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# TRANSIENT THERMAL RESPONSE OF TWO BODIES COMMUNICATING THROUGH A SMALL CIRCULAR CONTACT AREA

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**Abstract**—The transient thermal response of two semi-infinite bodies, initially at uniform but different temperatures, and which can communicate through a small circular contact area, is considered. The numerical solution procedure is formulated in oblate spheroidal coordinates. Full transient, numerical solutions are presented for materials combinations spanning the range of practical interest to thermal engineers. A correlation equation is presented which correlates the constriction resistance as a function of the Fourier modulus for the range of physical parameters considered. The correlation is within $\pm 7\%$ which is considered adequate for engineering calculations.

## NOMENCLATURE

$a$, contact radius;
$c_p$, specific heat at constant pressure;
C.V., control volume;
$C_{1-4}, C_5$, finite difference coefficients;
$D$, constant term in finite difference equation;
$Fo$, Fourier modulus, $Fo = \alpha_1 t/a^2$;
$i, j$, finite difference nodal indices;
$k$, thermal conductivity;
$P$, heat generation per unit volume and time;
$q$, thermal heat flux;
$Q$, total heat flow rate;
$r$, radial coordinate;
$R_c$, thermal constriction resistance;
$R_{c_\infty}$, steady-state thermal constriction resistance;
$t$, time;
$T$, temperature;
$T^o$, temperature evaluated at the 'old' time plane;
$V$, volume;
$X$, independent time variable, defined in text;
$z$, axial coordinate.

Greek symbols

$\alpha$, thermal diffusivity;
$\bar{\alpha}$, harmonic mean diffusivity, $\bar{\alpha} \equiv 2\alpha_1\alpha_2/(\alpha_1+\alpha_2)$;
$\Gamma$, contact area;
$\Delta$, denotes increment of the accompanying argument;
$\eta, \theta$, oblate spheroidal coordinates;
$\rho$, mass density.

Subscripts

1, 2, referring to body one or two;
$\bar{\alpha}$, based on harmonic mean diffusivity.

## INTRODUCTION

WHEN predicting the thermal resistance to heat transfer across lightly loaded joints placed in a vacuum environment, the thermal designer must be concerned with accounting for the effect of the microcontact areas. It is commonly assumed that the microcontact areas are circular in cross-section, and uniformly distributed over the apparent contact area. The validity of this assumption, of course, is dependent on such factors as the overall surface geometry and the loading state of the members. For lightly loaded joints, it is further assumed that the contact size is small when compared to typical contact-to-contact spacings or to the thickness of the contacting solids. Consequently the problem is reduced to the examination of a single contact area which can be considered to be located on the common boundary between two adjacent half-spaces. This is depicted schematically in Fig. 1. Outside of the contact area, on the boundaries of the two half-spaces, it is assumed that no heat transfer occurs, the radiative communication between the two bodies being neglected.

The steady-state problem has been previously analysed by several authors [1–6], each of whom examined the situation where the contact area is located on a single half-space and the boundary conditions are applied over the entire contact plane, including the contact area. In a recent paper [7], Schneider *et al.* examined the transient problem in a similar fashion, by

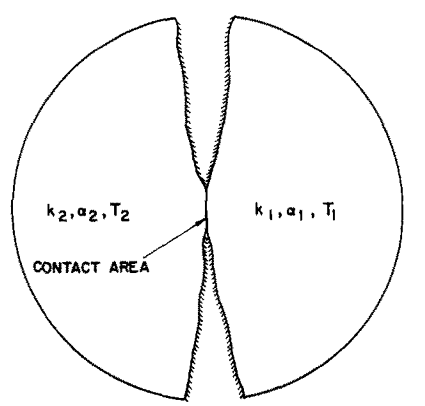


FIG. 1. Single contact formed between two bodies.

considering a single half-space and applying various boundary conditions directly over the contact area. This is consistent with previous investigations of the transient problem [8–17].

Consideration of the complete transient behavior, where the two contacting half-spaces are continuously allowed to communicate with each other through their mutual contact area, adds additional complication to the problem as a result of the differing thermal properties of the two contacting bodies. This is the situation of practical import, however, since only with completely mutual thermal communication between the two contacting bodies will any real process occur. Certain limiting cases will undoubtedly permit the isolation of the two bodies for analysis purposes, but in general this is not the case.

The purpose of this work, therefore, is to examine the complete transient thermal behavior of a system in which two bodies, having different temperatures and thermal properties, come into contact, thermally communicating through a small circular contact area.

## MATHEMATICAL STATEMENT OF THE PROBLEM

### *Problem statement*

We consider the thermal constriction resistance to transient heat conduction which occurs when two semi-infinite bodies, initially at uniform but different temperatures are brought into contact. The two bodies are allowed to communicate thermally through a small circular contact area, with the remaining surface of the contact plane of the two half-spaces assumed to be perfectly insulated for all time. Far from the contact area, deep within each of the two bodies, the temperature remains constant at its initial value.

