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TRANSIENT HEAT FLOW FROM A THIN CIRCULAR DISK

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Abstract

The transient thermal behavior of a half-space whose plane boundary is impervious to heat transfer, except for a circular contact area, has been considered. Full transient, numerical soltions are presented for a variety of disk boundary conditions. The problem was numerically formulated and solved using oblate spheroidal coordinates. A comparison is made with existing analyses for the special case of a constant, isothermal contact area. Results are also presented for various flux distributions in which case the surface flux rather than the disk temperature is prescribed for all time.

Nomenclature

a	contact radius
c.v.	control volume
C_{1-4}, C_s	finite difference (f.d) coefficients
D	constant term of f.d. equation
Fo	Fourier modulus, $Fo = \alpha t/a^2$
i,j	f.d. nodal indices
I	number of nodes in the η - direction
J	number of nodes in the θ - direction
P	heat generation rate per unit volume and time
q	thermal heat flux
Q	total heat flow rate
r	radial coordinate
R_c	thermal constriction resistance
R_c*	dimensionless thermal constriction resistance, $R_c{*} = R_c ka$
t	time
T	temperature
T_o	disk surface temperature
T°	temperature evaluated at the 'old' time step
V	volume
z	longitudinal coordinate

Greek Letters

α	thermal diffusivity
Γ	contact surface area
Δ	denotes increment of the accompanying argument
ε	transformed variable, $\varepsilon = \sinh \eta$
η,θ	oblate spheroidal variables
η'	transformed variable, $\eta' = \cos \theta$
λ	thermal conductivity
ρ	mass density

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Introduction

For almost four decades a number of engineers, scientists and mathematicians have attempted to determine the transient temperature distribution within a half-space whose plane boundary is impervious to heat transfer except for a circular contact area subjected to 1) a uniform, constant heat flux, or 2) a uniform, constant temperature.

The investigations can be divided into two categories: 1) approximate models [3,6,7,9] valid in a very limited range of Fourier modulus, and 2) rigorous solutions [1,2,4,5,8,10,11] which yield a limited amount of information about the entire temperature field.

Holm [3] appears to have initiated the method of replacing the circular contact by a hemispherical contact whose radius is chosen such that under a constant flux boundary condition and infinite time its thermal constriction resistance is equal to the known steady-state isothermal disk resistance. Davies [6], Clark and Powell [7], and Heasley [9] also replaced the actual contact area by a hemispherical contact area and solved the diffusion equation formulated in spherical coordinates. This method is approximately valid for long times, but is inappropriate for short times.

On the other hand, Blok [1] was the first to attempt to find a solution for the actual geometry and a constant flux. Posing the problem in circular cylinder coordinates, and making use of Green's function for a point source of unit strength, he obtained an integral expression for the transient temperature at the center of the contact. This integral was subsequently evaluated numerically and a plot of the temperature rise presented.

Lowan [2] used a Laplace transform to obtain a solution for the uniform flux condition. He did not attempt to evaluate the complex integralresulting from his analysis.

Oosterkamp [4] also examined the uniform flux condition. Thomas [5] followed the method of Blok [1] and obtained a more complex integral expression for the transient temperature at the center of the contact area which is exposed to a uniform flux as well as radiative cooling.

Norminton and Blackwell [8] and later Blackwell [10] were the first investigators to seek solutions to the diffusion equation formulated in oblate spheroidal coordinates. Blackwell recognized that this coordinate system was appropriate for the circular contact, because the mixed boundary conditions on the plane containing the contact area make it exceedingly difficult to obtain analytical solutions in the conventional coordinate systems. In his first paper [8] solutions were obtained only for a very long time (Fo > 4), and in the second paper [10], solutions for very short times (Fo < 0.1) were obtained. These solutions pertain to the constant temperature condition only. The reported expressions cannot be used to determine the transient thermal constriction resistance.

Blackwell [10] concluded that, "An exact solution in closed form of the complete problem seems impossibly difficult to obtain, and, if required, the obvious way to improve on these results is by means of a full numerical treatment".

Keltner [11] in a recent note reported on an approximate solution for the isothermal boundary condition using the heat balance integral method in oblate spheroidal coordinates.

The objective of the present paper is to obtain a complete solution of the transient field and the thermal constriction resistance of a circular contact area subjected to various boundary conditions. The resistance will be determined by means of a finite-difference solution of the diffusion equation expressed in oblate spheroidal coordinates.

Mathematical Description

Problem Statement

We consider the thermal constriction resistance to transient conduction from a circular area placed on the surface of a half-space. Initially the temperature everywhere within the half-space is zero. The surface of the half-space outside the contact area is impervious to heat transfer for all time, while the contact area will be subjected to various boundary conditions.

