CONSTRICTION RESISTANCE OF DOUBLY-CONNECTED CONTACT AREAS UNDER UNIFORM FLUX

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Abstract

Steady-state thermal constriction resistances of doubly-connected, planar, contact areas bounded by coaxial circles, squares and equilateral triangles under uniform flux conditions are examined. The Surface Element Method (SEM) is employed to obtain numerical values of the dimensionless constriction resistance $\lambda\delta R$ for the square and triangle for several values of the geometric parameter $\varepsilon = \sqrt{A_i/A_o}$ with $(A_o - A_i) = A_c$, the contact area. Several geometric parameters of the contact areas are examined as possible choices for the characteristic dimension, δ, and the dependence of $\lambda\delta R$ upon ε in the range $0 < \varepsilon < 0.9999$ is examined. Correlations of the dimensionless constriction resistance are developed for the square and triangle based upon the analytical solutions for the circle employing $\delta = \sqrt{A_c}$ and $\delta = P_o$, the outer perimeter.

Nomenclature

A_c = contact area $(A_o - A_i)$

A_i = inner projected area

A_o = outer projected area

2a = inner diameter or inner length dimension

2b = outer diameter or outer length dimension

C_i, i=1,...7 = correlation coefficients

P_o = outer perimeter of contact area

Q = total heat flow rate

R = thermal constriction resistance, Eq. (4)

R^* = dimensionless constriction resistance $(\lambda\delta R)$

r = distance from the source point to the field point

T = temperature

$T(\infty)$ = reference temperature

x,y,z = Cartesian coordinates

Greek Symbols

δ = characteristic dimension

λ = thermal conductivity

ε = square root of the area ratio $(\sqrt{A_i/A_o})$

η = point source coordinate

θ = temperature excess $(T - T(\infty))$

$\bar{\theta}$ = average temperature excess, Eq. (5)

ξ = point source coordinate

π = Pi

ρ = polar coordinate

∇^2 = Laplacian operator

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Introduction

Several papers have been published during the past several years concerning the steady-state, thermal constriction resistances of planar, singly-connected [2-7] and doubly-connected [3,5-7] contact areas on insulated, isotropic half-spaces. The two usual contact area boundary conditions which have been examined are the uniform Neumann condition [3-6] and the uniform Dirichlet condition [7]; although a paper on the effect of various flux distributions over a circular contact area [2] has been published.

A large variety of singly-connected contact geometries under uniform Neumann conditions have been studied: i) circular [1-5], ii) rectangular or square [1,3,4], iii) triangular [6], iv) semi-circular [6], and v) other non-symmetric shapes [6].

A single paper [7] has been published recently which examines the influence of uniform Dirichlet conditions over singly- and doubly-connected, circular and rectangular contact areas.

The concept of employing the square root of the singly-connected contact area as the characteristic dimension for the normalization of the constriction resistance proposed by Yovanovich [4,6] was used to great advantage by Schneider [7] when he examined doubly-connected contacts under uniform Dirichlet conditions.

The purpose of this paper is to obtain constriction resistances of doubly-connected, square and triangular contact areas under uniform Neumann boundary conditions, and to examine the effect of several other characteristic dimensions upon the normalization of the resistance, and to compare the results with those of the analytical solution obtained for the circular, annular contact area [5] as shown in Fig. 1.

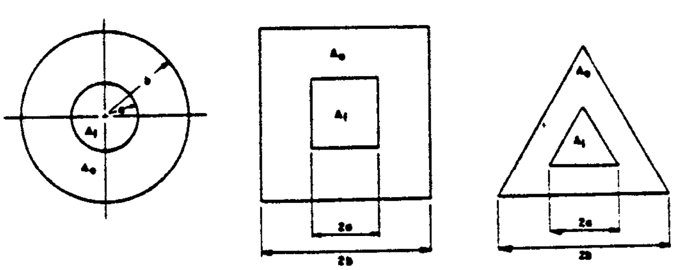

Fig. 1 Doubly-connected contact geometries

Problem Statement and Resistance Formulation

Steady heat conduction from a doubly-connected, planar, contact area through an isotropic thermal conductor is described by Laplace's equation:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} = 0 \qquad (1)$$

where the temperature excess $\theta = T - T(\infty)$, and $T(\infty)$ is some convenient reference temperature associated with points which are located at distances which are large relative to some characteristic dimension, δ, of the contact area.