When the temperature field has been determined at any time plane in the solution, the total instantaneous heat-transfer rate through the contact can be calculated. The instantaneous thermal constriction resistance can then be defined as the temperature difference between the two bodies, evaluated far from the contact, divided by the instantaneous total heat flow rate.

### *Differential equation and boundary conditions*

Following the arguments presented in [6] and [7], the problem is formulated mathematically using the oblate spheroidal coordinate system. The problem geometry and coordinate system are illustrated in Fig. 2. The oblate spheroidal coordinates readily accommodate the boundary condition discontinuity which exists over the contact plane. In Fig. 2, $\eta_1$ and $\theta_1$ are the oblate spheroidal coordinates within body one, and $\eta_2$ and $\theta_2$ those within body two.

The transformation which relates the oblate spheroidal coordinates to the circular cylinder coordinate system is given by [18]

$$\begin{aligned} r &= a\cosh\eta\sin\theta \\ z &= a\sinh\eta\cos\theta \end{aligned} \tag{1}$$

with axi-symmetric heat transfer considered. The governing differential equations describing the transient heat flow within the two bodies can then be written in

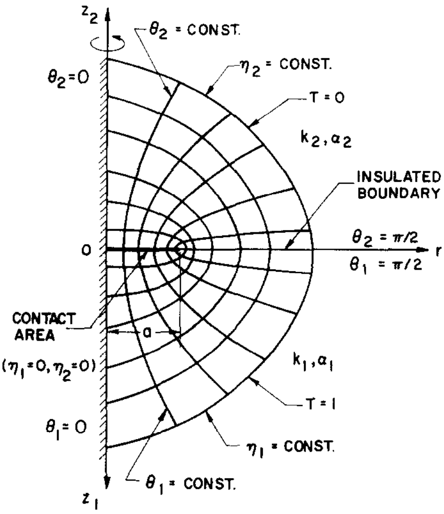


FIG. 2. Problem geometry and coordinate systems.

terms of the oblate spheroidal coordinates, $\eta$ and $\theta$. These equations are [18]

$$\frac{1}{(\cosh^2\eta_1-\sin^2\theta_1)}\left[\frac{\partial^2 T_1}{\partial\eta_1^2}+\tanh\eta_1\frac{\partial T_1}{\partial\eta_1}+\frac{\partial^2 T_1}{\partial\theta_1^2}+\cos\theta_1\frac{\partial T_1}{\partial\theta_1}\right]=\frac{\partial T_1}{\partial Fo} \tag{2}$$

valid within body one, and

$$\frac{1}{(\cosh^2\eta_2-\sin^2\theta_2)}\left[\frac{\partial^2 T_2}{\partial\eta_2^2}+\tanh\eta_2\frac{\partial T_2}{\partial\eta_2}+\frac{\partial^2 T_2}{\partial\theta_2^2}+\cos\theta_2\frac{\partial T_2}{\partial\theta_2}\right]=\frac{\alpha_1}{\alpha_2}\frac{\partial T_2}{\partial Fo} \tag{3}$$

which is valid within body two. The Fourier modulus, $Fo$, is defined by

$$Fo \equiv \alpha_1 t/a^2. \tag{4}$$

The initial boundary conditions are given by

$$\begin{aligned}
&\text{(i) } Fo = 0,\ 0 \leqslant \eta_1 \leqslant \infty,\ 0 \leqslant \theta_1 \leqslant \pi/2,\ T_1 = 1\\
&\text{(ii) } Fo = 0,\ 0 \leqslant \eta_2 \leqslant \infty,\ 0 \leqslant \theta_2 \leqslant \pi/2,\ T_2 = 0\\
&\text{(iii) } Fo > 0,\ \eta_1 = \infty,\ 0 \leqslant \theta_1 \leqslant \pi/2,\ T_1 = 1\\
&\text{(iv) } Fo > 0,\ \eta_2 = \infty,\ 0 \leqslant \theta_2 \leqslant \pi/2,\ T_2 = 0\\
&\text{(v) } Fo > 0,\ 0 < \eta_1 \leqslant \infty,\ \theta_1 = 0,\ \frac{\partial T_1}{\partial\theta_1} = 0\\
&\text{(vi) } Fo > 0,\ 0 < \eta_2 \leqslant \infty,\ \theta_2 = 0,\ \frac{\partial T_2}{\partial\theta_2} = 0\\
&\text{(vii) } Fo > 0,\ 0 < \eta_1 \leqslant \infty,\ \theta_1 = \pi/2,\ \frac{\partial T_1}{\partial\theta_1} = 0\\
&\text{(viii) } Fo > 0,\ 0 < \eta_2 \leqslant \infty,\ \theta_2 = \pi/2,\ \frac{\partial T_2}{\partial\theta_2} = 0\\
&\text{(ix) } Fo > 0,\ \eta_1 = \eta_2 = 0,\ 0 \leqslant \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} \leqslant \pi/2,\ \frac{\partial T}{\partial n_1} = \frac{-k_2}{k_1}\frac{\partial T}{\partial n_2}\\
&\text{(x) } Fo > 0,\ n_1 = \eta_2 = 0,\ 0 \leqslant \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} \leqslant \pi/2,\ T_1 = T_2
\end{aligned} \tag{5}$$

