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Simplified Explicit Elastoconstriction Resistance Expression for Ball/Race Contacts

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SIMPLIFIED EXPLICIT ELASTOCONSTRICTION RESISTANCE EXPRESSION FOR BALL/RACE CONTACTS

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Abstract

A novel elastoconstriction resistance relationship has been developed to aid in the rapid prediction of thermal resistance across spacecraft bearings. The development is based upon new definitions of effective contact curvatures and recently published approximations of the Hertz parameters. The elastoconstriction expression shows explicitly the influence of contact geometry upon the resistance. Differences between the approximate values and those predicted by the complex, exact Hertz equations are less than 1.7% for all ball/race contacts encountered in spacecraft design. A typical example is cited to demonstrate the simplicity and accuracy of the new expression.

Nomenclature

- A = geometric parameter, Eq. (14)
- a = semi-axis of elliptic contact area, Eq. (5)
- B = geometric parameter, Eq. (15)
- b = semi-axis of elliptic contact area, Eq. (6)
- $E(k')$ = complete elliptic integral of the second kind
- E_1, E_2 = modulii of elasticity of bodies 1 and 2
- $K(k')$ = complete elliptic integral of the first kind
- k = ellipticity of the contact area $(k = \frac{n}{m} = \frac{b}{a} < 1)$
- k' = complementary modulus $(k' = \sqrt{1 - k^2})$
- m = Hertz parameter for semi-major axis, Eq. (10)
- N = normal load on ball/race contact
- n = Hertz parameter for semi-minor axis, Eq. (11)
- R_e^T = constriction resistance of an isothermal elliptic contact, Eq. (1)
- x, y = cartesian coordinates

Greek Symbols

- α = ratio of minimum/maximum effective curvatures $(= \rho_{mn}/\rho_{mx})$, Eq. (20)
- β = ellipticity
- Δ = physical parameter of bodies 1 and 2, Eq. (7)
- λ_1, λ_2 = thermal conductivities of bodies 1 and 2
- λ_s = harmonic mean thermal conductivity of the contact, Eq. (4)
- ν_1, ν_2 = Poisson's ratios
- $\rho_1, \rho_1', \rho_2, \rho_2'$ = local curvatures of bodies 1 and 2
- ρ_x = effective curvature along x-axis, Eq. (18)
- ρ_y = effective curvature along y-axis, Eq. (19)
- ρ_{mn} = minimum effective curvature
- ρ_{mx} = maximum effective curvature
- ρ^* = total effective curvature, Eq. (8)
- τ = contact parameter, Eq. (16)
- χ = new thermal constriction parameter, Eq. (26)
- ψ^* = thermal constriction parameter, Eq. (17)

Introduction

There is considerable interest [1] in the resistance to heat transfer through the bearings employed in the numerous joints of the remote manipulating systems (RMS) being designed for the space shuttle. In a vacuum environment with negligible radiation heat transfer, the resistance is due to the constriction resistance at the ball/race contact areas. To calculate this constriction resistance, one must first use the classical Hertz theory to predict the shape and relative size of the contact areas; then employ the thermal constriction resistance theory for each contact.

These calculations are rather involved requiring the use of some iterative numerical procedures or tables. To facilitate these computations for ball/race contacts a new simplified, approximate elastoconstriction resistance expression has been developed and is presented in this paper.

Thermal Constriction Resistance

The total thermal constriction resistance of an isothermal elliptic contact area $(a > b)$ developed by Yovanovich [2,3] is

$$R_e^T = \psi_e^T / 2\lambda_s a \tag{1}$$

where ψ_e^T is the isothermal thermal constriction parameter given by

$$\psi_e^T = (2/\pi)\, K(k') \tag{2}$$

in which $K(k')$ is the complete elliptic integral of the first kind of modulus k':

$$k' = \sqrt{1 - (b/a)^2} \tag{3}$$

In Eq. (3), a and b are the semi-major and semi-minor axes of the contact area. Also λ_s is the harmonic mean thermal conductivity of the contact:

$$\lambda_s = 2\lambda_1\lambda_2/(\lambda_1 + \lambda_2) \tag{4}$$

whenever dissimilar materials are brought into contact.

