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THERMAL CONSTRICTION RESISTANCE OF CONTACTS ON A HALF-SPACE: INTEGRAL FORMULATION

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Abstract

Expressions have been developed for the determination of thermal constriction resistance of arbitrary planar contact areas subjected to arbitrary heat flux distributions. Local contact area temperatures under uniform flux conditions were determined for certain important shapes: triangular, rectangular, polygonal, circular, and annular ring. Dimensionless thermal constriction parameters are presented for the rectangular, circular, and annular ring contact areas.

Nomenclature

a = circular contact radius; inner radius of annular contact

b = outer radius of annular contact

B = complete elliptic integral, Eq. (50)

D = complete elliptic integral, Eq. (50)

E = complete elliptic integral of the second kind

F = incomplete elliptic integral of the first kind, omega function, Eq. (22)

K = complete elliptic integral of the first kind

M = arbitrary point in the contact area

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P = arbitrary point

r = radial position

Greek Symbols

ψ = constriction resistance parameter

Ω = omega function

Subscripts

a = annular contact

c = circular contact

r = rectangular contact

Introduction

A number of papers[1-5] have been published recently showing the importance of thermal constriction resistance in several thermal problems arising from aerospace applications. It has been shown that, whenever heat is contrained to flow through contact areas whose characteristic dimensions are small relative to the characteristic dimensions of the contacting bodies,[1,3-5] one observes a large local temperature drop, which is a manifestation of the thermal constriction resistance. This thermal constriction resistance is a function of the thermal conductivities of the contacting bodies, some characteristic dimension of the contact area, and a thermal constriction parameter that is dependent upon the shape of the contact area as well as the boundary condition over the contact area. Furthermore, a complex geometry such as a sphere in elastic contact with a race can be modeled with confidence as an elliptic contact area separating two half-spaces.[1] This model greatly simplifies the thermal analysis.

Up to the present, only a few publications have dealt with analytic solutions[1,2] or numerical solutions.[10] These papers have considered the elliptic and circular contact areas with either uniform temperature or uniform heat flux boundary conditions. There is, therefore, a great need for a theory for predicting the constriction resistance of arbitrary planar contact areas attached to half-spaces. This paper will develop

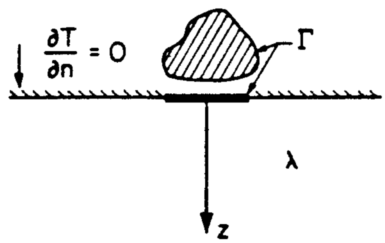

Fig. 1 Arbitrary contact on a half-space.

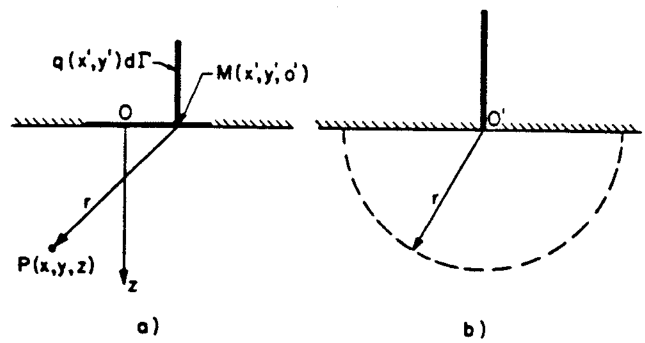

Fig. 2 Point source and coordinates.

an integral expression for determining the thermal constriction of arbitrary contact areas subjected to arbitrary heat fluxes.

Thermal Constriction Resistance

Plane Contact Area on a Half-Space: Integral Formulation

Consider the problem of steady heat transfer from an arbitrary plane contact area situated in the xoy plane as shown in Figs. 1 and 2. The heat flux q over the contact area Γ is a function of position. The region outside the contact is assumed to be perfectly insulated, whereas the thermal conductivity of the half-space $z > 0$ is λ.

The temperature field within the half-space $T(x,y,z)$ must satisfy Laplace's equation

$$\nabla^2 T = 0 \tag{1}$$

which, in Cartesian coordinates (x,y,z), takes the form

$$(\partial^2 T/\partial x^2) + (\partial^2 T/\partial y^2) + (\partial^2 T/\partial z^2) = 0 \tag{2}$$

The temperature field tends toward a uniform value $T(\infty)$ at points within the half-space which are far from the centroid of the contact area.