where $n_1$ and $n_2$ are the respective outward normal unit vectors to body one and body two over the contact area. In the above, the temperatures far from the contact within bodies one and two have been taken for convenience as 1 and 0 respectively.

Clearly, from examination of equations (2) and (3) and the boundary conditions (5), the solution will be a function of the three parameters, in addition to its spatial dependence,

$$T = T(k_1/k_2, \alpha_1/\alpha_2, Fo). \tag{6}$$

*Thermal constriction resistance*

The thermal constriction resistance, by our previous definition, can be expressed mathematically by

$$R_c = \frac{[T(\eta_1 = \infty) - T(\eta_2 = \infty)]}{\int_\Gamma q \, d\Gamma} \tag{7}$$

where $\Gamma$ is the contact area, assumed constant with time. The significance of this latter assumption is to restrict the analysis to situations in which the differential thermal stresses developed at the interface do not substantially alter the contact cross-section as the thermal field changes with time. Using equation (7), then, it can be shown that for a circular contact the expression for the thermal constriction resistance becomes

$$R_c = \frac{1}{2\pi a \int_0^{\pi/2} k_1 \left.\frac{\partial T_1}{\partial \eta_1}\right|_{\eta_1=0} \sin\theta \, d\theta} = \frac{-1}{2\pi a \int_0^{\pi/2} k_2 \left.\frac{\partial T_2}{\partial \eta_2}\right|_{\eta_2=0} \sin\theta \, d\theta} \tag{8}$$

where the temperature solution on either side of the interface can be used, as indicated, for evaluation.

## SOLUTION

*Coordinate selection*

It is evident from inspection of equations (2) and (3) with boundary conditions (5) and considering the previous work which considered only a single half-space [7–17], that an analytic solution to the problem will not likely be possible using the currently available methods. Consequently, a numerical solution was sought to describe the thermal behavior of the problem. The finite difference method is used in this analysis.

Following a line of reasoning which led to the analytic formulation of the problem in oblate spheroidal coordinates, this coordinate system will also be used in the numerical formulation of the problem. The arguments suggesting the use of this coordinate system in the numerical solution of the problem have been presented in detail in references [6] and [7] and will not be repeated here. Because of the conformability of the coordinate system to the anticipated thermal field behavior, it can more readily accommodate the flow field distortion near the boundary condition discontinuity on the contact plane than can other, more conventional coordinate systems. This flow field accommodation leads to a reduced absolute truncation error in the vicinity of the discontinuity. The expanding nature of the coordinate system as one moves away from the generating disk is also a feature to be desired in the analysis of the present problem.

A general formulation of the finite difference representation of the heat-conduction equation in general orthogonal curvilinear coordinates has been presented by Schneider *et al.* [19] using a Taylor series expansion of the temperature field to approximate the governing differential equation, and by the same authors [20] wherein an energy balance is applied to a control volume of finite size, and the resultant energy balance approximated using finite differences. The latter approach is adopted here owing to the greater physical interpretation that can be given the resultant finite difference coefficients and to the conservative nature of the formulation. The results of [20] will be used directly in this work.

*Finite difference equation and boundary conditions*

The discretized spatial domain is illustrated in Fig. 3 for one of the half-spaces. Considerations similar to those which will be discussed in the following section, will apply equally well to the other half-space, and for convenience the subscripts 1 and 2 will be dropped for purposes of this discussion.

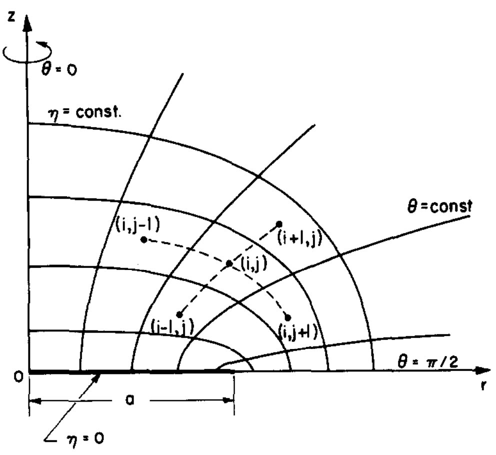


FIG. 3. Discretization of the solution domain.