The geometric parameters can be determined by the Hertzian theory of elastic contacts.

Hertzian Theory of Elastic Contacts

Whenever a ball is brought into contact with a race, the Hertzian theory predicts an elliptical contact whose semi-axes are related to the mechani-

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cal load, physical properties and geometry as follows [4-7]:

$$a = m[\frac{3 N \Delta}{2(A + B)}]^{1/3} \quad (5)$$

and

$$b = n[\frac{3 N \Delta}{2(A + B)}]^{1/3} \quad (6)$$

In Eqs. (5) and (6), N is the total normal load acting upon the contact area, and Δ is a physical parameter defined as

$$\Delta = \frac{1}{2} [\frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}] \quad (7)$$

when dissimilar materials form the contact. The physical parameters appearing in Eq. (7) are Young's modulus, E_1 and E_2, and Poisson's ratio, ν_1 and ν_2.

The geometric parameter in Eqs. (5) and (6) is

$$2(A + B) = \frac{1}{\rho_1} + \frac{1}{\rho_1'} + \frac{1}{\rho_2} + \frac{1}{\rho_2'} = \frac{1}{\rho^*} \quad (8)$$

and the local radii of curvature of the contacting solids are denoted as ρ_1, ρ_1', ρ_2 and ρ_2'.

For the ball, $\rho_2 = \rho_v' = D/2$ where D is the diameter. In the case of the inner race, the smaller radius of curvature is negative and the larger radius of curvature is positive. Therefore, $\rho_1 = \rho_i$ and $\rho_1' = -\rho_i'$. In the case of the outer race, both radii of curvature are negative, therefore $\rho_1 = -\rho_o$ and $\rho_1' = -\rho_o'$.

An additional relationship between A and B required in the Hertzian theory is

$$2(B - A) = \frac{1}{\rho_1} + \frac{1}{\rho_2} - \frac{1}{\rho_1'} - \frac{1}{\rho_2'} > 0 \quad (9)$$

The dimensionless parameters m and n which appear in Eqs. (5) and (6) are called the Hertz elastic parameters. They are determined by means of the following Hertz relationships [4-7]:

$$m = [\frac{2}{\pi} \frac{E(k')}{k^2}]^{1/3} \quad (10)$$

and

$$n = [\frac{2}{\pi} k\ E(k')]^{1/3} \quad (11)$$

where E(k') is the complete elliptic integral of the second kind of modulus k', and

$$k' = \sqrt{1 - k^2} \quad (12)$$

and $k = \frac{n}{m} = \frac{b}{a} \leq 1$.

The additional parameters k and k' are solutions of the transcendental equation [4-7]:

$$\frac{B}{A} = \frac{(1/k^2)E(k') - K(k')}{K(k') - E(k')} \quad (13)$$

where K(k') is the complete elliptic integral of the first kind of modulus k'. The ratio B/A can be determined by means of the following two reduced expressions valid for ball/race contacts:

$$2A = \frac{1}{\rho_2} + \frac{1}{\rho_1'} \quad (14)$$

and

$$2B = \frac{1}{\rho_2} + \frac{1}{\rho_1}$$

The Hertz solution requires the calculation of k, the ellipticity, K(k') and E(k'). This calls for the solution of Eq. (13) which relates k, K(k') and E(k') to the local geometry of the contacting solids. This is usually done by some iterative numerical procedure [4,5] or with the aid of tables [6] or graphs [7].

To this end, an additional parameter has been defined:

$$\cos\tau = \frac{B - A}{B + A} \quad (16)$$

and computed values of m and n, or (m/n) and n, are presented with τ as the independent parameter.

