals appearing in Eqs. (14) and (15) were evaluated by five-point Gaussian quadrature. The accuracy of the integration technique and the programming were checked by comparing the computed results for a circular contact area ($n = 2$, $\alpha = 1$), $\lambda a\bar{R} = 0.2701$ and $\lambda\sqrt{A}\ \bar{R} = 0.4787$, with the classical results, $\lambda a\bar{R} = (8/3\pi^2) = 0.2702$ and $\lambda\sqrt{A}\ \bar{R} = (8/3\pi^{3/2}) = 0.4789$.[1,4] These results were obtained for M = 50 and N = 80. Only slightly less accurate results were obtained for M = 25 and N = 40. The excellent agreement between the numerical results and the exact results for the circular contact area confirmed that the program was functioning properly and that it could be expected to yield accurate results for other values of n and α. Numerical results also were obtained for several values of n (1/4, 1/3, 1/2, 1, 2, 4, and 10) and several values of the aspect ratio α (1, 0.8, 0.6, 0.4, and 0.2).

Table 1 contains selected values of the dimensionless constriction resistances for the symmetric contact ($\alpha = 1$). The numerical results given in Table 1 show clearly that the dependence of the dimensionless resistance upon the contact area is reduced considerably when $\sqrt{A}$ is selected as the characteristic dimension of the system. If we use the results for the circular contact area ($n = 2$), it can be seen that, for $\lambda\sqrt{A}\ \bar{R}$, the difference between $n = 0.5$ and $n = 2$ is -7.25% and between $n = \infty$ and $n = 2$ is -1.15%. On the other hand, these differences reduce to -3.08% and -0.55%, respec-

Table 1 Selected values of the dimensionless constriction resistance for $\alpha = 1$ (M = 50, N = 80)

n	$\lambda a\overline{R}$	$\lambda a R_o$	$\lambda\sqrt{A}\,\overline{R}$	$\lambda\sqrt{A}\,R_o$
0.5	0.5424	0.6697	0.4440	0.5468
1.0	0.3341	0.3968	0.4728	0.5611
2.0	0.2701	0.3183	0.4787	0.5642
4.0	0.2477	0.2924	0.4770	0.5631
∞	0.2366	0.2805	0.4732	0.5611

tively, for the $\lambda\sqrt{A}\,R_o$ dimensionless resistance. If the average value of $\lambda\sqrt{A}\,R_o = 0.5593$ for the five cases given in Table 1 is used, it can be seen that the minimum value of $\lambda\sqrt{A}\,R_o$ corresponding to n = 0.5 differs by -2.23%.

To see whether this close agreement was due to the symmetry of these contact areas, two special nonsymmetric, singly connected, planar contacts were investigated. In the first case, the average and centroidal temperatures were computed for an equilateral triangle subjected to a uniform heat flux. The dimensionless constriction resistances were found to be $\lambda\sqrt{A}\,\overline{R} = 0.4600$ and $\lambda\sqrt{A}\,R_o = 0.5516$. In the second case, both average and centroidal temperatures were obtained for a semicircular contact area subjected to a uniform heat flux. For this shape, the dimensionless constriction resistances were found to be $\lambda\sqrt{A}\,\overline{R} = 0.4610$ and $\lambda\sqrt{A}\,R_o = 0.5456$.

The very close agreement between $\lambda\sqrt{A}\,R_o$ for the triangle and the semicircle and $\lambda\sqrt{A}\,R_o = 0.5593$ is remarkable. When these two nonsymmetric contacts are taken into account, the average value of $\lambda\sqrt{A}\,R_o$ becomes 0.5562 ($\approx$ 5/9). It can be shown that the mean value of $\lambda\sqrt{A}\,\overline{R}$ for the five cases given in Table 1 plus the triangle and semicircle is 0.4667. Since the constriction resistances are proportional to the average and centroidal temperatures, the ratio of the average value of the mean contact area temperatures to the average value of the centroidal temperatures is

$$\frac{(\overline{T})_{av}}{(T_o)_{av}} = \frac{(\lambda\sqrt{A}\,\overline{R})_{av}}{(\lambda\sqrt{A}\,R_o)_{av}} = 0.840 \qquad (26)$$

This is an important relationship, which can be used to estimate quickly the average contact area temperature given the centroidal temperature.