When the transient temperature field has been determined subject to a prescribed boundary condition over the contact area, the average temperature of the contact area as well as the total instantaneous heat transfer rate through the contact can be calculated. By means of the average temperature and the total heat flow rate, the dimensionless transient constriction resistance can be determined as a function of the Fourier modulus.

Differential Equation and Boundary Conditions

To put the problem into perspective, the governing differential equation and the boundary conditions will be given in both circular cylinder and oblate spheroidal coordinates.

If we assume the temperature field to be axisymmetric, the diffusion equation in circular cylinder coordinates can be written as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

Equation (1) will be subject to the following initial and boundary conditions:

i) $t = 0,\ z \geq 0,\ r \geq 0;\quad T = 0$

ii) $t > 0,\ z = 0,\ 0 \leq r \leq a;$

one of: a) temperature is uniform, $T = T_o$

b) $q = q(r)$ including the special case

$$\text{where } q = \frac{Q}{\pi a^2} \tag{2}$$

iii) $t > 0,\ z = 0,\ r > a$, surface is insulated, $\frac{\partial T}{\partial z} = 0$

iv) $t > 0,\ \sqrt{r^2 + z^2} \to \infty,\ T \to 0$

v) $t > 0,\ r = 0,\ \partial T/\partial r = 0.$

Since it is not possible to obtain a complete analytical solution to Eq. (1) because of the mixed boundary conditions, it is advisable to transform this equation to oblate spheriodal coordinates using the relationships:

$$r = a \cosh \eta \sin \theta$$
$$z = a \sinh \eta \cos \theta \tag{3}$$

Equation (1) then transforms into the equivalent diffusion equation [12]:

$$\frac{1}{a^2(\cosh^2\eta - \sin^2\theta)}\left[\frac{\partial^2 T}{\partial \eta^2} + \tanh\eta\,\frac{\partial T}{\partial \eta} + \frac{\partial^2 T}{\partial \theta^2} + \cos\theta\,\frac{\partial T}{\partial \theta}\right] = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{4}$$

Equation (4) could have been derived by means of a heat balance on a typical elemental volume formulated in oblate spheroidal coordinates.

The initial and boundary conditions now become:

i) $t = 0,\ 0 \leq \eta < \infty,\ 0 \leq \theta \leq \pi/2;\ T = 0$

ii) $t > 0,\ \eta = 0,\ 0 \leq \theta \leq \pi/2;$

one of a) $T = T_o$

b) $q = \frac{-k}{a \cos\theta}\frac{\partial T}{\partial \eta} = q(\theta)$ and including the

special case where $q = \frac{Q}{\pi a^2}$ (5)

iii) $t > 0,\ \theta = \pi/2,\ z > 0;\ \frac{\partial T}{\partial \theta} = 0$

iv) $t > 0,\ \eta \to \infty,\ 0 \leq \theta \leq \pi/2;\ T \to 0$

v) $t > 0,\ \theta = 0,\ \eta > 0;\ \frac{\partial T}{\partial \theta} = 0$

Thermal Constriction Resistance

The transient constriction resistance is defined as the average temperature of the contact area divided by the instantaneous total heat transfer rate through the contact:

$$R_c = \frac{1}{\Gamma} \frac{\int_\Gamma T \, d\Gamma}{\int_\Gamma q \, d\Gamma} \qquad (6)$$

where Γ is the contact area. For the circular contact, $\Gamma = \pi a^2$ and the elemental area is

$$d\Gamma = 2\pi r dr = 2\pi a^2 \sin\theta \cos\theta d\theta \qquad (7)$$

If we further define a dimensionless constriction resistance $R_c^* = R_c ka$, then Eq. (6) with Eq. (7) yields two expressions:

$$R_c^* = \frac{1}{\pi a} \frac{\int_0^a T r dr}{\int_0^a -\frac{\partial T}{\partial z} r dr} \qquad (8)$$

or equivalently,

$$R_c^* = \frac{1}{\pi} \frac{\int_0^{\pi/2} T \sin\theta \cos\theta d\theta}{\int_0^{\pi/2} -\frac{\partial T}{\partial \eta} \sin\theta d\theta} \qquad (9)$$

Since equation (9) is valid for the solution obtained by means of oblate spheroidal coordinates, it will be used in all subsequent calculations.

Coordinate Selection for Numerical Solution

Due to the inability of present analytic methods for providing a solution to the governing differential equation (1) or (4) and boundary conditions (2) or (5) of the problem under consideration, a numerical solution was sought to describe the thermal behavior of the problem. The finite difference method of solution was used in this analysis.