With an arbitrary control volume centered about a point in space characterized by the notation $(i,j)$, Fig. 3, $i$ and $j$ shall be used to indicate the finite discretization of space in the $\eta$ and $\theta$ directions respectively. For uniform spacing in each direction independently, successive nodes in the $\eta$ and $\theta$ directions are incremented by amounts $\Delta\eta$ and $\Delta\theta$ respectively. These increments do not represent the physical distances separating adjacent nodes but merely the change in the coordinate value between these nodes.

The energy balance for this control volume, having total source strength $P\Delta V$, and centered about node $(i,j)$ can be written for the axisymmetric situation considered here as

$$Q_{i-\frac{1}{2},j}-Q_{i+\frac{1}{2},j}+Q_{i,j-\frac{1}{2}}-Q_{i,j+\frac{1}{2}}+P\Delta V = \frac{\partial}{\partial t}\int\int\int_{\text{C.V.}} \rho c_p T\,\mathrm{d}V. \quad (9)$$

Adopting the results of [20], the finite difference approximation of equation (9) can be written in the form

$$C_s T_{i,j} = C_1 T_{i-1,j}+C_2 T_{i+1,j}+C_3 T_{i,j-1}+C_4 T_{i,j+1}+D \quad (10)$$

where after some manipulation and division by common factors, the coefficients can be reduced to

$$\begin{aligned} C_1 &= 1/(\Delta\eta)^2-\tanh\eta_i/2\Delta\eta \\ C_2 &= 1/(\Delta\eta)^2+\tanh\eta_i/2\Delta\eta \\ C_3 &= 1/(\Delta\theta)^2-\cot\theta_j/2\Delta\theta \\ C_4 &= 1/(\Delta\theta)^2+\cot\theta_j/2\Delta\theta \\ D &= (\cosh^2\eta_i-\sin^2\theta_j)T^o_{i,j}/\Delta Fo + \frac{P\Delta V}{2\pi ka\cosh\eta_i\sin\theta_j\Delta\eta\Delta\theta} \\ C_s &= \sum_{n=1}^{4} C_n+(\cosh^2\eta_i-\sin^2\theta_j)/\Delta Fo. \end{aligned} \quad (11)$$

Since the time base normalization has been based on body one, i.e. $Fo = \alpha_1 t/a^2$, when applying the above equations to body two, the Fourier modulus should be replaced by the product, $(\alpha_2/\alpha_1)Fo$. Similarly for the source term (not required here) the appropriate value for the thermal conductivity must be used. The superscript $o$ has been used to denote quantities which are evaluated at the immediately preceding time.

The boundary condition application in the finite difference solution for the adiabatic and isothermal boundaries has been discussed fully in two previous papers by the present authors [6, 7] and will not be repeated here. The application is straightforward. As in these previous examinations, a value of $\eta \doteq 8.5$ was taken to be the numerical equivalent of $\eta = \infty$. This has been verified by the previous numerical results and can be supported from theoretical considerations. The remaining boundary condition, that of flux continuity across the interface was incorporated through the use of one-sided differences adjacent to the interface. These one-sided differences, when multiplied by the corresponding thermal conductivity and control volume interfacial surface area yield interfacial heat flow rates for each of the bodies. The resultant interfacial heat flow rates are equated for all control volumes adjacent to the contact area to provide incorporation of the final boundary condition.

*Numerical solution*

In the numerical solutions obtained for the transient thermal response of the two half-spaces which come into contact over a circular contact area, a grid arrangement was selected which has 20 spatial increments in the $\eta$-direction and 10 spatial increments in the $\theta$-direction within each of the bodies. Experience with both the steady-state solution of the single body problem [6] and the transient solution of the single body problem [7] has indicated that the spatial truncation error inherent in this grid arrangement will be less than 2%.

The tridiagonal matrix algorithm was applied on a line-by-line basis throughout the discretized spatial domain to effect the field solutions. Successive over-relaxation was used with a relaxation factor of 1.5 at each time step in the solution. Experience with the steady-state solution of the single body problem has indicated extremely rapid and stable convergence characteristics when this method is used [6].

An estimate of the time-dependent error is difficult to establish for this problem, because at present no analytic solutions are available for comparison. To minimize the propagation of time-dependent errors, therefore, a check was incorporated into the solution algorithm. The excellent agreement obtained between the asymptotic steady-state results and the steady-state analytic solution suggests that this check was successful in preventing the propagation of significant time-dependent errors.

