The shape factor of the capacitance of a conductor

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This paper shows, analytically and numerically, that the capacitance is a slowly changing function of the conductor shape. This slow change can be monitored by a "shape factor" which is independent of the conductor size. Because of this slow change, from the tabulation of the shape factor, the capacitance of many conductors of arbitrary sizes and shapes with both convex and concave surfaces, can be estimated to an error of 5% or less.

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I. INTRODUCTION

The capacitance of many arbitrarily shaped conducting bodies have to be computed numerically through the point-matched boundary integral methods,[1,2] simulated charge methods (optimized[3] or nonoptimized[4]), or the finite element method.[5] Normally the full computation has to be carried out in these numerical methods, each time the shape of the conductor body is changed by even a small amount.

This paper corrects this situation. Based on the inequality of Payne and Weinbeger,[6] it proves that the capacitance is, in fact, a slowly changing function of the conductor shape for a constant conductor surface area. Using the variational formulation[7] it also shows that the capacitance is proportional to the square root of the conductor surface area for a constant conductor shape. From these two properties, a capacitance shape factor is defined so that the capacitance of any conducting body is simply proportional to the product of the slow-changing factor and the square root of the surface area.

The shape factor of capacitance of a sphere is defined as unity. A series of the shape factor of different conductor shapes are tabulated to show that they normally do not change beyond $\pm$ 10% from unity. They also show that the shape factor between similar conductors, such as a prolate and a right cylinder of the same length-to-width ratio, are observed to be within 5% of each other. This means that if the capacitance of a conductor is known, the capacitance of a whole family of similar conductors is also known within a 5% error.

One can derive a slow-changing shape factor not only for the convex-shaped conductors, such as the above, but also for conductors with concave surfaces. For the latter case, the effective surface area of the concave conductor is not the actual surface area, but the surface area of the smallest convex conductor that can circumscribe the concave conductor. The justification and examples of the effective area are included in this paper.

II. THE SLOWLY CHANGING CAPACITANCE OF CONDUCTORS OF DIFFERENT SHAPES BUT OF THE SAME SURFACE AREA

Assume that there is a conductor S_0 of boundary potential V_s, assume also that the potential field Φ_0 around the conductor and its capacitance C_0 are known exactly. As a simple illustration, let the conductor be a sphere as shown in Fig. 1(a). The potential field of a sphere can be considered to come from an equivalent point change at the center of the sphere. As a result the potential field Φ_0 of the sphere need not stop at the boundary of the sphere but continues into the interior of the sphere. It has been demonstrated that this concept of equivalent interior charges and continuation of potential field across the conductor boundary can be applied to not only the sphere but many other conductor shapes including bodies with edges.[8]

Assume now that there is a new conductor S_1 of the same boundary potential V_s and the *same* total surface area. Having the same total area, the surface of the new conductor would normally intercept that of the original conductor. For illustration, let the new conductor be a cube and intercepts the original sphere as shown in Fig. 1(b).

Let the exact but unknown capacitance of the new conductor be C_1 ($= C_0 + \Delta C$). Now because of the concept of continuation of potential across the conducting boundary, even with the surface interceptions as in Fig. 1(b), we can allow the exact potential distribution Φ_0 of the original conductor to be the *approximate* potential distribution of the new conductor. Then the capacitance inequality of Payne and Weinberger[6] can be written as

$$(\Delta C)^2 \leqslant \epsilon_0 C_1 \iint_{S_1} h_1^{-1}\left(\frac{\Phi_0}{V_s}-1\right)^2 ds, \tag{1}$$

where

$$h_1 = r_1 \frac{dr_1}{dn_1}, \tag{2}$$

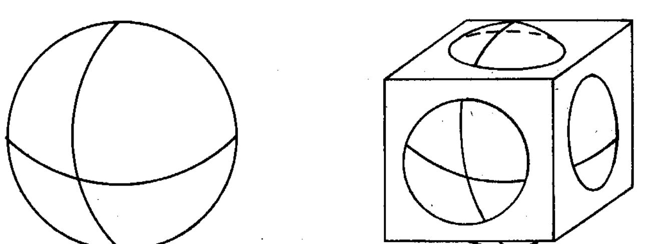


FIG. 1. A sphere and its intercepting cube of the same surface area. (a) A sphere; (b) The intercepting cube.

the projected distance from the origin inside S_1 to a particular point on the surface S_1. The surface integral in Eq. (1) represents the h^{-1} weighted squared potential error (normalized by V_s) over S_1.

Equation (1) can be written as

$$\left(\frac{\Delta C}{C_1}\right) \leqslant \left[\frac{\epsilon_0}{C_1}\iint_{S_1} h_1^{-1}\left(\frac{\Phi_0}{V_s}-1\right)^2 ds\right]^{1/2}. \tag{3}$$

As seen in Fig. 1(b), there are lines where the new conductor intercepts the original conductor. Along these lines the potential error $\Phi_0/V_s - 1$ in Eq. (3) is zero. Hence it is easy to see that the integral of the square of the weighted potential error at the right-hand side of Eq. (3) is usually small. This means that the ratio $\Delta C/C_1$ at the left-hand side of Eq. (3) is also small, or that ΔC is only a slowly changing function of the shape of the conductor.

It may be noted that the property of slow-changing capacitance is ensured by the presence of interceptions of the new and the original conducting surfaces. The interceptions of the conducting surfaces are usually obtained by choosing the total surface areas of the two conductors to be the same. The interceptions are ensured when the two conductors are convex.