If the finite difference spatial discretization of the governing differential equation is performed in either the circular cylinder or cartesian coordinate system, the discontinuity occurring in the thermal boundary condition specification at the disk outer edge for all time $t > 0$ gives rise to considerable numerical difficulty. This is due to the second order truncation error inherent in the expansion of the conduction equation which cannot adequately approximate the large adjustment of the thermal flow field in the vicinity of this discontinuity. Extremely fine numerical detail must be used to adequately describe the thermal behavior in this region. While a sufficiently large computing facility may allow for a sufficient degree of detail at the expense of considerable additional and unjustified computer time, our experience indicates that the numerical solution of this problem is not feasible at present.

It has been shown [12] that for the special case of steady-state heat transfer from an isothermal disk the problem and solution become one-dimensional if oblate spheroidal coordinates are used in the formulation of the problem. While this degree of simplification cannot be achieved for other boundary conditions and indeed not even for an isothermal boundary condition if transient behavior is considered, it is expected that the flow of heat into the semi-infinite region will maintain a dominant flow direction which will be near that of the steady-state heat transfer. This hypothesis has been used successfully to advantage in reference [13] for steady-state, finite-difference solutions with arbitrary flux distributions over the contact area.

The general formulation of the finite difference representation of the heat conduction equation in general orthogonal curvilinear coordinates developed in references [14] has been used in this analysis to determine the finite difference equations governing the heat transfer within the discretized spatial domain. For the reasons mentioned above, attention has been directed to the oblate spheroidal coordinate system.

Finite Difference Formulation

With an arbitrary control volume centered about a point in space characterized by the notation (i,j), figure 1, i and j shall be used to indicate the finite discretization of space in the η and θ directions respectively. For uniform spacing in each direction independently, successive nodes in the η and θ directions will be incremented by amounts $\Delta\eta$ and $\Delta\theta$ respectively. Note that these increments do not represent the physical distances separating adjacent nodes but merely the change in the respective coordinate value between these nodes.

An energy balance for an arbitrary control volume with a total source strength $P\Delta V$, and centered about nodes (i,j) can be written as

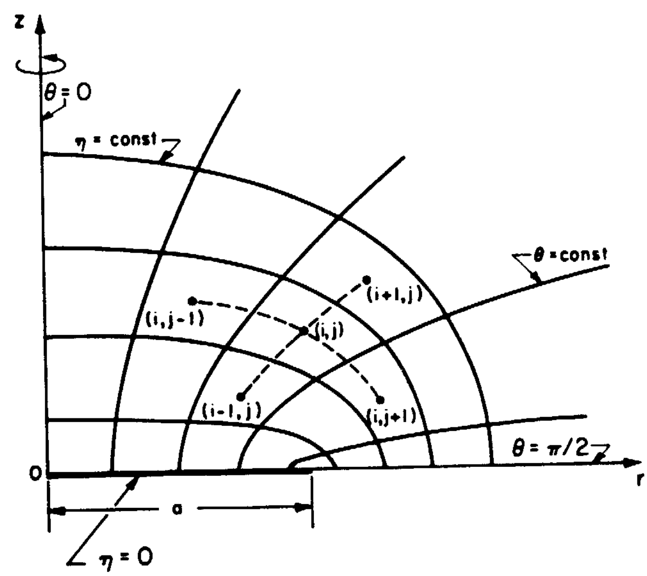

Fig. 1. Problem Geometry in Oblate Spheroidal Coordinates.

$$Q_{i-\frac{1}{2},j} - Q_{i+\frac{1}{2},j} + Q_{i,j-\frac{1}{2}} - Q_{i,j+\frac{1}{2}} + P\Delta V$$

$$= \frac{\partial}{\partial t} \iiint_{c.v} \rho C_p \, T dV \qquad (10)$$

where due to the axi-symmetric nature of the problem only four heat flow rates have been considered.

According to reference [14] this energy balance can be expressed in terms of the nodal temperatures neighboring node (i,j) by the equation

$$C_s T_{i,j} = C_1 T_{i-1,j} + C_2 T_{i+1,j} + C_3 T_{i,j-1} + C_4 T_{i,j+1} + D \qquad (11)$$

where after some manipulation and division by common factors the coefficients can be reduced to

$$C_1 = 1/(\Delta\eta)^2 - \tanh \eta_i/2\Delta\eta)$$

$$C_2 = 1/(\Delta\eta)^2 + \tanh \eta_i/2\Delta\eta)$$

$$C_3 = 1/(\Delta\theta)^2 - \cot \theta_j/(2\Delta\theta)$$

$$C_4 = 1/(\Delta\theta)^2 + \cot \theta_j/(2\Delta\theta)$$

$$D = (\cosh^2\eta_i - \sin^2 \theta_j)T^\circ_{i,j}/(\Delta Fo) \qquad (12)$$

$$+ \frac{P\Delta V}{2\pi \ ka \cosh \eta_i \sin \theta_j \ \Delta\eta \ \Delta\theta} \quad \text{or equivalent source term}$$

$$C_s = \sum_{n=1}^{4} C_n + (\cosh^2 \eta_i - \sin^2 \theta_j)/\Delta Fo$$

where $\Delta Fo = \frac{\alpha\Delta t}{a^2}$ and $\alpha = \frac{k}{\rho C_p}$.