The check was arranged such that for each time sub-interval considered, a tentative solution was obtained for the 'new' Fourier modulus. This sub-interval was then halved and a new estimate obtained for the new Fourier modulus through the use of an intermediate time plane. If the two solutions thus obtained agreed favorably, the latter one was accepted as the solution for the new time plane. If, conversely, an unacceptable agreement was obtained, the latter solution was maintained for comparison purposes with the solution resulting from a further halving of the time scale increment. This procedure was repeated until an acceptable agreement was obtained and the most recent solution was accepted as the solution for the time plane of interest. The criterion used for solution acceptance was that the maximum difference in temperature at any point in the field between the most recent two trial solutions be less than 5.0% when normalized with respect to the change in temperature occurring at that location between the two time planes under consideration.

## RESULTS

*Numerical results*

A finite difference program was written to solve the algebraic equations resulting from the finite difference formulation presented earlier in this paper. Solutions were obtained for five material combinations of practical interest to the engineering community. These material combinations are: (1) copper/stainless steel (304), (2) copper/glass, (3) copper/steel, (4) steel/glass, and (5) identical materials on both sides of the contact plane. As will be seen, these combinations are sufficient to describe a practical range of material combinations, scanning the range of thermal conductivity ratios given by $1 \leqslant k_1/k_2 \leqslant 370$ and the range of thermal diffusivity ratios given by $1 \leqslant \alpha_1/\alpha_2 \leqslant 220$. The materials combinations and the pertinent thermal properties used in the investigation are presented in Table 1.

Table 1

| Materials | $k$(W/mK) | | | $\alpha \times 10^5$ (m$^2$/s) | | |
|---|---|---|---|---|---|---|
| (Mat'l)$_1$/(Mat'l)$_2$ | $k_1$ | $k_2$ | $k_1/k_2$ | $\alpha_1$ | $\alpha_2$ | $\alpha_1/\alpha_2$ |
| Identical materials | N/A | N/A | 1.0 | N/A | N/A | 1.0 |
| Copper/steel | 381 | 43 | 8.86 | 13.2 | 1.36 | 9.70 |
| Steel/glass | 43 | 0.87 | 49.4 | 1.36 | 0.06 | 22.7 |
| Copper/S.S. 304 | 381 | 15.6 | 24.4 | 13.2 | 0.49 | 26.9 |
| Copper/glass | 381 | 1.03 | 370 | 13.2 | 0.06 | 220.0 |

Consistently throughout the investigation, the normalization contained within the Fourier modulus, $Fo \equiv \alpha_1 t/a^2$, was chosen to correspond to the material having the larger thermal diffusivity. This was done because the field within the high thermal diffusivity material will develop more quickly than that in the lower diffusivity material. Since all solutions for increasing time depend on the preceding solutions, it is important that the initial field development be adequately described. This motivation prompted the normalization with respect to the high diffusivity material.

The thermal resistance as a function of the Fourier modulus, normalized as described above, is presented in Fig. 4 for the five materials combinations considered in this paper. For the case of identical materials the diffusivity was assumed to be unity since here a direct scaling relationship exists to apply the results to materials having a thermal diffusivity different from unity. The ordinate of Fig. 4 is the ratio of the transient thermal resistance of the system at a given value of the Fourier modulus, $Fo$, divided by the steady-state value of the resistance, $R_{c_{ss}}$. The steady-state value of the resistance is given by

$$R_{c_{ss}} = \frac{1}{4k_1 a} + \frac{1}{4k_2 a}. \tag{12}$$

This result, based on an isothermal condition over the disk at steady-state, can be determined using analytic arguments, and moreover is supported by the steady-state asymptote of the numerical solution.

The numerical solutions yield asymptotic long time solutions which agree to within 1% of the analytic values without exception. As a comment on the error control used in the generation of the numerical results we note here the accuracy of the steady-state solution, the fact that the solution satisfies the initial condition that $R_c/R_{c_{ss}} \to 0$ for $Fo \to 0$, and thirdly the monotonic smoothness of the intermediate solution. As noted earlier the anticipated error of solution due to spatial truncation is less than 2%. The maximum expected time-dependent error of 5% of the maximum inter-time-plane change (this maximum change being approximately 15% of the steady-state value) yields a value of approximately 0.75% of the steady-state solution. Adding these two influences, it is expected that the solution error will be less than 3% of the steady-state solution. Noting that the time-dependent error will be cumulative in nature, and that the steady-state solution agreement is within 1%, it is concluded that the intermediate error in the resistance at any Fourier modulus will not exceed the 1% of the steady-state value.