Definition of Thermal Constriction Resistance

The thermal constriction resistance R is defined as the difference between the average temperature of the contact area $\bar{T}$ minus the temperature far from the contact area divided by the total heat flow rate through the contact area Q. Mathematically, we can state the definition as

$$R = [\bar{T} - T(\infty)]/Q \qquad (3)$$

The average contact area temperature is determined by means of the following expression:

$$\bar{T} = \frac{1}{\Gamma} \iint_{\Gamma} T(x,y,o)\ d\Gamma \qquad (4)$$

The total heat flow rate through the contact area is simply the integrated value of the product of the local heat flux and corresponding contact area. Therefore,

$$Q = \iint_{\Gamma} q d\Gamma \qquad (5)$$

If, for convenience, we take the temperature far from the contact area to be zero, Eq. (3) becomes

$$R = \frac{1}{\Gamma} \iint_{\Gamma} T(x,y,o)\ d\Gamma / \iint_{\Gamma} q d\Gamma \qquad (6)$$

It is now necessary to determine the local contact area temperature $T(x,y,o)$ as a function of the prescribed heat flux distribution over the contact area.

Superposition of Heat Sources

Consider the effect of a heat source $q(x',y')d\Gamma$ located at $M(x',y',o)$ upon a point $P(x,y,z)$ located a distance r from the source (Fig. 2). The effect of $q(x',y')d\Gamma$ at P can be determined by means of Fourier's equation. If we place the ori-

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gin o at the heat source and consider the heat transfer through the hemispherical surface located a distance r from the source, we can write

$$q(x',y')\,d\Gamma = -\lambda\,2\pi r^2 (dT/dr) \tag{7}$$

Rearranging Eq. (7) and integrating with respect to T and r, $q(x',y')\,d\Gamma$ being constant, we obtain

$$T_1(r) - T_2(r) = [q(x',y')\,d\Gamma/2\pi\lambda][1/r_1 - 1/r_2] \tag{8}$$

We can let r_1 be an arbitrary point r and $T_1(r)$ be T. For convenience, we choose $T_2(r)$ to be $T(\infty) = 0$ and $r_2 = \infty$.

The temperature at P due to a heat source at o' is therefore

$$T(x,y,z) = q(x',y')\,d\Gamma/2\pi\lambda r \tag{9}$$

We now can consider the effect of heat input into the half-space due to the entire contact area. The temperature at P is simply the expression in Eq. (9) integrated over the entire contact area. Thus,

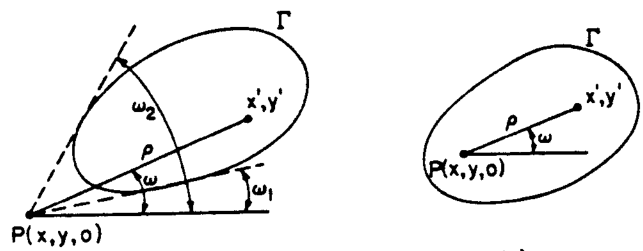
$$T(x,y,z) = \frac{1}{2\pi\lambda}\iint_{\Gamma} \frac{q(x',y')\,d\Gamma}{r} \tag{10}$$

Since $d\Gamma$ can be written as $dx'dy'$ and the radial distance is given by $r = [(x'-x)^2 + (y'-y)^2 + z^2]^{1/2}$, Eq. (10) in a Cartesian system becomes

$$T(x,y,z) = \frac{1}{2\pi\lambda}\iint_{\Gamma} \frac{q(x',y')\,dx'dy'}{[(x'-x)^2 + (y'-y)^2 + z^2]^{1/2}} \tag{11}$$

As a result of the definition of the thermal constriction resistance, it can be seen in Eq. (6) that the contact area temperature is required. This being the case, it is recommended that the following method to be used for the evaluation of Eq. (11). The projection of P(x,y,z) upon the xoy plane is shown in Fig. 3. Introduce the polar coordinates ρ,ω with this point as origin. Therefore,

$$x' - x = \rho\cos\omega, \quad y' - y = \rho\sin\omega \tag{12}$$



a) b)

Fig. 3 Polar coordinates.

An elemental area in Γ becomes $d\Gamma = \rho d\rho d\omega$. Equation (10), with $z = 0$, now becomes

$$T(x,y,0) = \frac{1}{2\pi\lambda} \iint_{\Gamma} q d\omega d\rho \tag{13}$$

If the point P lies outside Γ, the angle ω has minimum and maximum values ω_1 and ω_2 (Fig. 3a). For $z = 0$, Eq. (11) becomes

$$T(x,y,o) = \frac{q}{2\pi\lambda} \int_{\omega_1}^{\omega_2} [\rho_2(\omega) - \rho_1(\omega)] d\omega \tag{14}$$

where ρ_1 and ρ_2 are clearly functions of ω, and q is uniform. If, on the other hand, the point P lies inside Γ, as shown in Fig. 3b, the angle ω goes from 0 to 2π, and Eq. (11) reduces to

$$T(x,y,o) = \frac{q}{2\pi\lambda} \int_{o}^{2\pi} \rho(\omega) d\omega \tag{15}$$

when q is uniform.