Thermal Elastoconstriction Resistance

The results of the thermal analysis, Eq. (1), and the results of the Hertz analysis can be combined to give us the thermal elastoconstriction resistance expression. After substitution of Eqs. (5) - (8) into Eq. (1), and re-arranging we obtain

$$\lambda_s [24\ N\Delta\rho^*]^{1/3} R_e^T = (2/\pi)[K(k')/m] \equiv \psi^* \quad (17)$$

where the effective radius of the contact is denoted by $\rho^* = [2(A + B)]^{-1}$. The left hand side of Eq. (17) is dimensionless; it contains the known total mechanical load, the thermal, physical and geometric properties of the contact, as well as the unknown constriction resistance. The right hand side of Eq. (17) is denoted by ψ^* which is called the thermal elastoconstriction parameter.

Tables of m, n and ψ^* versus the parameter τ have been developed [11]. Some typical values are presented in Table 1.

New Thermal Elastoconstriction Parameter

Effective Radii of Curvature

The effective geometry of the contacting solids will be redefined following the discussion presented in Refs. [8,9]. The effective radii of curvature in the x- and y-planes are defined as

$$\frac{1}{\rho_x} = \frac{1}{\rho_1} + \frac{1}{\rho_2} \quad (18)$$

and

$$\frac{1}{\rho_y} = \frac{1}{\rho_1'} + \frac{1}{\rho_2}$$

with $\rho_2' = \rho_2$ for the ball. The smaller value of ρ_x and ρ_y will be denoted ρ_{mn}, the minimum effective radius of curvature. The larger value will be denoted ρ_{mx}.

Next we define the radius ratio,

$$\alpha = \frac{\rho_{mn}}{\rho_{mx}} < 1 \quad (20)$$

It can be demonstrated that the Hertz geometric parameters A, B and τ are related to α, ρ_{mn} and

ρ_{mx} as follows:

$$\alpha = \frac{A}{B} \quad (21)$$

and

$$2(A + B) = \frac{1}{\rho^*} = (1 + \alpha)/\rho_{mn} \quad (22)$$

and

$$2(B - A) = (1 - \alpha)/\rho_{mn} \quad (23)$$

Therefore,

$$\frac{(B - A)}{(B + A)} = \frac{1 - \alpha}{1 + \alpha} \quad (24)$$

and

$$\alpha = \frac{1 - \cos\tau}{1 + \cos\tau} \quad (25)$$

Elastoconstriction Parameter

Returning at this point to Eq. (17), let us replace ρ^* by $\rho_{mn}/(1 + \alpha)$, and rewrite Eq. (17) generating the new expression,

$$\lambda_s [24\ N\Delta\rho_{mn}]^{1/3}\ R_e^T = \frac{2}{\pi}\ (1 + \alpha)^{1/3}\ \frac{K(k')}{m} \equiv \chi \quad (26)$$

where χ is the new thermal elastoconstriction parameter depending upon α only, because the transcendental expression, Eq. (13), is a function of α through Eq. (21).

Equation (26) is exact because, to this juncture, no approximations have been made. The exact values of the parameter χ can be computed from Table 1 or by means of the transcendental expression.

Approximate Solution Applicable to Ball/Race Contacts

The various terms appearing in χ, Eq. (26), will be approximated. The modulus of $K(k')$ is

$$k' = \sqrt{1 - (\frac{b}{a})^2} = \sqrt{1 - (\frac{n}{m})^2} \quad (27)$$

For most ball/race contacts, the ellipticity $k = \frac{b}{a} = \frac{n}{m} < 0.20$, therefore $k' \geq 0.98$.

Let $k' = \sin w$; therefore $\cos w = n/m$. For $k' \geq 0.98$ the complete elliptic integral of the first kind can be approximated by

$$K(k') \doteq \ln\ [\frac{2(1 + \sin w)}{\cos w}] = \ln\ (4\ \frac{m}{n}) \quad (28)$$

In Refs. [8,9] it was reported that the ellipticity k can be approximated by

$$k = \frac{b}{a} = \frac{n}{m} \doteq \alpha^{0.636} \quad (29)$$

with an error not exceeding 3%. In the ball/race contact range of α, the error is less than 1%.