Examination of Fig. 4 does not, unfortunately, allow immediate interpolation to material combinations that have not been considered in this work. This is due to the normalization of the time dependence with respect to the single thermal diffusivity, $\alpha_1$, even though this was convenient for performing the numerical solution. It was found that while the use of the harmonic mean diffusivity considerably narrowed the variation between solutions, an appreciable separation of solutions remained. This separation can be reduced to within

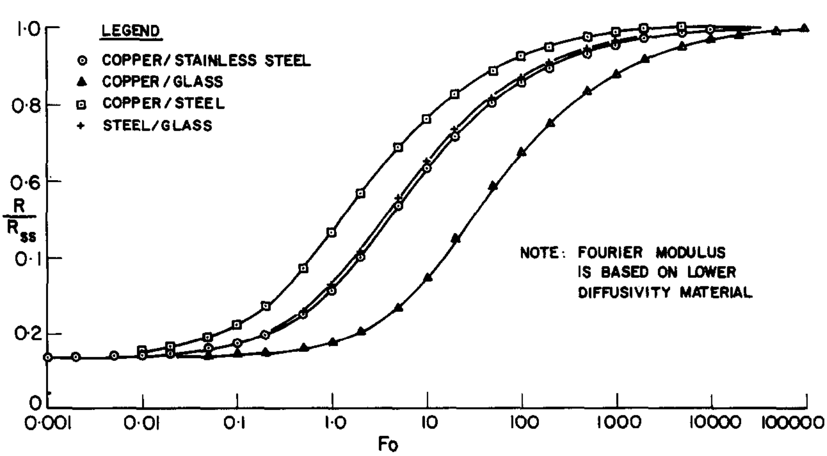


FIG. 4. Constriction resistance vs Fourier modulus based on low diffusivity material.

acceptable limits through the introduction of a parameter, $X$, defined by

$$X \equiv \tfrac{1}{2}[1+\sqrt{(\alpha_2/\alpha_1)}]Fo_{\bar{\alpha}}; \quad \alpha_2 \leqslant \alpha_1 \tag{13}$$

where the thermal diffusivity used in the determination of $Fo_{\bar{\alpha}}$ is

$$\bar{\alpha} \equiv 2\left[\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right]. \tag{14}$$

Through the use of the time variable, $X$, all of the solutions collapse to essentially a single curve.

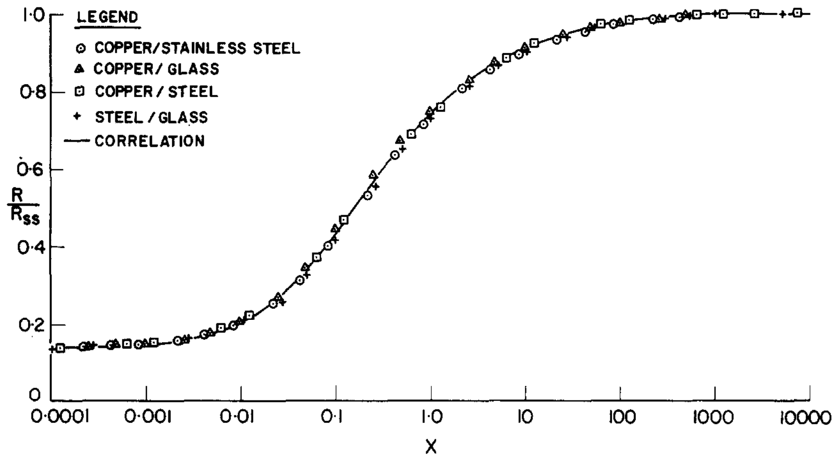


FIG. 5. Constriction resistance vs the correlation parameter $X$.

The parameter $X$ was determined primarily by inspection of the numerical solutions and as a result contains no rigorous physical or mathematical foundation. The many orders of magnitude of variation in the time dependence which can be removed through its use, however, certainly attaches significant utility to this combination of the thermal diffusivities and warrants its use as a correlation parameter. Further, the restriction that $\alpha_2$ be less than $\alpha_1$, as implied in equation (13), poses absolutely no restriction on the material combinations which may be considered since the designation of $\alpha_1$ and $\alpha_2$ to either of the contacting bodies is arbitrary, and consequently $\alpha_2$ can always be chosen to satisfy this requirement. The numerical results of $R_c/R_{c_{ss}}$ plotted as a function of the correlation parameter $X$ are presented in Fig. 5 for all of the material combinations considered.

*Correlation*

A correlation was sought which would adequately describe the thermal characteristics presented in Fig. 5 in order to be of direct application in thermal analysis. The correlation will consider the range of variation of $X$ that is indicated in Fig. 5.

The hyperbolic tangent functional behavior was chosen for use in the correlation as a result of the close similarity of the numerical results to the characteristics of this function. It was found that the correlation equation

$$R_c/R_{c_{ss}} = 0.43\tanh[0.37\ln(4X)]+0.57 \tag{15}$$

suitably describes the numerical data throughout the range $10^{-4} \leqslant X \leqslant \infty$. The maximum error of the correlation with the data is 7% occurring for a value of $X = 0.097$. The correlation equation results are also presented in Fig. 5. It is felt that the above correlation equation (15) will be of considerable utility in applying the results of this work.