With the origin of the Cartesian coordinates located at the center of the inner projected area, A_i, the boundary conditions are

$$\text{i)}\quad z=0,\quad (x,y)\text{ within } A_i,\quad \frac{\partial\theta}{\partial z} = 0$$

$$(x,y)\text{ within } A_c,\quad \frac{\partial\theta}{\partial z} = \frac{-Q}{\lambda A_c}$$

$$(x,y)\text{ outside } A_o,\quad \frac{\partial\theta}{\partial z} = 0 \qquad (2)$$

$$\text{ii)}\quad \rho = \sqrt{x^2 + y^2} = 0,\quad \frac{\partial\theta}{\partial x} = \frac{\partial\theta}{\partial y} = 0$$

$$\text{iii)}\quad \sqrt{x^2 + y^2 + z^2} \to \infty,\quad \theta \to 0$$

The solution to equation (1) which also satisfies the boundary conditions, equations (2), is [3,7]

$$\theta = \frac{q}{2\pi\lambda} \iint_{A_c} \frac{dA_c}{r} \qquad (3)$$

with $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}$ provided the heat flux, q is uniform over the contact area, A_c.

The thermal constriction resistance due to steady conduction from an arbitrary contact area subjected to arbitrary Dirichlet or Neumann boundary conditions is defined as the average temperature of the contact area minus the reference temperature divided by the total heat flow rate, Q, through the conductor. The mathematical statement of this definition is, therefore,

$$R = \frac{\bar{\theta}(z=0)}{Q} \qquad (4)$$

with

$$\bar{\theta} = \frac{1}{A_c} \iint_{A_c} \theta(z=0)\, dA_c \qquad (5)$$

and

$$Q = \iint_{A_c} q dA_c \qquad (6)$$

For the Neumann problem with uniform heat flux, equation (3) with equations (5) and (6) becomes [3]:

$$2\pi\lambda R = \frac{1}{A_c^2} \iint_{A_c} \left\{ \iint_{A_c} \frac{dA_c}{r} \right\} dA_c \qquad (7)$$

An analytical solution to equation (7) is available for the circular annular contact area [5]; but it is very difficult or impossible to obtain analytical solutions for arbitrary, doubly-connected contact areas such as the square or the triangle. Numerical solutions are therefore required.

Numerical Solutions and Results

The numerical method used here is one which has proved to be very satisfactory for solving a variety of thermal contact problems. The method consists of dividing the contact area into a finite number of surface elements over each of which the heat flux is assumed to be uniform. The temperature excess at the centroid of each surface element is then determined by

$$\theta_i = \sum_{j=1}^{N} C_{ij}\, q_j \qquad (8)$$

with the influence coefficients, C_{ij}, defined by

$$C_{ij} = \frac{1}{2\pi\lambda} \iint_{A_{cj}} \frac{dA_{cj}}{r_{ij}} \qquad (9)$$

and r_{ij} is the distance from the centroid of the ith surface element to an arbitrary location within the jth surface element.

Analytical expressions for the influence coefficients have been developed for circular and annular surface elements [5], for rectangular surface elements [7] and for triangular surface elements [8].

The average temperature of the contact area is given by

$$\bar{\theta} = \frac{1}{A_c} \sum_{i=1}^{N} \sum_{j=1}^{N} C_{ij}\, q_j\, dA_{ci} \qquad (10)$$

and the constriction resistance for arbitrary flux distributions is given by

$$R = \frac{1}{A_c} \sum_{i=1}^{N} \sum_{j=1}^{N} C_{ij}\, q_j\, dA_{ci} \Big/ \sum_{j=1}^{N} q_j\, dA_{cj} \qquad (11)$$

For the uniform heat flux problem considered here, equation (11) simplifies to

$$R = \frac{1}{A_c^2} \sum_{i=1}^{N} \sum_{j=1}^{N} C_{ij}\, dA_{ci} \qquad (12)$$

Equation (12) is the discretized equivalent of the analytical expression, equation (7).