Alternate Expressions of the Thermal Constriction Resistance

Equation (6) now can be written in the following manner when we use the expression for the temperature given by Eq. (10):

$$R = \frac{1}{2\pi\lambda\Gamma} \left[\iint_{\Gamma} \left[\iint_{\Gamma} \frac{q d\Gamma}{r} \right] d\Gamma \Big/ \iint_{\Gamma} q d\Gamma \right] \tag{16}$$

This expression is the general equation of the thermal constriction resistance due to an arbitrary heat flux distribution over an arbitrary contact area Γ. The inner integral in the numerator represents the local contact area temperature due to the total heat input into the half-space. The outer integral is the average temperature of the contact area. It is obvious that the integral in the denominator is the total heat flow rate through the contact area.

For the special case of uniform heat flux, q can be taken out of the double integration in the numerator as well as the integration in the denominator and canceled. Thus Eq. (16) reduces to

$$R = \frac{1}{2\pi\lambda\Gamma^2}\iint_\Gamma \left[\iint_\Gamma \frac{d\Gamma}{r}\right] d\Gamma \qquad (17)$$

which is clearly dependent upon the geometry of the contact only. According to Eq. (15), Eq. (17) can be expressed as

$$R = \frac{1}{2\pi\lambda\Gamma^2}\iint_\Gamma \left[\int_0^{2\pi} \rho(\omega)\ d\omega\right] d\Gamma \qquad (18)$$

Several special cases will be considered next to demonstrate the applicability of Eqs. (17) and (18).

Contact Area Temperatures due to Uniform Heat Flux

Triangular Contact. The temperature at the vertex of a triangular contact area (Fig. 4) due to uniform heat flux over the entire area now will be determined. In Fig. 4a, the perpendicular from the vertex P(x,y) to the opposite side AB intersects it at C and divides the triangular area into two

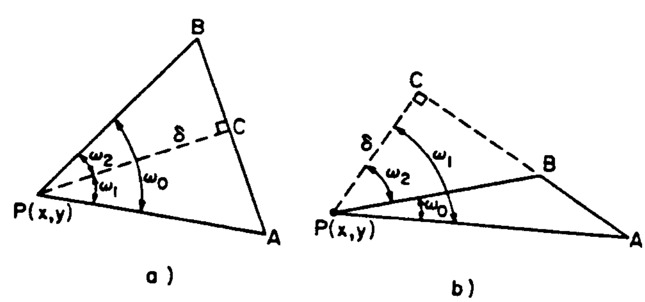


Fig. 4 Triangular contacts.

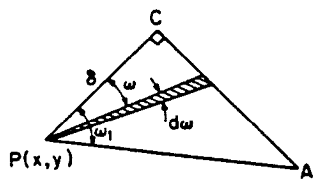


Fig. 5 Triangular element.

right-angle triangles PAC and PBC with angles ω_1 and ω_2 subtended at the vertex P. The length PC will be denoted δ.

Consider the triangle PAC alone (Fig. 5). The effect of uniform heat flux distributed over the shaded elemental area is, according to Eq. (13),

$$T(x,y) = \frac{q}{2\pi\lambda}\int_0^{\omega_1} \frac{\delta\, d\omega}{\cos\omega} = \frac{q}{2\pi\lambda}\int_0^{\omega_1} \frac{\delta\, d\omega}{\sqrt{1-\sin^2\omega}} \qquad (19)$$

where $\omega_1 = \tan^{-1}(AC/\delta)$ Equation (19) integrates readily and is

$$T(x,y) = (q/2\pi\lambda)\delta \ln \tan\left[(\pi/4) + (\omega_1/2)\right] \qquad (20)$$

For convenience, we introduce the omega function:

$$\Omega(\omega) = \ln \tan\left[\frac{\pi}{4} + \frac{\omega}{2}\right] = \frac{1}{2}\ln\left[\frac{1+\sin\omega}{1-\sin\omega}\right] \qquad (21)$$

The second expression in Eq. (19) is recognized as the incomplete elliptic integral of the first kind of ω_1 and modulus equal to unity. Thus an alternative expression for the omega function is

$$\Omega(\omega_1) = F(\omega_1, 1) \qquad (22)$$

The values of $\Omega(\omega_1)$ can be read directly from a table of elliptic integrals.[7-9]

In a similar manner, the effect of a unform heat flux dis- over the right-angle triangle PBC can be obtained:

$$T(x,y) = (q\delta/2\pi\lambda)\Omega(\omega_2) \qquad (23)$$

with $\Omega(\omega_2) = \ln \tan\left[(\pi/4) + (1/2)\tan^{-1}(BC/\delta)\right]$.