In the above the superscript (o) has been used to indicate that the quantity of interest should be evaluated at the time step immediately preceding the one under consideration.

Boundary Conditions

Having determined the nodal coefficients for the generalized internal node, we consider now those special control volumes which have one or more surfaces contacting physical boundaries of the total thermal system. The boundary control volumes of concern in this analysis and their associated thermal boundary conditions are illustrated in figures 2(a) - 2(d). Only those coefficients corresponding to inter-nodal links crossing a boundary will be considered here since only they will have their values altered by the presence of the boundary.

(a) $\theta = 0$

Due to the problem symmetry about the vertical axis, the line defined by the equation $\theta = 0$ will represent a zero flux boundary. This condition is reflected by the statement that

$$Q_{i,j-\frac{1}{2}}\Big|_{j=1} = 0 \qquad (13)$$

The effect of this condition on the finite difference equation (11) is that we must set

$$C_3 = 0 \qquad (14)$$

(b) $\theta = \pi/2$

This boundary is also a zero flux boundary by its definition in the problem statement. For this boundary then we set

$$C_4 = 0 \qquad (15)$$

(c) $\eta \to \infty$ ($\eta \simeq 8$)

For this boundary, illustrated in figure 2(c), a temperature is prescribed and maintained for all time for $\eta \to \infty$. It was found numerically and can be illustrated analytically that it is sufficient, for the accuracy of the present solution to take $\eta = 8$ to be the numerical equivalent of $\eta \to \infty$. For convenience a value of $\eta = 8.5$ was used in this analysis. In this manner this boundary condition requires no special treatment other than to ensure that the temperature for all nodes (I + 1, j) are assigned and maintained at the appropriate value.

(d) $\eta = 0$

(i) Specified flux

The heat flux from the disk surface into the control volume is given by

$$Q_{\frac{1}{2},j} = \pi a^2 \int_{\theta_j - \frac{\Delta\theta}{2}}^{\theta_j + \frac{\Delta\theta}{2}} q\ (\theta) \sin\ (2\theta)\ d\theta \qquad (16)$$

This represents a rate of heat addition to the control volume and its direction need not be specified. The influence of this boundary condition can then be included simply by the assignment to these control volumes of a total source strength of equivalent magnitude. Thus the assignment is made

$$P\Delta V = \pi a^2 \int_{\theta_j - \frac{\Delta\theta}{2}}^{\theta_j + \frac{\Delta\theta}{2}} q\ (\theta) \sin\ (2\theta)\ d\theta \qquad (17)$$

and the constant D of equation (11) becomes

$$D = (\cosh^2\eta_i - \sin^2\theta_j)\ T^\circ_{i,j}/(\Delta Fo)$$

$$+ \frac{a}{2k\cosh(\Delta\eta/2)\sin(\theta_j)\Delta\eta\Delta\theta} \int_{\theta_j-\Delta\theta/2}^{\theta_j+\Delta\theta/2} q(\theta)\sin(2\theta)d\theta \qquad (18)$$

Including the effect of this condition in this manner does not involve directly the temperature gradients at the disk surface so that a further inclusion of the coefficient C_1 in our nodal equation would entail a redundant specification of $Q_{\frac{1}{2},j}$. To avoid this we set

$$C_1 = 0 \tag{19}$$

for these boundary nodes.

(ii) Temperature specified

In a similar fashion to the condition for $\eta \to \infty$, this condition requires no special consideration except to determine the appropriate value of C_1 for the corresponding nodal equations. For this set of control volumes the coefficient C_1 can be taken as

$$C_1 = 2\left[\frac{1}{\Delta\eta^2} - \frac{\tanh\,(\Delta\eta/2)}{2\Delta\eta}\right] \tag{20}$$

In each of the cases considered above, the central nodal coefficient, C_s, is to be determined from equation (12).