The numerical procedure and the convergence study are similar to that employed by Schneider [7] and, therefore, will not be given here.

Numerical Results and Correlations

The numerical values of the constriction resistance obtained by means of equations (9) and (12) were normalized with respect to the thermal conductivity, λ, and the characteristic dimension,

δ, resulting in a dimensionless constriction resistance $R^* = \lambda\delta R$.

Several geometric characteristics of the doubly-connected contact areas were examined. Some proved to be poor candidates because the normalized resistance, R^*, as a function of $\varepsilon = \sqrt{A_i/A_o}$ was not monotonic and diverged yielding large values of R^*. It became apparent that $\delta = \sqrt{A_o}$ which was an excellent characteristic dimension for singly-connected contact areas [4,6] was not an appropriate choice for doubly-connected contact areas over the full range of the parameter, ε. Based upon the analytical solution for the annular contact area [5] it was decided to use $\delta = \sqrt{A_c}$ and $\delta = P_o$.

The normalized numerical results for $\lambda \sqrt{A_c}\, R$ are presented in Fig. 2 for the full range of ε and in tabular form in Table 1. For very thin contact areas corresponding to $\varepsilon \geq 0.995$, $\lambda \sqrt{A_c}\, R$ is presented in Fig. 3.

Two correlation equations were developed for $\lambda \sqrt{A_c} R$; one for ε running from 0 to 0.995 and the other running from 0.995 to 0.9999. In the range $0 \leq \varepsilon \leq 0.995$:

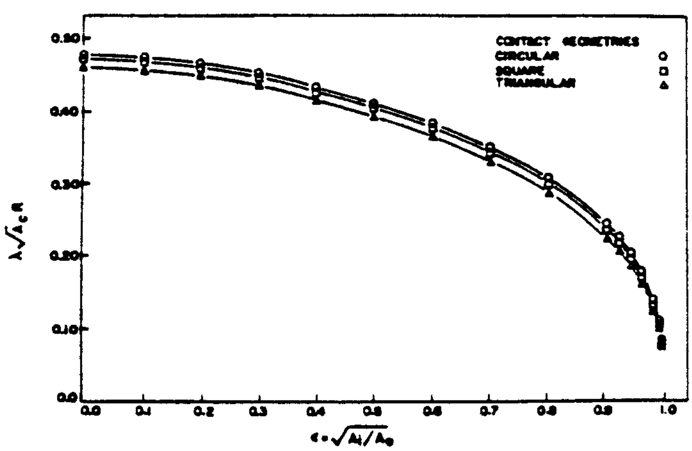

Fig. 2 Dimensionless constriction resistances vs geometric parameter ε

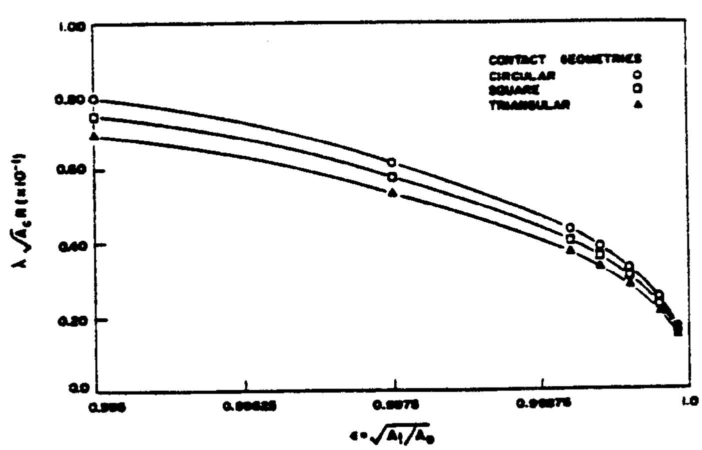

Fig. 3 Dimensionless constriction resistances vs geometric parameter ε

Table 1 Numerical values of $\lambda\sqrt{A_c}R$ vs ε

ε	Circle	Square	Triangle
0.0	0.4789	0.4732	0.4602
0.1	0.4752	0.4695	0.4566
0.2	0.4655	0.4597	0.4466
0.3	0.4509	0.4449	0.4318
0.4	0.4321	0.4259	0.4125
0.5	0.4092	0.4025	0.3890
0.6	0.3815	0.3744	0.3607
0.7	0.3476	0.3399	0.3262
0.8	0.3040	0.2957	0.2824
0.9	0.2402	0.2317	0.2195
0.92	0.2223	0.2139	0.2022
0.94	0.2009	0.1928	0.1817
0.96	0.1739	0.1661	0.1560
0.98	0.1350	0.1282	0.1197
0.99	0.1041	0.0983	0.0914
0.995	0.0798	0.0750	0.0694
0.9975	0.0608	0.0569	0.0525
0.999	0.0421	0.0392	0.0360
0.99925	0.0375	0.0349	0.0320
0.9995	0.0317	0.0295	0.0270
0.99975	0.0238	0.0221	0.0202
0.9999	0.0162	0.0150	0.0137

$$R^*(\varepsilon) = R^*(\varepsilon = 0)\left[1.0 - \left(\frac{\varepsilon}{c_1}\right)^{c_2}\right]^{c_3} \qquad (13)$$

where $R^*(\varepsilon = 0)$ is obtained from Table 1 and the correlation coefficients c_1, c_2 and c_3 are obtained from Table 2. The maximum difference between the values predicted by equation (13) and the computed values are 0.49% at $\varepsilon = 0.2$ for the circle, 0.58% at $\varepsilon = 0.995$ for the square and 0.93% at $\varepsilon = 0.995$ for the triangle.

In the range $0.995 \leq \varepsilon \leq 0.9999$ based upon the analytical solution and its correlation [5] we recommend

$$R^*(\varepsilon) = c_4 \sqrt{1 - \varepsilon^2}\, \ln\left[\frac{c_5}{\frac{1}{\varepsilon} - 1}\right] \qquad (14)$$

where the coefficients c_4 and c_5 are also obtained from Table 2.

The excellent agreement between equation (14) and the computed values can be seen in Tables 4, 5 and 6 where the maximum difference is -0.32% which occurs at $\varepsilon = 0.9999$.

The alternate choice of the characteristic dimension $\delta = P_o$ which is recommended for very thin contact areas yields the following correlation equation:

$$\lambda P_o R = c_6 \ln\left[\frac{c_5}{\frac{1}{\varepsilon} - 1}\right] \qquad (15)$$

Table 2 Correlation coefficients for equations (13) and (14)

Geometry	c_1	c_2	c_3	c_4	c_5
Circle	0.99957	1.5056	0.35931	0.08915	39.66
Square	0.9998	1.5150	0.37302	0.078845	68.59
Triangle	1.0001	1.5101	0.38637	0.069157	115.91

Table 3 Correlation coefficients for equations (15) and (16)

Geometry	c_6	c_7
Circle	0.31604	40
Square	0.31538	69
Triangle	0.31529	116

where the correlation coefficients c_5 and c_6 for the three geometries are given in Tables 2 and 3, respectively. It can be seen in Tables 4, 5 and 6 that there is excellent agreement between the computed values of $\lambda P_o R$ and those predicted by equation (15).

Noting that the correlation coefficient c_6 is a very weak function of geometry and that it is also approximately equal to $\pi/10$, an alternate approximate correlation equation was developed:

$$\lambda P_o R = \frac{\pi}{10} \ln \left[\frac{c_7}{\frac{1}{\varepsilon} - 1}\right] \quad (16)$$

where c_5 of equation (15) is replaced by the rounded values c_7 of equation (16). From Tables 4, 5 and 6 it can be seen that equation (16) predicts values of $\lambda P_o R$ which are in good engineering agreement with the numerical values of $\lambda P_o R$ for the three geometries examined.