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By superposition, we now can write, for the temperature at the vertex P of triangle PAB due to uniform heat flux distribution q,

$$T(x,y) = (q\delta/2\pi\lambda)\{\Omega(\omega_1) + \Omega(\omega_2)\} \tag{24}$$

By means of superposition, the temperature at the vertex P of the triangle PAB shown in Fig. 4b can be shown to be

$$T(x,y) = (q\delta/2\pi\lambda)\{\Omega(\omega_1) - \Omega(\omega_2)\} \tag{25}$$

with

$$\Omega(\omega_1) = \ln \tan\left[\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{AC}{\delta}\right] \tag{26a}$$

$$\Omega(\omega_2) = \ln \tan\left[\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{BC}{\delta}\right] \tag{26b}$$

Regular Polygon Contact. The results of the previous section now can be applied to the determination of the temperature at an arbitrary point P lying inside a polygon of n sides (Fig. 6). If the point P is joined to the n vertices of the n-sided polygon, n triangles are formed, $\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n$, with common vertices at point P, the bases of which form the sides of the polygon. The temperature T(x,y) at P consists of the evaluation of integrals over the triangles Δ_i and their subsequent summation. For the point P lying within the polygon, the temperature is

$$T(x,y) = \frac{q}{2\pi\lambda}\sum_{i=1}^{n} \delta_i\,\Omega_i \tag{27}$$

where the δ_i are the perpendiculars from P to the n sides or their projection, and the Ω_i are the corresponding omega functions defined by Eqs. (21) and (22).

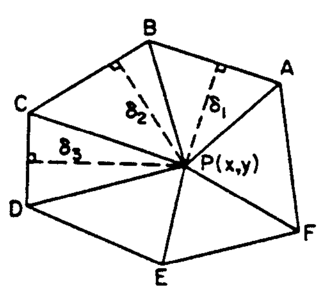

Fig. 6 Polygon contact.

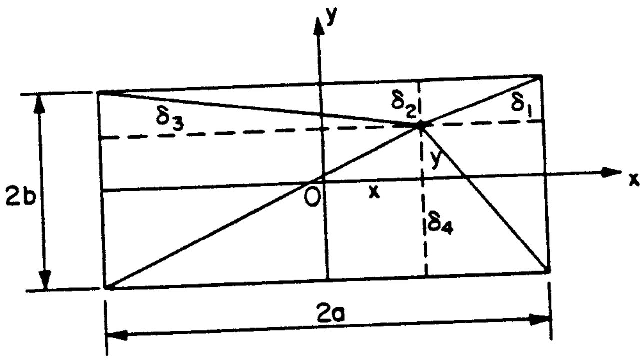

Fig. 7 Rectangular contact.

Rectangular Contact. The temperature at an internal point P(x,y) of a rectangular contact area (2a x 2b) can be determined readily by the superposition of solutions of eight triangular areas. Place the origin at the center of the contact area, with the x and y axes running parallel to the sides 2a and 2b, respectively (Fig. 7). The four perpendiculars from the point P to the four sides of the rectangular area are

$$\delta_1 = a - x = a(1 - \xi) \quad (28a)$$

$$\delta_2 = b - y = b(1 - \eta) \quad (28b)$$

$$\delta_3 = a + x = a(1 + \xi) \quad (28c)$$

$$\delta_4 = b + x = b(1 + \eta) \quad (28d)$$

with $\xi = x/a$ and $\eta = y/b$.

There are eight triangles whose vertices have the common point P. The eight angles subtended at the point P are

$$\omega_1 = \tan^{-1}\frac{b - y}{\delta_1} = \tan^{-1}\frac{b(1 - \eta)}{a(1 - \xi)} \quad (29a)$$

$$\omega_2 = \tan^{-1}\frac{a - x}{\delta_2} = \tan^{-1}\frac{a(1 - \xi)}{b(1 - \eta)} \quad (29b)$$

$$\omega_3 = \tan^{-1}\frac{a + x}{\delta_2} = \tan^{-1}\frac{a(1 + \xi)}{b(1 - \eta)} \quad (29c)$$

$$\omega_4 = \tan^{-1}\frac{b - y}{\delta_3} = \tan^{-1}\frac{b(1 - \eta)}{a(1 + \xi)} \quad (29d)$$

THERE

$\omega_5 = \tan^{-1}$

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$$\omega_5 = \tan^{-1}\frac{b + y}{\delta_3} = \tan^{-1}\frac{b(1 + \eta)}{a(1 + \xi)} \tag{29e}$$

$$\omega_6 = \tan^{-1}\frac{a + x}{\delta_4} = \tan^{-1}\frac{a(1 + \xi)}{b(1 + \eta)} \tag{29f}$$

$$\omega_7 = \tan^{-1}\frac{a - x}{\delta_4} = \tan^{-1}\frac{a(1 - \xi)}{b(1 + \eta)} \tag{29g}$$

$$\omega_8 = \tan^{-1}\frac{b + y}{\delta_1} = \tan^{-1}\frac{b(1 + \eta)}{a(1 - \xi)} \tag{29h}$$

By superposition of solutions for eight triangular areas, the temperature at P is

$$T(x,y) = \frac{q}{2\pi\lambda}\sum_{i=1}^{8} \delta_i\, \Omega_i \tag{30}$$

where the omega functions are determined by means of Eqs. (29), and the δ_i are to be determined by means of Eq. (28). It should be noted that δ_1 is common to Δ_1 and Δ_8, δ_2 is common to Δ_2 and Δ_3, etc.

