Thermal Resistance of a Convectively Cooled Plate with Nonuniform Applied Flux
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Two-dimensional steady conduction within a plate of rectangular cross section is considered. One of the surfaces is convectively cooled on a uniform ambient temperature while the opposite face is exposed to a nonuniform flux distribution over a portion of its boundary. An analytic solution is presented for general flux distribution and the specific cases are solved. The solution depends upon the flat model, the plate thickness, and the extent and character of the flux distribution. Graphical results are presented for the thermal conduction resistances over the range of parameters of practical interest.

Nomenclature

- width of object in typical cell
- width of the typical cell for analysis
- film modulus
- thickness of typical cell
- constants, defined in text
- function dependent upon flux distribution
- heat-transfer coefficient
- thermal conductivity
- length of typical cell normal to cross section
- integral
- new flux
- constant in heat flux distribution
- total heat flow
- thermal resistance
- non-dimensional reistance
- non-dimensional conduction resistance
- temperature
- non-dimensional temperature
- average constant area temperature
- field temperature
- non-dimensional coordinate x/a
- Cartesian coordinates
- non-dimensional thickness c/b
- non-dimensional coordinate y/b
- eight-place analytical solution
- parameter in flux distribution
- non-dimensional coordinate x/b
- function, defined in text

Introduction

In many engineering situations, a thermal conductor convectively cooled on one face may be subject to a nonuniform heat flux over its opposite face. In particular, this nonuniform flux distribution may be prescribed only over a portion of the surface, with the adjacent surface area remaining essentially adiabatic for thermal analysis purposes. This problem is examined in this work for an arbitrary flux distribution and for the case of steady two-dimensional heat transfer.

The analysis presented in this work [finds important application to the design and analysis of solar collectors and collector plates in both the design of solar collectors and in the evaluation of experimental facilities utilized to estimate collector efficiencies due to true convection within the collector. The collector plate is frequently constructed of a thermal conductor having evenly spaced coolant tubes secured to its lower surface. A second important application of this work is to the cooling of disks of electronic circuitry as can be found where extensive use is made of integrated circuits (IC) devices. In order to maintain the temperature of the electronic circuitry below its maximum reliable operating temperature, a knowledge of the thermal resistance of the mounting plate is required.

Restricting the analysis to the case of evenly spaced tubes or IC banks in the aforementioned examples, a single typical cell can be extracted for analysis purposes. This typical cell is delineated by the plates of thermal symmetry existing at the center of the flux distribution and as the midpoint between neighboring tubes or IC banks. The thermal influence of the containment of interest, tube, IC bank, etc., will be modeled as a flux distribution over a portion of the surface of the device with its mating surface. The problem geometry is then that shown in Fig. 1 for a typical cell. A uniform heat transfer coefficient is considered in this work. The portion of the surface on the device opposite plate being outside of the device contact area is assumed to be impervious to heat transfer.

The problem geometry of Fig. 1 has been examined by several investigators, each of which considered restrictive thermal boundary conditions. Van Sant [1] considered the case of a uniform flux contact region and an isothermal contact region conducting to a convectively cooled lower surface. For the isothermal contact case a numerical procedure was required and a plot of the results presented. His interest, however, was only in extending the temperature variation over the convectively cooled surface and thus of limited utility in the evaluation of the thermal resistance. For the uniform flux case, only the series representation of the maximum temperature variation over this surface was presented.

Schmitz [2] considered the special case of an isothermal contact with conducting to a second, isothermal, surface. He used separation of variables in his solution to this problem but presented the series representation for only the heat flow

Footnotes

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through the conducting medium. This same problem was later solved by Consolino,1 who used the theory of conformal transformations to obtain a single-form solution for this special case. In his solution the conductance for the plate section is determined in terms of complete elliptic integrals of the first kind. The complex computational procedure required to evaluate the pertinent solution parameters, however, makes his results very difficult to use in practice.

Gavrilii and Voroti2 examined the case where both the contact strip and the lower plate surface are convectively coupled through film coefficients to two external fluid temperatures. Again, separation of variables was used and a contraction coefficient was determined. They presented a solution for the uniform flux contact and their results agree with those of Van Saas.1 The solution where the contact region is convectively coupled to a second fluid temperature, however, produced some unexplained abnormal behavior. The lack of explanation for this somewhat erratic behavior suggests that there are as yet unresolved questions concerning the convective-contact solution.

It is the purpose of this paper to examine the flux prescribed contact case with the heat conducted to a second surface which is convectively coupled to an external fluid. This will be done in a general fashion for arbitrary flux distributions and for arbitrary combinations of the geometric parameters. Two specific flux distributions in addition to the uniform flux case are examined in detail. One of these provides a very close approximation to the uniform temperature contact situation over a very wide range of the geometric parameters, while the second distribution provides flux concentrations near the contact centerline. The three distributions examined will be useful in estimating limits for the thermal resistance by which most cases of practical interest will be bounded. The thermal resistance is presented for all three cases of practical interest as a function of the geometric parameters and the heat-transfer coefficient \( h \) through the Biot modulus \( B_i \).