Conclusions

The Surface Element Method has been employed to obtain the thermal constriction resistance of doubly-connected, square and triangular, contact areas which are subjected to a uniform heat flux. Numerical values of the normalized constriction resistance, R^*, are presented in tabular form over the range $0 \leq \varepsilon \leq 0.9999$ which covers most engineering applications for $\delta = \sqrt{A_c}$ and $\delta = P_o$. Several correlation equations are presented for $\lambda\sqrt{A_c}R$ and $\lambda P_o R$ which are in excellent agreement with the numerical values obtained here. An approximate correlation equation is recommended for engineering applications to very thin contact areas, $\varepsilon \geq 0.995$.

Table 4 Normalized constriction resistances for circular contact areas

ε	$\lambda\sqrt{A_c}R$ num.	$\lambda\sqrt{A_c}R$ eq. (14)	% diff.	$\lambda P_o R$ num.	$\lambda P_o R$ eq. (15)	% diff.	$\lambda P_o R$ eq. (16)
0.995	0.0798	0.0799	-0.13	2.8324	2.8218	-0.37	2.8360
0.9975	0.0608	0.0609	-0.19	3.0500	3.0404	-0.32	3.0559
0.999	0.0421	0.0422	-0.24	3.3380	3.3287	-0.27	3.3459
0.99925	0.0375	0.0375	-0.12	3.4330	3.4192	-0.40	3.4369
0.9995	0.0317	0.0318	0.31	3.5540	3.5466	-0.21	3.5652
0.99975	0.0238	0.0239	-0.29	3.7733	3.7645	-0.23	3.7843
0.9999	0.0162	0.0163	-0.32	4.0608	4.0524	-0.21	4.0739

Table 5 Normalized constriction resistances for square contact areas

ε	$\lambda\sqrt{A_c}R$ num.	$\lambda\sqrt{A_c}R$ eq. (14)	% diff.	$\lambda P_o R$ num.	$\lambda P_o R$ eq. (15)	% diff.	$\lambda P_o R$ eq. (16)
0.995	0.0750	0.0750	-0.03	3.0038	3.0029	-0.36	2.9931
0.9975	0.0569	0.0569	-0.05	3.2208	3.2223	-0.28	3.2117
0.999	0.0392	0.0393	-0.13	3.5070	3.5117	-0.19	3.5000
0.99925	0.0349	0.0349	-0.07	3.6051	3.6025	-0.40	3.5905
0.9995	0.0295	0.0295	-0.04	3.7320	3.7305	-0.37	3.7179
0.99975	0.0221	0.0221	-0.11	3.9536	3.9492	-0.45	3.9358
0.9999	0.0150	0.0150	-0.11	4.2427	4.2382	-0.45	4.2237

Table 6 Normalized constriction resistances for triangular contact areas

ε	$\lambda\sqrt{A_c}R$ num.	$\lambda\sqrt{A_c}R$ eq. (14)	% diff.	$\lambda P_o R$ num.	$\lambda P_o R$ eq. (15)	% diff.	$\lambda P_o R$ eq. (16)
0.995	0.0694	0.0694	-0.02	3.1679	3.1563	-0.36	3.1674
0.9975	0.0525	0.0525	-0.01	3.3870	3.3749	-0.36	3.3868
0.999	0.0360	0.0361	-0.14	3.6709	3.6632	-0.21	3.6761
0.99925	0.0320	0.0320	-0.02	3.7675	3.7537	-0.37	3.7669
0.9995	0.0270	0.0270	-0.05	3.8930	3.8811	0.31	3.8948
0.99975	0.0202	0.0202	-0.13	4.1187	4.0990	0.48	4.1135
0.9999	0.0137	0.0137	-0.32	4.4166	4.3869	0.67	4.4024

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