The temperature at the center of the rectangular area can be evaluated easily because, by symmetry, there are two sets of triangles which are identical. Thus,

$$T(0,0) = (q/2\pi\lambda)\ 4\ [\delta_1\Omega_1 + \delta_2\Omega_2] \tag{31}$$

where

$$\delta_1 = a,\ \delta_2 = b$$

$$\omega_1 = \tan^{-1}(b/a),\quad \omega_2 = \tan^{-1}(a/b)$$

Thus Eq. (31) reduces to

$$T(0,0) = \frac{2}{\pi}\frac{q}{\lambda}\left\{a \ln \tan\left[\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{b}{a}\right] + b \ln \tan\left[\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{a}{b}\right]\right\} \tag{32}$$

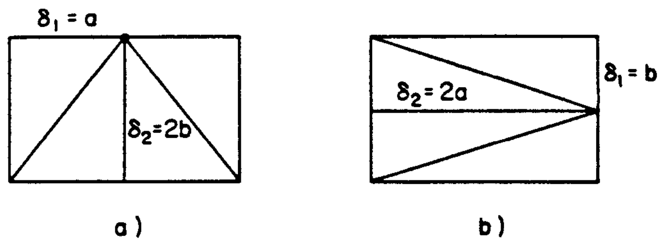

Fig. 8 Midpoint temperature.

For the special case of a square contact area, the temperature at the center can be shown to be

$$T(0,0) = \frac{2}{\pi}\frac{qa}{\lambda}\left\{2 \ln \tan\left[\frac{\pi}{4} + \frac{1}{2}\frac{\pi}{4}\right]\right\} = 1.122 \frac{qa}{\lambda} \tag{33}$$

The temperatures at the midpoints of the two sides of the rectangular contact areas (Fig. 8) also can be determined easily, and they are

$$T(0,b) = \frac{q}{2\pi\lambda}\left\{2 a \ln \tan\left[\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{2b}{a}\right] + 4b \ln \tan\left[\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{a}{2b}\right]\right\} \tag{34}$$

$$T(a,0) = \frac{q}{2\pi\lambda}\left\{2 b \ln \tan\left[\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{2a}{b}\right] + 4a \ln \tan\left[\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{b}{2a}\right]\right\} \tag{35}$$

At the corners, the temperature is

$$T(a,b) = \frac{q}{2\pi\lambda}\left\{2 a \ln \tan\left[\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{b}{a}\right] + 2 b \ln \tan\left[\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{a}{b}\right]\right\} \tag{36}$$

For the square contact area, the temperatures at the midpoints and corners are, respectively,

$$T(a,0) = T(0,a) = 0.7659(qa/\lambda) \tag{37}$$

$$T(a,a) = 0.561(qa/\lambda) \tag{38}$$

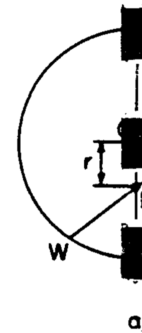

Fig. 9

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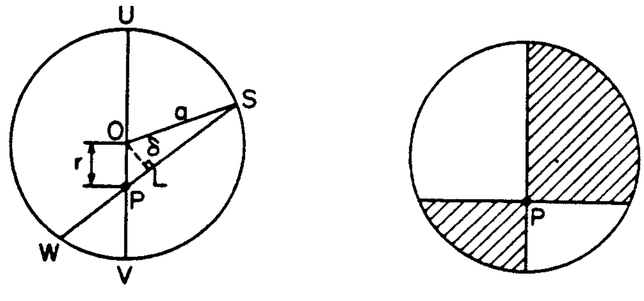

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Fig. 9 Circular contact with internal point.

The maximum temperature occurs at the center of the contact and is twice the temperature at the corners.

<u>Circular Contact</u>. In this section, the temperature inside and outside the contact area will be determined for the case of uniform heat flux distributed over a circular contact of radius a (Fig. 9). Figure 9a shows the internal point P located at any distance r from the center of the contact area. A straight line drawn through P intersects the circumference of the contact area at points W and S, making an angle ω with the diameter UV. The perpendicular from O to WS intersects it at L.

We wish to determine the temperature at P due to unform heat flux distribution along the line WS as it is rotated through an angle π. Equation (13) can be used to determine the temperature:

$$T = \frac{q}{2\pi\lambda}\int_{o}^{\pi/2} 2(\rho_1 + \rho_2)d\omega \qquad (39)$$

In Eq. (39), we have used symmetry to simplify the calculations. It can be seen in Fig. 9b that, as the radius vectors ρ_1 and ρ_2 are rotated through $\pi/2$ rad, the effect of the uniform heat flux distribution over the shaded area is taken into account. Since the unshaded area is identical to the shaded area by reason of symmetry, we need only multiply by a factor of 2 to account for all heat fluxes.