Table 2 Effect of upon $\lambda\sqrt{A}\,\bar{R}$, $\lambda\sqrt{A}\,R_o$, and $\bar{R}/R_o$

	n = 1/2			n = 1		
α	$\lambda\sqrt{A}\,\bar{R}$	$\lambda\sqrt{A}\,R_o$	$\bar{R}/R_o$	$\lambda\sqrt{A}\,\bar{R}$	$\lambda\sqrt{A}\,R_o$	$\bar{R}/R_o$
1	0.4440	0.5468	0.8100	0.4728	0.5611	0.8420
0.8	0.4428	0.5458	0.8100	0.4713	0.5597	0.8419
0.6	0.4376	0.5420	0.8074	0.4651	0.5540	0.8400
0.4	0.4237	0.5310	0.7963	0.4487	0.5385	0.8328
0.2	0.3860	0.5005	0.7701	0.4052	0.4957	0.8170

	n = 2			n = 4		
α	$\lambda\sqrt{A}\,\bar{R}$	$\lambda\sqrt{A}\,R_o$	$\bar{R}/R_o$	$\lambda\sqrt{A}\,\bar{R}$	$\lambda\sqrt{A}\,R_o$	$\bar{R}/R_o$
1	0.4787	0.5642	0.8482	0.4770	0.5631	0.8471
0.8	0.4772	0.5624	0.8483	0.4756	0.5611	0.8476
0.6	0.4711	0.5551	0.8486	0.4696	0.5530	0.8492
0.4	0.4548	0.5360	0.8486	0.4537	0.5317	0.8536
0.2	0.4112	0.4845	0.8488	0.4111	0.4762	0.8636

The effect of the aspect ratio α upon $\lambda\sqrt{A}\,\bar{R}$, $\lambda\sqrt{A}\,R_o$, and their ratio is shown in Table 2 for a few selected values of the geometric parameter n. It is apparent from an examination of Table 2 that both $\lambda\sqrt{A}\,\bar{R}$ and $\lambda\sqrt{A}\,R_o$ decrease slightly as α goes from 1.0 to 0.4. For $\alpha < 0.4$, the change is somewhat greater. The ratio of the average and centroidal is a relatively weak function of the aspect ratio. It also is interesting to note that this ratio is very nearly the average value of 0.840, Eq. (26).

Conclusions

An integral-numerical method is proposed for calculating the constriction resistance of singly connected, arbitrary contact areas. The agreement with certain exact solutions was found to be very good. The square root of the constant area was found to be the characteristic dimension, which resulted in a dimensionless constriction resistance that did not vary much with the shape of the contact areas considered in this study. An alternate definition of the constriction resistance based upon the centroidal temperature, which is computed easily, resulted in a dimensionless resistance that was rather insensitive to the shape of the contact as well as its aspect ratio.

For the shapes considered in this study, it was found that the dimensionless constriction resistance based upon the centroidal temperature is $\lambda\sqrt{A}\,R_o = 5/9$, with an error not exceeding 5% for $\alpha \geq 0.4$. The average and centroidal temperatures were found to be related in the following manner: $\bar{T}/T_o = 0.84$, with an error not exceeding 4% for all of the shapes considered. These relationships allow one to estimate quickly the magnitude of the constriction resistance. Also, most contact geometries can be modeled as circular contacts having the same area with an error not exceeding 5%.

Acknowledgement

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References

[1]Carslaw, H.S. and Jaeger, J.C., Conduction of Heat in Solids, Oxford University Press, London, 1959.

[2]Yovanovich, M.M., "Thermal Constriction Resistance between Contacting Metallic Paraboloids: Application to Instrument Bearings," AIAA Progress in Astronautics and Aeronautics: Heat Transfer and Spacecraft Thermal Control, Vol. 24, edited by J.W. Lucas, The MIT Press, Cambridge, Mass., 1970, pp. 337-358.

[3]Mikic, B.B. and Rohsenow, W.M., "Thermal Contact Resistance," Massachusetts Institute of Technology, Heat Transfer Lab. Rept. 4542-41, Sept. 1966.

[4]Yovanovich, M.M., "Thermal Constriction Resistance of Contacts on a Half-Space: Integral Formulation," AIAA Paper 75-708, Denver, Colo., May 27-29, 1975; also AIAA Progress in Astronautics and Aeronautics: Radiative Transfer and Thermal Control, Vol. 49, edited by A.M. Smith, New York, 1976, pp. 397-418.

[5]Gradshteyn, I.S. and Ryzhik, I.M., Table of Integrals, Series, and Products, Academic Press, New York, 1975.