Problem Solution

Mathematical Statement of the Problem

The geometry to be analyzed is that shown in Fig. 1. A Cartesian coordinate system is set up as shown in the figure with the origins coinciding with the center of the contact region. The typical cell half-width is denoted by \( b \) with the contact having a half-width of \( a \). The plate thickness is denoted by \( e \). Over the upper surface, the plate communicates thermally with a fluid at a temperature \( T_f \) through the convective film coefficient \( h \), while over the contact a flux distribution \( q(x) \) symmetric about the vertical axis, is prescribed. The remaining boundary is assumed to be impermeable to heat transfer.

Considering steady-state heat transfer with no internal heat generation, the governing differential equation is Laplace's equation in the two directions considered,

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0
\]

with the boundary conditions given by

\[
\begin{align*}
\frac{\partial T}{\partial x} & = q(x) & (2a) \\
\frac{\partial T}{\partial y} & = 0 & (2b)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial T}{\partial x} & = -k\left[ (T(x) - T_j) / k ight] & (2b) \\
\frac{\partial T}{\partial x} & = 0 & (2c)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial T}{\partial x} & = 0 & (2d)
\end{align*}
\]

In order to maintain simplicity in the analysis it is useful to non-dimensionalize the governing differential equation, Eq. (1), and boundary conditions, Eqs. (2). To effect this non-

1. van Saas.

Using the above definitions the governing differential equation can be written as

\[
\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} = 0
\]

with boundary conditions.

\[
\begin{align*}
\bar{x} = 0 & \quad \frac{\partial \bar{T}}{\partial \bar{x}} = -\frac{\bar{q}(1)\bar{L}}{\bar{Q}} & (6a) \\
\bar{x} < \bar{x} & \quad \frac{\partial \bar{T}}{\partial \bar{x}} = 0 & (6b)
\end{align*}
\]

\[
\begin{align*}
\bar{T} & = \bar{a} & \frac{\partial \bar{T}}{\partial \bar{y}} = -\bar{B}T^* & (6b)
\end{align*}
\]

\[
\begin{align*}
\bar{T} & = 1 & \bar{y} < \bar{a} & \frac{\partial \bar{T}}{\partial \bar{y}} = 0 & (6c)
\end{align*}
\]

\[
\begin{align*}
\bar{T} & = \bar{l} & \bar{y} > \bar{a} & \frac{\partial \bar{T}}{\partial \bar{y}} = 0 & (6d)
\end{align*}
\]

where the additional parameters have been introduced

\[
\bar{a} = \frac{a}{b} & \quad \bar{L} = \frac{L}{b} & \quad \bar{Q} = \frac{q}{b} & \quad \bar{B} = \frac{h}{k} & \quad \bar{T}^* = \frac{T - T_j}{T_f - T_j}
\]

as well as the Biot modulus defined by

\[
\bar{B} = \frac{h}{k}\frac{T_f - T_j}{T_f - T_j}
\]

It can be readily seen now by examination of Eqs. (5) and (6) that the solution for the temperature field will be dependent upon the four non-dimensional parameters \( \bar{a}, \bar{L}, \bar{Q}, \bar{B}, \bar{e}, \), and \( \bar{a} \), in addition to the two trivial coordinates.

Analytical Solution

In solving the thermal problem described above, solutions are sought in Eq. (5) that satisfy the appropriate boundary conditions. Eq. (6). Following the classical method of
which is the average dimensionless temperature over the contact region. Using the expression developed earlier for the temperature distribution in Eq. (14) yields the result

\[ R^* = \frac{a}{1 + \frac{1}{Bi}} \]

where \( g_e \) is defined in Eq. (15).

A dimensionless thermal conductivity resistance can be obtained by subtracting from the thermal resistance of Eq. (15), the resistance of the slab which results from one-dimensional heat conduction from the surface \( \xi = \beta \) to the field at temperature \( T_e \). The one-dimensional resistance is given in non-dimensional form by

\[ R_{1D} = \frac{a}{1 + \frac{1}{Bi}} \]

### Table 1: Influence of \( \beta \) on \( J_e \)

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( J_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>( J_e(0) )</td>
</tr>
<tr>
<td>0</td>
<td>( J_e(0) )</td>
</tr>
<tr>
<td>0.5</td>
<td>( J_e(0) )</td>
</tr>
</tbody>
</table>

### Results

The thermal conductivity resistance can be obtained for the range of Biot modulus from \( Bi = 0.01 \) through \( Bi = 100 \).