From simple geometric arguments, we have that

$$\rho_1 = WP = WL - PL, \quad \rho_2 = PS = LS + PL \qquad (40)$$

Since OL bisects WS, WL = LS and both are equal to $\sqrt{a^2 - OL^2}$. We also note that OL = r sin ω. Adding the two expressions in Eq. (40) yields

$$\rho_1 + \rho_2 = 2LS = 2a\sqrt{1 - (r/a)^2 \sin^2 \omega} \tag{41}$$

Upon substitution of Eq. (41) into Eq. (39), we have an expression of the temperature at P:

$$T(r) = \frac{2}{\pi}\frac{qa}{\lambda}\int_0^{\pi/2} \sqrt{1 - \left(\frac{r}{a}\right)^2 \sin^2 \omega}\, d\omega$$

$$= \frac{2}{\pi}\frac{qa}{\lambda} E\left(\frac{r}{a}\right) \tag{42}$$

where E is the complete elliptic integral of the second kind of modulus κ' = r/a. The temperature at the center of the contact area is qa/λ, whereas the temperature at the edge r = a is (2/π) qa/λ. As expected, the temperature at the center is in excess of the temperature at the edge (approximately 63.7% 57.1 % greater).

The temperature outside the contact area will be determined by means of Eq. (13) and the geometric relationships shown in Fig. 10. The external point P is located a distance r from the center of the contact area. The effect of uniform heat fluxes distributed along LM rotated through an angle $2\omega_o$ can be determined by Eq. (14), which, by reason of symmetry, is written as

$$T(r) = \frac{2q}{2\pi\lambda}\int_0^{\omega_o} LM\, d\omega \tag{43}$$

But we know that

$$LM = 2UM = 2[OM^2 - OU^2]^{1/2}$$

and

$$OM = a, \quad OU = r \sin \omega$$

Thus, Eq. (43) can be written as

$$T(r) = \frac{2}{\pi}\frac{qa}{\lambda}\int_{o}^{\omega_o}\sqrt{1 - \left(\frac{r}{a}\right)^2 \sin^2 \omega \, d\omega} \tag{44}$$

where $\sin \omega_o = a/r$. Equation (44) can be put into a more manageable form if we use the following transformation:

$$\sin \omega = \sin \omega_o \sin\psi, \qquad d\omega = \frac{\sin \omega_o \cos\psi \, d\psi}{\sqrt{1 - (a/r)^2 \sin^2\psi}}$$

After substitution Eq. (44) becomes

$$T(r) = \frac{2}{\pi}\frac{qa}{\lambda}\frac{a}{r}\int_{o}^{\pi/2}\frac{\kappa^2 \cos^2\psi \, d\psi}{\kappa^2\sqrt{1 - (a/r)^2 \sin^2 \psi}} \tag{45}$$

where the numerator and demonimator have been multiplied by κ^2. The numerator inside the integral can be rewritten as

$$(\kappa^2 - 1) + (1 - \kappa^2 \sin^2\psi) \tag{46}$$

where $\kappa = a/r$. With Eq. (46), Eq. (45) reduces to

$$T(r) = \frac{2}{\pi}\frac{qa}{\lambda}\left\{\int_{o}^{\pi/2}\frac{1}{\kappa^2}\frac{(\kappa^2 - 1)\, d\psi}{\sqrt{1 - \kappa^2 \sin^2\psi}} + \int_{o}^{\pi/2}\frac{1}{\kappa^2}\sqrt{1 - \kappa^2 \sin^2\psi} \, d\psi\right\} \tag{47}$$

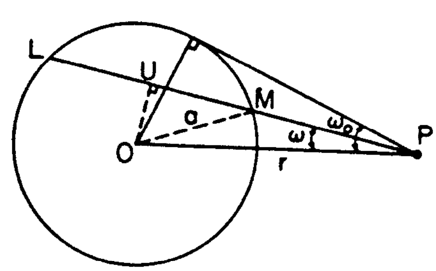

Fig. 10 Circular contact with external point.

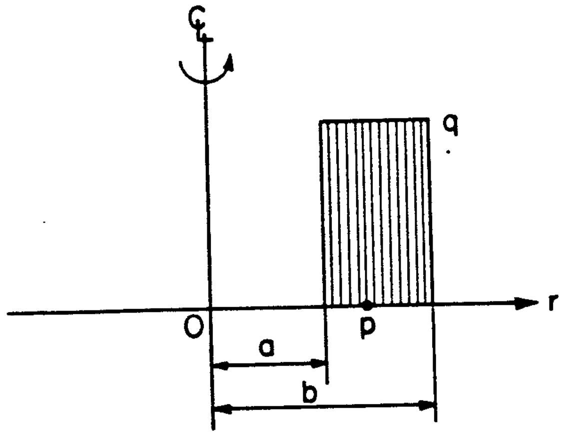

Fig. 11 Coordinates and heat flux distribution for annular contact.