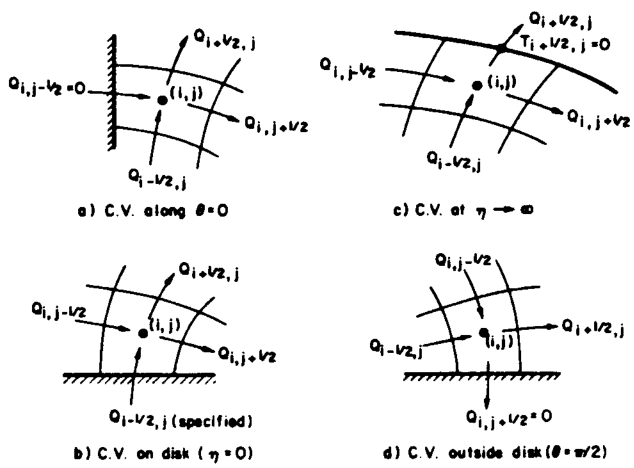

Fig. 2. Boundary Conditions

Numerical Solutions

In the numerical solutions of the transient thermal behavior for heat flow from a circular disk conducting to a semi-infinite medium, a grid arrangement was selected which has 20 spatial increments in the η-direction and 10 spatial increments in the θ-direction for the cross-section of figure 1. Experience with the steady-state solution of this problem [13] has indicated that the error inherent in this grid arrangement will be less than two per cent for the boundary conditions under consideration.

The tridiagonal matrix algorithm was applied on a line-by-line basis throughout the discretized spatial domain to effect the field solutions. Successive over-relaxation was used with a relaxation factor of 1.5 at each time step in the solution. Experience with the steady-state solution has indicated extremely rapid and stable convergence characteristics when this method is used [13].

An estimate of the time-dependent error is difficult to establish for this problem, however, since there are presently no rigorous solutions available encompassing the complete range of the Fourier modulus considered to be relevant to the application of these results. A check was incorporated into the solution algorithm to minimize the propagation of time-dependent errors and the excellent agreement obtained between asymptotic steady-state results and the exact solution for these two boundary conditions suggests that this check was successful in preventing the propogation of significant time-dependent errors.

The check was arranged such that for each time sub-interval considered, a tentative solution was obtained for the 'new' Fourier modulus. This sub-interval was then halved and a new estimate obtained for the new Fourier modulus through the use of an intermediate time plane. If the two solutions thus obtained agreed favorably the latter one was accepted as the solution for the new time plane. If, on the other hand, an unacceptable agreement was obtained the latter solution was maintained and compared with the solution resulting from a further halving of the time scale increment. This procedure was continued until an acceptable agreement was obtained and the most recent solution was accepted as the solution for the time plane of interest. The criterion used for solution acceptance was that the maximum difference in temperature at any point in the field between the most recent two trial solutions be less than 5.0 percent when normalized with respect to the change in temperature occurring at that location between the two time planes under consideration.

Results

Figure 3 shows thermal resistance as a function of Fourier modulus for five different disk heat flux specifications, namely,

i) the isothermal disk

ii) uniform heat flux ($q = 100$)

iii) $q = 200\cos^2(\theta)$

iv) $q = 63.67\tan(\theta)$

and v) $q = 50\sec(\theta)$

All flux distributions were normalized so that the total thermal loading was the same. The first two cases were chosen because they represent the two classical solutions of this problem. In both cases the steady-state resistance is known analytically and in the case of the isothermal disk considerable attention has been given to obtaining approximate analytical solutions for the transient resistance. The steady-state resistance for case three is also known [13], but more importantly it represents a thermal loading such that the steady-state resistance is above that of the constant flux case. The fourth flux distribution yields a thermal resistance below that of the isothermal disc for steady-state conditions. The sec θ distribution is the steady-state flux distribution for the isothermal disk.