## DISCUSSION AND CONCLUSIONS

The complete transient solution for two semi-infinite bodies, initially at uniform but different temperatures, coming into thermal contact over a small circular contact region has been formulated and solved numerically. The solution is presented in terms of a transient thermal constriction resistance, defined in the text of this paper.

To the best of the present authors' knowledge, this is the first analysis directed at the analysis of the complete, composite problem, where the materials on each side of the contact area are not identical. For this reason, comparison with other solutions is not possible. The comparisons which can be made are for the steady-state asymptotic value of the present solution and for the transient behavior of the case for which the materials on each side of the interface are identical. In both cases the comparison fully supports the results of the present solutions. The 'S-curve' characteristics observed in the previous, single body transient solution [7] are also observed in the present solutions.

The thermal resistance, in general depends upon three parameters, the conductivity ratio, the thermal diffusivity ratio, and a Fourier modulus representing the time dependence. It was found, however, that the influence of the conductivity ratio can be embodied within a dimensionless dependent variable by normalizing the thermal constriction resistance with respect to the steady-state value, which depends upon the thermal conductivities of the two materials.

Further examination of the numerical results revealed that the influence of the thermal diffusivity ratio for the two materials can be accommodated for engineering purposes by the introduction of a new independent variable, which explicitly contains the two thermal diffusivities. The property combination employed in this new independent variable does not have rigorous physical or mathematical foundations but nevertheless is useful as a correlation parameter. Indeed, by normalizing the solution in the above-mentioned fashion, the thermal constriction resistance becomes dependent upon a single independent variable, irrespective of the materials combination considered. This is supported by the five cases of practical interest which were examined in this work, with the thermal conductivity ratio ranging from unity to 370 and the thermal diffusivity ratio ranging from unity to 220. The situations for which the diffusivity ratio is less than unity are easily accommodated by merely changing the reference material to that on the opposite side of the interface.

Based on these normalized numerical results, a simple correlation equation was obtained to determine the thermal constriction resistance over the range of the independent variable $10^{-4} \leqslant X \leqslant \infty$. The maximum error of correlation of the numerical data is 7%. It is felt that this will be adequate for engineering analysis.

The finite difference solution procedure made use of the oblate spheroidal coordinate system as the basis for modelling the heat flow. In this system the steady-state, isothermal disk, single body problem becomes one-dimensional. Although this is not the case for transient heat flow, the coordinate system is 'quasi-natural' and considerable savings can be realized through its use. This is reflected in the numerical solution of the problem by a decrease in the computational time required for solution. In addition, the coordinate system can naturally account for the discontinuity of heat flow occurring at the disk outer edge and will ultimately lead to a smaller truncation error since the coordinate system follows more naturally the heat flow near this discontinuity. The solution time required for the complete transient solutions was less than 10 min of IBM 360/75 computational time per solution.