Thus, for $r > a$, we have

$$T(r) = \frac{2}{\pi}\frac{qa}{\lambda}(\kappa)\frac{[E(\kappa) - (1 - \kappa^2)\,K(\kappa)]}{\kappa^2} \tag{48}$$

where K and E are the complete elliptic integrals of the first and second kind of modulus κ. But Eq. (48) can be written in terms of other complete elliptic integrals defined by Jahnke and Emde[9] as

$$T(r) = (2/\pi)(qa/\lambda)\ \kappa\ B(\kappa) \tag{49}$$

where

$$B = K - D, \quad D = (K - E)/\kappa^2 \tag{50}$$

According to Jahnke and Emde, $B \to \pi/4$ as $\kappa \to 0$; therefore, the temperature of $z = 0$ for points far from the center can be approximated as

$$T(r \to \infty) = qa^2/2\lambda r \tag{51}$$

Annular Contact. The temperature at any internal point P within an annular contact area of radii a, b ($a < b$) due to a uniform heat flux distribution over the area $\pi(b^2 - a^2)$ can be determined by superposition of two solutions corresponding to the circular contact area. Figure 11 shows the resultant heat flux distribution due to +q placed over the area πb^2 and −q placed over the area πa^2. The temperature at a point P ($a \leq$

$r \leq b$) is required. ... bution over πb^2, an... of Eq. (42):

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where $\kappa = a/r$.

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Constriction Resist... Distributions

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Rectangular Co... determined by means... average temperature...

$\bar{T} = \frac{1}{ab}\int_0^a\int$

$r \leq b$) is required. P is an internal point for the +q distribution over πb^2, and the temperature can be determined by means of Eq. (42):

$$T(+q) = (2/\pi)(qb/\lambda)E(r/b) \qquad (52)$$

P is an external point for the −q distribution over πa^2, and the temperature can be determined by means of Eq. (48) or Eq. (49) to be

$$T(-q) = \frac{-2}{\pi}\frac{qa}{\lambda}(\kappa)\frac{[E(\kappa) - (1-\kappa^2)K(\kappa)]}{\kappa^2} \qquad (53)$$

or

$$T(-q) = (-2/\pi)(qa/\lambda)\,\kappa\, B(\kappa) \qquad (54)$$

where $\kappa = a/r$.

Adding Eqs. (52) and (53), we have, as the temperature at P,

$$T(r) = \frac{2}{\pi}\frac{qb}{\lambda}\left\{E\left(\frac{r}{b}\right) - \left(\frac{r}{b}\right)E\left(\frac{a}{r}\right) + \left(\frac{r}{b}\right)\left[1 - \left(\frac{a}{r}\right)^2\right]K\left(\frac{a}{r}\right)\right\} \qquad (55)$$

where $a \leq r \leq b$.

An equivalent expression obtained by adding Eqs. (52) and (54) is

$$T(r)\ \frac{2}{\pi}\frac{qb}{\lambda}\left\{E\left(\frac{r}{b}\right) - \left(\frac{a}{b}\right)\left(\frac{a}{r}\right)B\left(\frac{a}{r}\right)\right\} \qquad (56)$$

Constriction Resistances Due to Uniform Heat Flux Distributions

In this section, the thermal constriction resistance of three important shapes (rectangular, circular, and an annular ring) will be evaluated for uniform heat flux. From our definition of constriction resistance, we must evaluate the average contact temperature.

Rectangular Contact. The local contact temperature can be determined by means of Eqs. (28-30). Because of symmetry, the average temperature can be written as

$$\overline{T} = \frac{1}{ab}\int_0^a\int_0^b T(x,y)\,dx\,dy \qquad (57)$$

Table 1 Some values of ψ_r as a function of ε for a rectangular contact

ε	ψ_r
1	0.2366
1.5	0.1919
2	0.1625
3	0.1272
5	0.09128
10	0.05617
100	0.00925

It can be shown that the average contact temperature is[6]

$$\overline{T} = \frac{2}{\pi}\frac{qa}{\lambda}\left\{ \sinh^{-1}\frac{b}{a} + \frac{b}{a}\sinh^{-1}\frac{a}{b} + \frac{1}{3}\frac{a}{b}\left[1 + \left(\frac{b}{a}\right)^3 - \left(1+\left(\frac{b}{a}\right)^2\right)^{3/2}\right]\right\} \tag{58}$$

With the total heat flow rate $Q = 4\,qab$, the dimensionless constriction resistance can be shown to be

$$\psi_r = \frac{1}{2\pi}\left\{ \sinh^{-1}\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\sinh^{-1}\varepsilon + \frac{\varepsilon}{3}\left[1 + \frac{1}{\varepsilon^3} - \left(1 - \frac{1}{\varepsilon^2}\right)^{3/2}\right]\right\} \tag{59}$$

where $\psi_r = \lambda b R_r$ and $\varepsilon = a/b \geq 1$. Some values of ψ_r are given in Table 1 for an interesting range of ε.