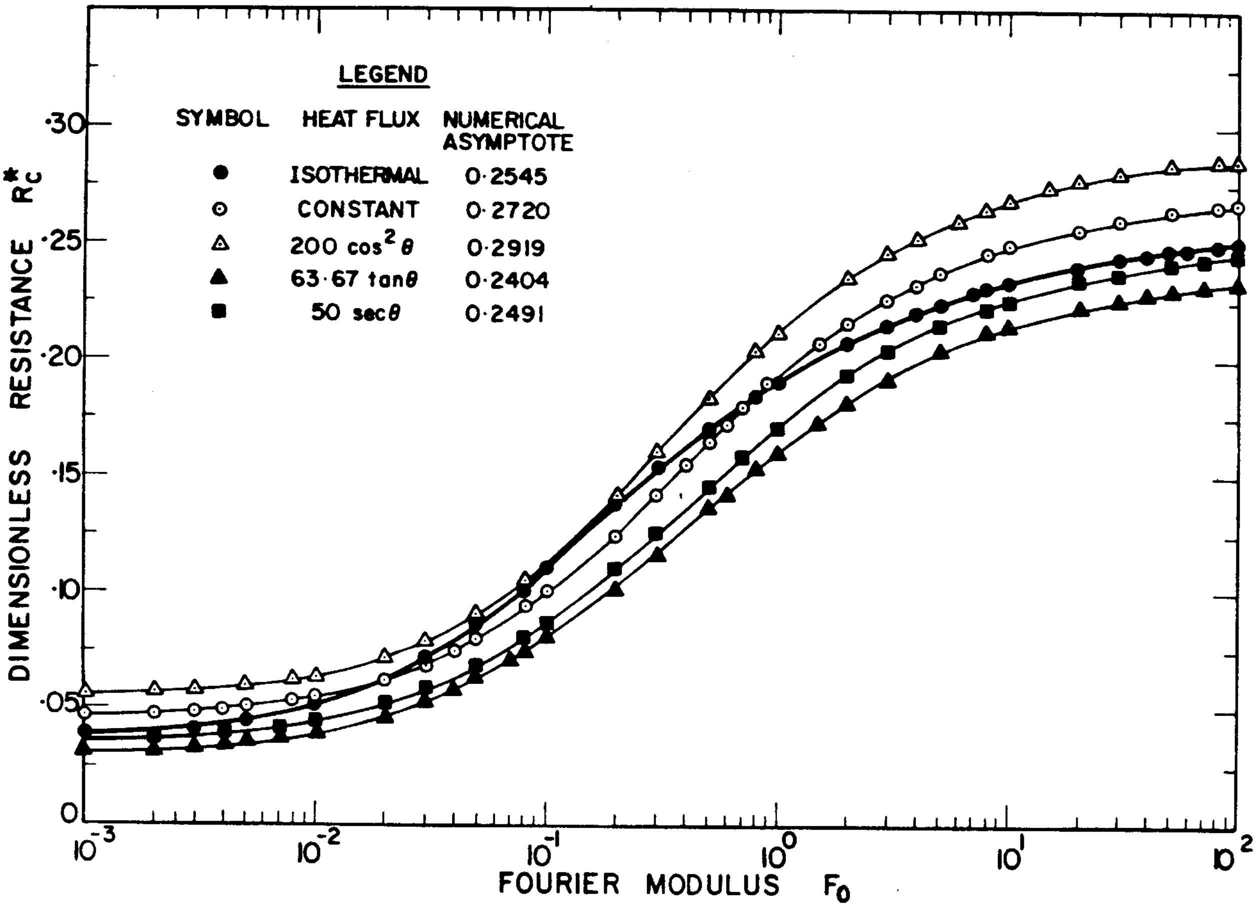

Fig. 3. Dimensionless Resistance vs. Fourier Modulus for the Various Disk Boundary Conditions

The numerical solutions yield asymptotic long time thermal resistances in the first three cases which are $R_c^* = 0.2545$, 0.2720 and 0.2919, respectively. These correspond to steady-state values which are within 1.8% 0.7% and 1.3% of the exact values respectively. The fourth case yields $R_c^* = 0.2402$ which is within 0.7% of the numerical value obtained from a purely steady-state solution, and the fifth case yields $R_c^* = 0.2490$ which is within 0.5% of the true value. We note that solutions extend over a large number of decades and we have presented the solution only for the range of practical importance. As a comment on the methods of error control used in the generation of the numerical results we note the accuracy of the steady-state solution, the fact that the solution satisfies the initial condition $T = 0$, $R_c^* = 0$ and thirdly the monotonic smoothness of the intermediate solution. We expect that the local error at any Fourier modulus will be less than 1.8%.

Discussion

The comparison of the present results with the results of the workers discussed in the introduction are now considered. Unfortunately, comparisons with field temperatures are limited since in many cases no direct method is given to calculate the field temperatures at specific spatial locations. Conversely, Blackwell's analysis yields field temperatures at arbitrary spatial points but the results cannot be used to calculate the thermal resistance.

Norminton and Blackwell's [8] long time solution is shown in figures 4 and 5 compared to the present numerical results. Four field positions have been chosen corresponding to node points (η, θ) given by (0.2125, $\pi/40$), (1.4875, $\pi/40$) in figure 4 and by (0.2125, 19 $\pi/40$), (1.4875, 19 $\pi/40$) in figure 5. The results of Norminton and Blackwell were computed by using the equations given in their paper noting the difference in coordinate definition where their coordinates ε, η' are related to the present η, θ by the relations $\varepsilon = \sinh(\eta)$ and $\eta' = \cos\theta$.