Equation (28) can be written approximately as

$$K(k') \doteq \ln\ [4/\alpha^{0.636}] \quad (30)$$

Using the results of [9], it can be demonstrated [10] that the Hertz parameter n is related to α in the following approximate way:

$$n = [0.6175\ \alpha^{0.636} + 0.3674\ \alpha^{1.636}]^{0.333} \quad (31)$$

with errors of 0.5%, 0.03% and 1% at $\alpha = 1.0$, 0.10 and 0.01, respectively.

Taking advantage of these approximations, the new thermal elastoconstriction parameter χ can be written as

$$\chi = \frac{2}{\pi}\ \frac{(1 + \alpha)^{1/3}\ \alpha^{0.636}\ \ln\ [4/\alpha^{0.636}]}{[0.6175\ \alpha^{0.636} + 0.3674\ \alpha^{1.636}]^{0.333}} \quad (32)$$

in terms of the new parameter α.

A comparison of the approximate values of χ with the exact values of $(2/\pi)[K(k')/m]$ shows that the maximum difference is about 1.7% when $\alpha = 0.06$. For other values of $0.01 \leq \alpha \leq 0.15$, the error is less than 1.7%.

Equation (32) can be written as

$$\chi = \frac{2}{\pi}\ \frac{(1 + \alpha)^{1/3}\ (\alpha^{1.908})^{1/3}\ \ln[4/\alpha^{0.636}]}{[0.6175\ \alpha^{0.636} + 0.3674\ \alpha^{1.636}]^{1/3}} \quad (33)$$

The last expression can be factored to yield

$$\chi = \frac{2}{\pi}\ \frac{\ln\ [4/\alpha^{0.636}]}{[\frac{0.6175 + 0.3674\ \alpha}{(1 + \alpha)\ \alpha^{1.272}}]^{1/3}} \quad (34)$$

If we limit our approximation to values of α corresponding to ball/race contacts, i.e. $\alpha \leq 0.2$, Eq. (34) can be further simplified to

$$\chi = \frac{2}{\pi}\ (1.174)\ \alpha^{0.424}\ \ln\ [4/\alpha^{0.636}] \quad (35)$$

The factor $(2/\pi)(1.174) = 0.7476$. For convenience, the factor 0.750 will be used in Eq. (35).

A comparison of the approximate values of χ, Eq. (35), with the exact values of $(2/\pi)[K(k')/m]$ shows that the maximum difference is about 1.7% when $\alpha = 0.04$. For other values of $0.01 \leq \alpha \leq 0.15$, the difference is less than 1.7%.

The new simplified approximate thermal elastoconstriction resistance expression is

$$\lambda_s [24\ N\Delta\rho_{mn}]^{1/3}\ R_e^T = 0.750\ \alpha^{0.424}\ \ln[4/\alpha^{0.636}] \quad (36)$$

where $0.01 \leq \alpha \leq 0.15$. This covers the range of most ball/race contacts.

If $\alpha > 0.15$, the exact solution, Eq. (26) must be used. Table 1 is based upon the exact solution.

For $\alpha \leq 0.01$, the ellipticity k becomes small and it becomes necessary to use another parameter

parameter $\beta = \frac{n}{m} = k$ to obtain additional approximations.

The transcendental equation can be written in terms of β and α,

$$\frac{\beta^2 \, [K(\sqrt{1-\beta^2}) - E\,\sqrt{(1-\beta^2)}]}{E\,(\sqrt{1-\beta^2}) - \beta^2\, K\,(\sqrt{1-\beta^2})} = \alpha \qquad (37)$$

As $\sqrt{1-\beta^2} \to 1$, we can write approximately,

$$E(\sqrt{1-\beta^2}) \doteq 1 \qquad (38)$$

and

$$K(\sqrt{1-\beta^2}) \doteq \ell n\,(4/\beta) \qquad (39)$$

For $\beta \leq 0.01$, we can neglect the second term in the denominator of Eq. (37) and write

$$\beta^2\,[\ell n\,(\frac{4}{\beta}) - 1] = \beta^2\,\ell n\,(\frac{1.4715}{\beta}) \doteq \alpha \qquad (40)$$

as the approximation to the transcendental equation.