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## REFERENCES


1. J. H. Jeans, *The Mathematical Theory of Electricity and Magnetism*, pp. 356–357. Cambridge University Press, Cambridge (1963).
2. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, pp. 216–217. Oxford University Press, Oxford (1959).
3. L. C. Roess, Theory of spreading conductance, in Thermal resistance measurement of joints formed between stationary metal surfaces, Appendix A, by N. D. Weills and E. A. Ryder, *Trans. Am. Soc. Mech. Engrs* **71**, 259–267 (1949).
4. B. B. Mikic and W. M. Rohsenow, Thermal contact resistance, M.I.T. Heat Transfer Laboratory Report No. DSR 74542-41, Contract No. NGR-22-009-065 (September 1966).
5. M. G. Cooper, B. B. Mikic and M. M. Yovanovich, Thermal contact conductance, *Int. J. Heat Mass Transfer* **12**, 279–300 (1969).
6. A. B. Strong, G. E. Schneider and M. M. Yovanovich, Thermal constriction resistance of a disk with arbitrary heat flux, in *Progress in Astronautics and Aeronautics*, Vol. 39, *Heat Transfer with Thermal Control Applications*, edited by M. M. Yovanovich, pp. 65–78. AIAA/MIT Press, Cambridge, MA (1975).
7. G. E. Schneider, A. B. Strong and M. M. Yovanovich, Transient heat flow from a thin circular disk, AIAA Paper No. 75-707 presented at the AIAA 10th Thermophysics Conference, Denver, Colorado (27–29 May 1975).
8. H. Blok, Theoretical study of temperature rise at surfaces of actual contact under oiliness conditions, *Inst. Mech. Engng Gen. Disc. Lubric.* **2**, 225–235 (1937).
9. R. Holm, Calculation of the temperature development in a contact heated in the contact surface, and application to the problem of the temperature rise in a sliding contact, *J. Appl. Phys.* **19**, 361–366 (1948).
10. W. J. Oosterkamp, Calculation of the temperature development in a contact heated in the contact surface, *J. Appl. Phys.* **19**, 1180–1181 (1948).
11. P. H. Thomas, Some conduction problems in the heating of small areas on large solids, *Q. J. Mech. Appl. Math.* **10**, 482–493 (1957).
12. W. Davies, Thermal resistance in graphite–copper contacts, *Br. J. Appl. Phys.* **10**, 516–522 (1959).
13. W. D. Clark and R. W. Powell, Measurement of thermal conduction by the thermal comparator, *J. Scien. Instrum.* **39**, 545–551 (1962).
14. E. J. Norminton and J. H. Blackwell, Transient heat flow from constant temperature spheroids and the thin circular disk, *Q. J. Mech. Appl. Math.* **17**, 65–72 (1964).
15. J. H. Heasley, Transient heat flow between contacting solids, *Int. J. Heat Mass Transfer* **8**, 147–154 (1965).
16. J. H. Blackwell, Transient heat flow from a circular disk—small-time solution, *J. Aust. Math. Soc.* **14**, 433–442 (1972).
17. N. R. Keltner, Transient heat flow in a half-space due to an isothermal disk on the surface, *J. Heat Transfer* **95**, 412–414 (1973).
18. M. M. Yovanovich, *Advanced Heat Conduction*. Hemisphere, Washington, D.C. (1976).
19. G. E. Schneider, A. B. Strong and M. M. Yovanovich, Finite difference modelling of the heat conduction equation in general orthogonal curvilinear coordinates using Taylor series expansion, in *Proceedings of the AICA International Symposium on Computer Methods for Partial Differential Equations, Lehigh University, Bethlehem, Pennsylvania*, edited by R. Vichnevetsky, pp. 312–317. AICA, New Brunswick, NJ (June 1975).
20. G. E. Schneider, A. B. Strong and M. M. Yovanovich, A physical approach to the finite difference solution of the conduction equation in orthogonal curvilinear coordinates, ASME Paper No. 75-WA/HT-94. Presented at ASME Winter Annual Meeting, Houston, Texas (30 November–4 December 1975).


## REPONSE THERMIQUE TRANSITOIRE POUR DEUX CORPS EN COMMUNICATION PAR UNE PETITE AIRE CIRCULAIRE DE CONTACT

**Résumé**—On considère la réponse thermique transitoire de deux corps semi-infinis, initialement à des températures différentes et uniformes, en communication à travers une petite aire circulaire de contact. La solution numérique est formulée en coordonnées sphériques. Elle est présentée pour des combinaisons de matériaux dans le domaine d'intérêt pratique pour les thermiciens. Une équation est proposée pour la résistance de striction, fonction du module de Fourier dans le domaine considéré des paramètres physiques. La dispersion inférieure à $\pm 7\%$, est considérée comme bonne pour les calculs d'ingéniérie.

## DIE THERMISCHE ÜBERGANGSFUNKTION FÜR ZWEI KORPER, DIE AN EINER KLEINEN, KREISFÖRMIGEN KONTAKTFLÄCHE MITEINANDER IN BERÜHRUNG STEHEN

**Zusammenfassung**—Es wird die thermische Übergangsfunktion für zwei halbunendliche Körper mit einheitlicher, jedoch verschiedener Anfangstemperatur betrachtet; die beiden Körper berühren sich an einer kleinen, kreisförmigen Kontaktfläche. Das numerische Lösungsschema wird in abgeflachten Kugelkoordinaten formuliert. Vollständige numerische Lösungen für das Übergangsverhalten werden für solche Materialkombinationen wiedergegeben, die von praktischem Interesse für den Ingenieur sind. Für den betrachteten Bereich der physikalischen Parameter wird eine Korrelationsgleichung für den Kontraktionswiderstand in Abhängigkeit von der Fourier-Zahl angegeben. Mit einer Fehlergrenze von $\pm 7\%$ kann diese Korrelation als ausreichend genau für Ingenieurberechnungen angesehen werden.

## ПЕРЕХОДНЫЙ ТЕПЛОВОЙ ПРОЦЕСС ДЛЯ ДВУХ ТЕЛ, ИМЕЮЩИХ НЕБОЛЬШУЮ КРУГОВУЮ ОБЛАСТЬ КОНТАКТА

**Аннотация — Рассматривается переходный тепловой процесс для двух полуограниченных тел с постоянными, но не одинаковыми температурами, имеющих небольшую круговую область контакта. Численное решение формулируется в специальных сферических координатах. Приводятся полные решения задачи для различных сочетаний материалов, представляющих практический интерес для теплотехников. Представлены корреляционные зависимости для сопротивления сжатию как функция модуля Фурье для рассматриваемого диапазона параметров. Погрешность составляет $2 \pm 7\%$, что вполне соответствует инженерным расчетам.**