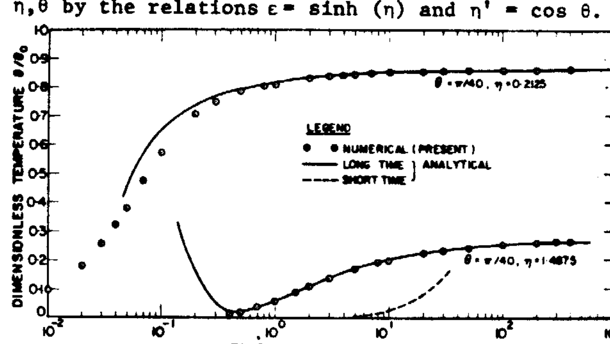

Fig.4. Comparison of Present Results with Ref. [8, 10] for $\theta = \pi/40$, $\eta = .2125$ and $\eta = 1.4875$.

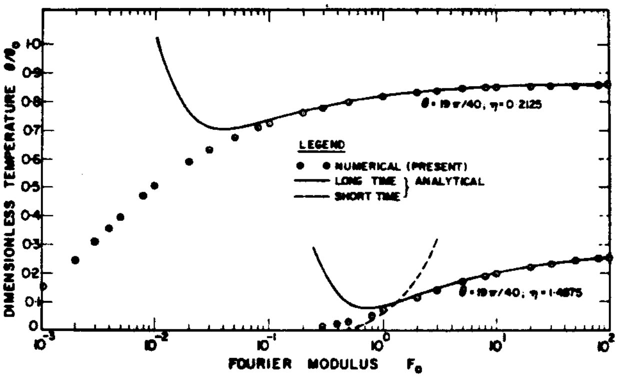

Fig. 5. Comparison of Present Results with Ref. [8, 10] for $\theta = 19\pi/40$, $\eta = .2125$ and $\eta = 1.4875$.

It can be seen that the long time approximation yields excellent agreement as $Fo \to \infty$ and that the range of agreement depends on the distance from the contact. The authors suggested that when $Fo/\varepsilon^2 \simeq 1.0$ the error of their approximation should be less than 10%. The present results essentially confirm this and show a difference of $\simeq$ 10% for (0.2125, 19 /40) and $\simeq$ 1% for (1.4873, 19 π/40) when the condition $Fo/\varepsilon^2 \simeq 1$ is imposed. The difference is slightly less for the results of figure 5 when $\theta \to 0$. In general the stated validity of the long time expansion for $Fo \geq 4$ is supported by the present results.

Also shown on these figures (dashed line) is Blackwell's short time solution [10]. It is readily apparent that although it must be asymptotically correct as $Fo \to 0$ it does not agree with the present results for the range of variables shown. Our present results showed that agreement was possible only for Fourier modulii much less than 0.001 which corresponds to temperature ratios of a similar order. The validity of the numerical results are certainly in question for these orders of magnitude of temperature and since such temperatures are beyond any conceivable practical use the matter was not pursued.

We now turn our attention to the variation of dimensionless resistance with Fourier modulus, figure 6. Direct comparisons for these reults are difficult since, in spirit, the apparent attempt of the several workers discussed at the outset was to force their resistance to an asymptotic long time value of that corresponding to an isothermal disk. However, the analyses generally rely on specifying the heat flux on some intermediate boundary; on a hemisphere for example. The resulting transient solution is therefore representative of a hybrid isothermal-constant heat flux problem.

The true fully transient isothermal solution is the case where temperature is specified on the disk and the heat flux is transient. The comparison most representative of the hybrid analytical approximation is case v) where the heat flux equivalent to that of the isothermal disk in the steady-state is prescribed on the disk. The application of such a heat flux for all time, however, will not permit the disk to be isothermal except in the steady state.

Both results are compared to the results of Oosterkamp [4], Holm [3] and Heasley [9]. As far