![Fig. 2 Flux distribution](image)
Dimensionless thickness ratio $\alpha$ has been considered over the range $0.05 < \alpha < 2.0$, and the dimensionless contact size considered over the range of $0.1 < \varepsilon < 1.0$. These results are presented in Figs. 3 through 7. These figures correspond to various values of the thickness ratio $\alpha$ with each figure indicating the dependence of $R^*$ on the Biot modulus $B_i$ and on the relative contact size $\varepsilon$.

It can be observed from these figures that the conduction resistance is largest for the parameter combination of small $\varepsilon$, small $\alpha$, and small $B_i$. The increase in $R^*$ with decreasing values of $\varepsilon$ is monotone for all cases as is expected. The dependence of $R^*$ on $\alpha$, however, is dependent upon the particular Biot modulus under consideration.

For small values of the Biot modulus, corresponding to poorly conducting solid/fluid interface behavior, the heat flow is forced to spread more uniformly over the larger, conductive surface. This was only achieved by the lateral heat flow in the positive $x$ direction and consequently the small relative thicknesses $\alpha$ there is a large conduction influence. As the thickness increases and the lateral conductive resistance therefore decreases, the thermal conduction resistance also decreases as observed by comparison of the $B_i = 0.01$ curves for example as the thickness ratio is increased from 0.05 to 2.0.

For large values of the Biot modulus, however, the dependence of $R^*$ on $\alpha$ is the opposite of that described above. For a very large Biot modulus, for example, the appraxis, convectively coupled surface remains very nearly uniform in temperature at the fluid temperature. In this case very large fluxes can be tolerated at the convective surface with only a slight temperature rise above $T_f$. For very thin members, $\alpha$ small, the solid region constitute a thermal short circuit and the conduction resistance is small. As the thickness of the member increases, the conductive resistance of the solid portion increases and this causes a larger fraction of the total heat flow to pass laterally prior to leaving the region at the convective surface. Therefore, for a large Biot modulus, the thermal conduction resistance increases as the thickness ratio increases.
When the thickness ratio is increased beyond $\alpha = 0.5$, the dependence of the Biot modulus vanished and the trends established above for large and small Biot modulus converge to a single value. The dependence of the indentation resistance on $\alpha$ also vanished in this limit and the thermal conduction resistance becomes dependent on the single parameter $\varepsilon$, the relative contact size.

The influence of the three different flux distributions shown in Fig. 2 is presented in Fig. 3 for the case where $\varepsilon = 0.05$. The departures from the uniform flux distribution curves are larger for this case and it can be seen from the figure that the influence is small despite the widely differing nature of the flux distributions. The insensitivity to the nature of the flux distribution is attributed to the averaging procedure and in having the thermal conduction resistance on the average contact temperature. This is a desirable effect since in adding components chemically is series, it is the average temperature which two contacting surfaces will have in common. While the percentage difference is results due to the different flux distributions increases as $\varepsilon = 1$ is approached, this is of little consequence since here the onedimensional resistance is the dominant one. It is interesting to note, however, that the convection resistance approaches zero for all flux distributions considered as $\varepsilon$ approaches unity. This result is apparent upon examination of Eq. (17) but not an obvious one based solely on physical reasoning. This peculiarity is also attributed to the averaging process carried out in deriving it Eq. (17).

Conclusions and Discussion

The thermal conduction resistance corresponding to a partially flux prescribed surface conducting as a secondary surface which is convectively coupled to a fluid environment temperature has been determined in this work. The problem has been solved in a general fashion for an arbitrary flux distribution over the contact region and three flux distributions have been specifically examined. Solutions have been obtained for the range of the Biot modulus 0.01 $\leq k < 100.0$ and for the thickness ratio 0.05 $\leq \varepsilon \leq 2.0$. For each of the above variable combinations the relative contact size influence was determined and plotted for the range 0.01 $\leq \varepsilon \leq 1.0$.

It was found that the thermal conduction resistance, when based on the average contact temperature, is relatively insensitive to the precise nature of the applied flux distribution over the range of primary interests of $\varepsilon$. This insensitivity is attributed to the averaging of the contact temperature.

It was also found that the maximum thermal conduction effect is obtained for the parameter combination where both the Biot modulus and the width ratio are smallest. For small values of the Biot modulus, the influence of increasing the thickness ratio is to decrease the thermal conduction resistance. Conversely the maximum thermal conduction resistance is obtained for situations where the Biot modulus is large and the thickness ratio is small. For large values of the Biot modulus, the influence of increasing the thickness ratio is also to increase the thermal conduction resistance. For thickness ratio larger than 0.5, the above two trends approach they common asymptote and an insensitivity of $R_{\varepsilon}$ to both $\alpha$ and $R_{\varepsilon}$ is exhibited.

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