The Hertz parameter m, Eq. (10), reduces to

$$m \doteq [\frac{2}{\pi\beta^2}]^{1/3} \qquad (41)$$

For the elastoconstriction parameter, Eq. (17), we can write approximately,

$$\psi^* = [\frac{2\beta}{\pi}]^{2/3}\,\ell n[4/\beta] \qquad (42)$$

A comparison of the values computed by means of the approximations, Eqs. (40), (41) and (42), for $\alpha \leq 0.01$ with the values given in Table 1 shows clearly how good these approximations are.

Illustrative Example

To demonstrate the simplicity and accuracy of the new expression consider the following typical example:

Inner Race Contact	Outer Race Contact
$\rho_i = 41.68$ mm	$\rho_o = -46.44$ mm
$\rho_i' = -2.48$ mm	$\rho_o' = -2.48$ mm
$\rho_2 = \rho_2' = 2.38$ mm	$\rho_2 = \rho_2' = 2.38$ mm

Exact Method

$A + B = 0.2301$	0.2074
$B - A = 0.2140$	0.1913
τ[Eq. (16)] = 25.56^o	22.73^o
α[Eq. (25)] = 0.0363	0.0404
ψ^* [11] or Table 1 = 0.6246	0.6409
χ[Eq. (26)] = 0.6321	0.6494

Approximate Method

$\rho_x = \rho_{mn} = 2.2514$	2.509
$\rho_y = \rho_{mx} = 62.01$	62.00
$\alpha = 0.0363$	0.0405
χ[Eq. (35)] = 0.6426	0.6597
% Difference = 1.67	1.59

Conclusions

A new simplified approximate thermal elastoconstriction resistance expression applicable to most ball/race contacts has been developed. Its range of applicability is $0.01 \leq \alpha \leq 0.15$ and $11.4^o \leq \tau \leq 42.3^o$ with an error less than 1.7%.

Other approximate expressions are presented for values of $\alpha \leq 0.01$ or $\tau \leq 11.4^o$. For $\alpha > 0.15$ and $\tau > 42.3^o$, the exact solution must be used. Tabular values of the Hertz parameters and the thermal elastoconstriction parameter are presented for $\alpha > 0.15$.

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Table 1 Hertz Contact Parameters and Elastoconstriction Parameter

	k	m	n	ψ^*
0.001	0.0147	14.316	0.2109	0.2492
0.002	0.0218	11.036	0.2403	0.3008
0.004	0.0323	8.483	0.2743	0.3616
0.006	0.0408	7.262	0.2966	0.4020
0.008	0.0483	6.499	0.3137	0.4329
0.010	0.0550	5.961	0.3277	0.4581
0.020	0.0828	4.544	0.3765	0.5438
0.040	0.1259	3.452	0.4345	0.6397
0.060	0.1615	2.935	0.4740	0.6994
0.080	0.1932	2.615	0.5051	0.7426
0.100	0.2223	2.391	0.5313	0.7761
0.200	0.3460	1.813	0.6273	0.8757
0.300	0.4504	1.547	0.6969	0.9261
0.400	0.5441	1.386	0.7544	0.9557
0.500	0.6306	1.276	0.8045	0.9741
0.600	0.7117	1.1939	0.8497	0.9857
0.700	0.7885	1.1301	0.8911	0.9930
0.800	0.8618	1.0787	0.9296	0.9972
0.900	0.9322	1.0361	0.9658	0.9994
1.000	1.0000	1.0000	1.0000	1.0000

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