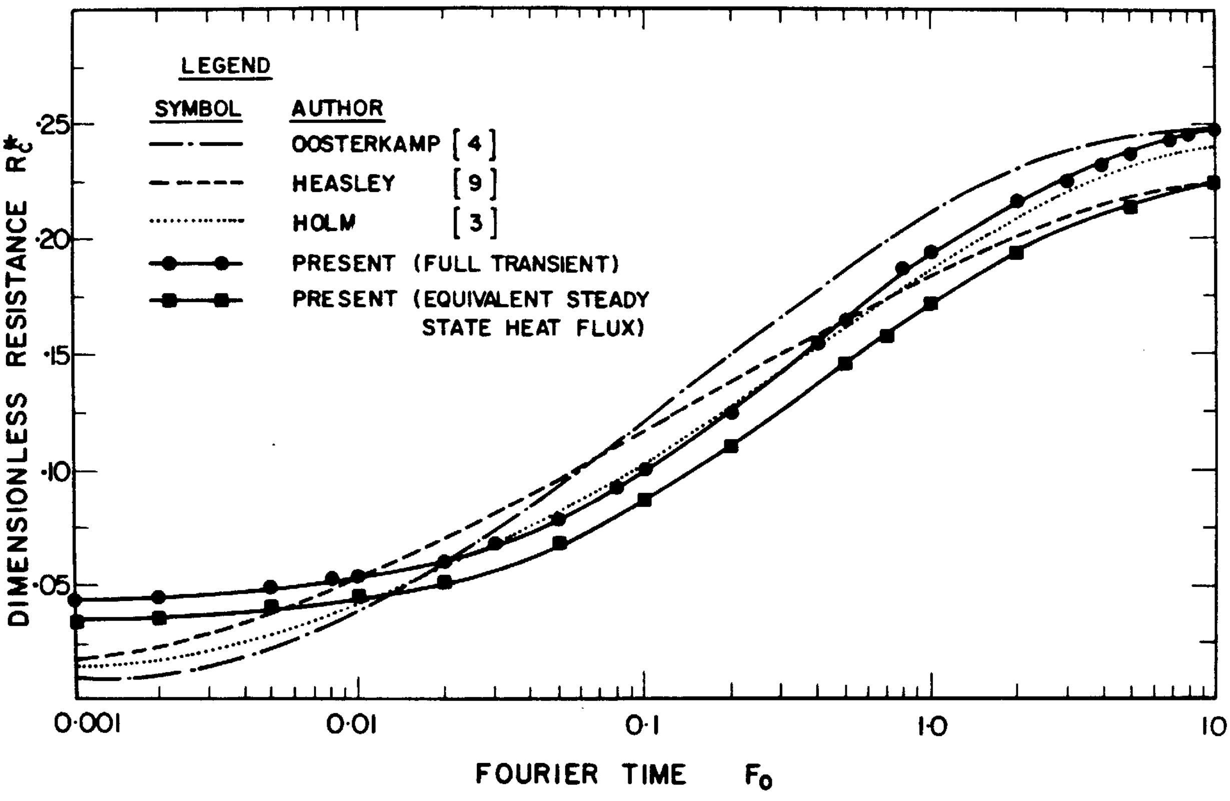

Fig. 6. Comparison of Present Results with the Approximate Solutions for Isothermal Contact.

as known, solutions for the transient resistance for the other cases studied here have not been reported by other workers so that additional comparisons are not possible.

Conclusions

The full transient solution for the resistance to heat flow of a circular disk supplying heat to a half-space has been formulated and solved numerically for a variety of disk heat flux distributions. The results are presented in figure 3.

The solution for the classical case of an isothermal disk has been compared to several approximate solutions available in the literature. It is concluded that the general 's-curve' characteristics of the true solution are displayed by each of the approximate analyses considered. However, in predicting the dimensionless thermal resistance corresponding to any Fourier modulus using these analyses may lead to considerable error. This error varies depending on the Fourier modulus under consideration and the approximate solution used.

In the region where $Fo \leq .01$ all the approximate solutions are poor and errors of greater than 100% can easily be found, while moderate errors, in the order of 20%, are found in the mid Fourier modulus range. Since solutions are hinged at the known steady-state values, lower errors are found for $Fo > 10$, going to zero as steady-state is approached.

An examination of the available solutions indicates that Holm has provided the approximate solution of greatest utility. Figure 6 illustrates the close agreement of Holm's results with the present solution for all $Fo \geqslant .02$. Agreement is within approximately 5% throughout this range of practical utility.

The present analysis, in addition to determining the dimensionless thermal constriction resistance will also provide the field temperatures of the system at each Fourier modulus considered. This information, though not reported here, can then be used to establish 'penetration depth' versus Fourier modulus relationships, or 'time-to-temperature' versus temperature relationship for fixed field locations. This information is of importance in the analysis of composite systems.

Also presented in this work, in addition to the isothermal disk full transient solution, are solutions corresponding to four disk heat flux distributions. These distributions were applied over the full Fourier range and represent cases whose steady-state solutions exhibit considerable variation from the classical solutions.

A unique feature of the finite difference solution technique is the use of oblate spheroidal coordinates as the basis for modelling the heat flow. In this system the steady-state, isothermal disc problem becomes one-dimensional. Although all other cases (including transient isothermal disk) are two-dimensional, the coordinate system is 'quasi-natural' and considerable savings can be realized through its use. This is reflected in the numerical solution of the problem by a decrease in the computational time required for solution. In addition, the coordinate system can naturally account for the discontinuity of heat flux occurring at the disc outer edge and will ultimately lead to a smaller truncation error since the coordinate system flows more naturally with the heat near this discontinuity.

Acknowledgements

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