<u>Circular Contact</u>. The average temperature of a circular contact subjected to a uniform heat flux is

$$\overline{T} = \frac{1}{\pi a^2}\int_o^a T\, 2\pi r\, dr \tag{60}$$

After substitution of Eq. (42) into Eq. (60), we obtain

$$\bar{T} = \frac{4}{\pi}\frac{qa}{\lambda}\int_0^1 E(\kappa')\,\kappa' d\kappa' \qquad (61)$$

The integral can be evaluated readily by means of Refs. 8 and 9:

$$\int_0^1 E(\phi,\kappa)\,\kappa d\kappa = \frac{1}{3}\left[\frac{\sin^2\phi + 1 - \cos\phi}{\sin\phi}\right] \qquad (62)$$

Since we are considering the complete elliptic integral of the second kind, $\phi = \pi/2$, and by Eq. (62) we have 2/3 as the value of the integral. Thus,

$$\bar{T} = (8/3\pi)(qa/\lambda) \qquad (63)$$

and, therefore, the dimensionless constriction resistance with $Q = q\pi a^2$ becomes[6,10]

$$\psi_c = 8/3\pi^2 \qquad (64)$$

with $\psi_c \equiv \lambda a R_c$.

Annular Contact. The average contact area temperature of an annular contact with uniform heat flux is given by

$$\bar{T} = \frac{2}{(b^2-a^2)}\int_a^b T\,r dr \qquad (65)$$

where T is given by either Eq. (55) or Eq. (56).

If we substitute Eq. (55) into Eq. (65), we obtain the expression[11]

$$\bar{T} = \frac{4}{\pi}\frac{qb}{\lambda}\frac{1}{(b^2-a^2)}\left[I_1 + I_2 + I_3 + I_4\right] \qquad (66)$$

where[11]

$$I_1 = \int_a^b E\left(\frac{r}{b}\right) r\,dr = a^2 \int_1^{\varepsilon} \frac{E(\varepsilon/\kappa)\,d\kappa}{\kappa^3} \qquad (67a)$$

$$I_2 = -\int_a^b \left(\frac{r}{b}\right) E\left(\frac{r}{b}\right) r\,dr = \varepsilon \int_1^{\varepsilon} \frac{K(\kappa)\,d\kappa}{\kappa^2} \qquad (67b)$$

$$I_3 = \int_a^b \left(\frac{r}{b}\right) E\left(\frac{a}{r}\right) r\,dr = -\varepsilon \int_1^{\varepsilon} \frac{K(\kappa)\,d\kappa}{\kappa^4} \qquad (67c)$$

$$I_4 = -\int_a^b \left(\frac{r}{b}\right) \left(\frac{a}{r}\right)^2 K\left(\frac{a}{r}\right) r\,dr = \varepsilon \int_1^{\varepsilon} \frac{E(\kappa)\,d\kappa}{\kappa^4} \qquad (67d)$$

The evaluation of these integrals is given in Ref. 11.

If we let $\varepsilon = a/b < 1$ and write $Q = q\pi b^2 (1-\varepsilon^2)$, the dimensionless constriction resistance ψ_a can be expressed as

$$\psi_a = (8/3\pi^2)(1-\varepsilon^2)^{-2}\left[1+\varepsilon^3-(1+\varepsilon^2)E(\varepsilon)+(1-\varepsilon^2)K(\varepsilon)\right] \qquad (68)$$

Table 2 ψ_a as a function of ε for an annular contact

ε	ψ_a
0.0	0.2702
0.1	0.2695
0.2	0.2680
0.3	0.2667
0.4	0.2660
0.5	0.2666
0.6	0.2691
0.7	0.2746
0.8	0.2858
0.9	0.3109
0.938	0.3306
0.995	0.4506

where $\psi_a \equi \lambda b R_a$. I
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where $\psi_a \equiv \lambda b R_a$. In Eq. (68), $K(\varepsilon)$ and $E(\varepsilon)$ are the complete elliptic integrals of the first and second kind, respectively, having modulus ε. It should be noted that, as $a \to o$ ($\varepsilon \to o$), Eq. (68) yields $\psi_a = 8/3\pi^2$, which is identical to Eq. (64). Some typical values of ψ_a for several values of ε are presented in Table 2. For a detailed discussion of these results, the reader is referred to Ref. 11.

Conclusions

Superposition of heat sources due to arbitrary heat flux distributions over planar contact areas has been used to derive general expressions for evaluating local contact area temperatures and thermal constriction resistances as a function of heat flux distributions over the contact. The special case of uniform heat flux was considered, and a number of important contact area shapes were examined. Thermal constriction parameters were obtained for the rectangular, circular and annular ring contacts.

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