III. THE CAPACITANCE OF CONDUCTORS OF DIFFERENT SURFACE AREAS AND THE CAPACITANCE SHAPE FACTOR

The last section considered conductors of the same total surface area. If a new conductor is of different shape and different total area from the original conductor of known capaciatance, then the change from the original conductor to the new conductor can be considered in two steps. The first step is to change the shape while keeping the area constant, as seen from the change of "a sphere to a cube" in Fig. 1. The second step is to change the area while keeping the shape similar, as the change from the cube to a bigger cube. We shall consider this latter aspect in this section.

For a known charge distribution σ_0, on the surface S_0 of a conductor, the capacitance is[7]

$$C_0 = \frac{\left[\iint_{S_0} \sigma_0\, ds\right]^2}{\iint_{S_0} \sigma_0 \left[\iint_{S_0} \frac{\sigma_0}{4\pi\epsilon_0 r_0}\, ds'\right] ds}, \tag{4}$$

where r_0 is the distance from the field point to the source point, on the surface of the conductor. If the shape of the conductor remains similar but its linear dimensions are increased by a factor a, then the charge distribution σ_0 on the new surface S_1 can remain the same, but the new distance is $r_1 = a\, r_0$ and the new area is $S_1 = a^2 S_0$. As a result, one can easily see from Eq. (4) that the new capacitance C_1 is related to the original C_0 by the ratio

$$C_1 = \sqrt{S_1}\left(\frac{C_0}{\sqrt{S_0}}\right). \tag{5}$$

Equation (5) is exact if the shapes of the conductors remain similar. However, Eq. (5) is still a good approximation even if the shapes of the conductors are changed, based on the fact that the capacitance is a slowly changing function of the shape.

It will be desirable to get a dimensionless capacitive shape factor to monitor the slowly changing capacitance with respect to the shape. The normalized parameter $C/\sqrt{S}$ in Eq. (5) is not dimensionless, although it is independent of the conductor area. The parameter $C/\sqrt{S}$ can be made truly dimensionless by a division by $K\epsilon_0$, where K is a constant. Based on this we may define the shape factor as

$$c_f = \frac{C}{\epsilon_0 (S\, 4\pi)^{1/2}}, \tag{6}$$

where the constant K has been chosen to be $(4\pi)^{1/2}$. This constant is chosen so that the shape factor is equal to unity when the conductor is a sphere.

With c_f known for a conductor shape, the capacitance is simply

$$C = c_f \epsilon_0 (4\pi S)^{1/2}. \tag{7}$$

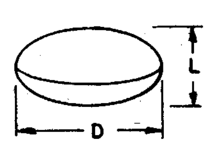
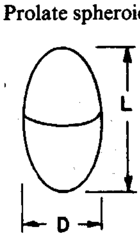
From the tables in the next two sections we shall see that c_f can have values both larger and smaller than unity.

The values of c_f tabulated could have errors up to 3%. The details of these errors, however, are incomplete and difficult to get as many of the c_f values are gathered from the capacitance computed numerically by different workers. Therefore they are not included in the tables.

IV. THE SHAPE FACTOR OF CONVEX CONDUCTORS

The sphere is the simplest convex shape with a shape factor of unity. Simple compressing and elongation of the sphere give, respectively, the oblate and prolate spheroids. Table I gives the shape factor as a function of a chosen aspect ratio γ (i.e., length of the axis of symmetry to the cross-section diameter) of the spheroids. The table shows that the shape factor ranges only from 0.90 (for $\gamma = 0.0$, a circular disk) to 1.14 (for $\gamma = 8.0$, a long prolate spheroid) over a vast change in shape. Thus, this table shows two properties: (i) the shape factor is a slowly changing facator of the shape, and (ii) in the process of elongation (i.e., γ increasing) the shape factor increases.

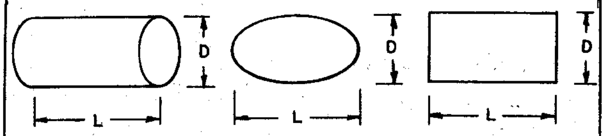
TABLE I. Shape factors c_f of spheroids.

Oblate spheroid		Prolate spheroid	
			
$\gamma = L/D$	c_f	$\gamma = L/D$	c_f
1[a]	1.000	1[a]	1.000
0.8	1.000	2	1.006
0.6	0.998	3	1.023
0.4	0.991	4	1.045
0.2	0.968	5	1.069
0.0[b]	0.900	6	1.093
		7	1.117
		8	1.140

[a] a sphere
[b] a circular disk

c_f is calculated from the formula for ellipsoid in Ref. 9.

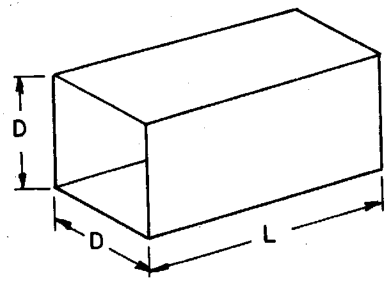
TABLE II. Shape factors of three elongated bodies.

	Cylinder	Ellipt. disk	Rect. disk
			
$\gamma=\frac{L}{D}$	c_f[a]	c_f[b]	c_f[c]
1	0.922	0.900*	0.904**
2	0.950	0.928	0.931
3	0.983	0.969	0.970
4	1.013	1.010	1.002
5	1.041	1.048	*a circular disk
6	1.067	1.085	**a square disk
7	1.091	1.115	
8	1.114	1.151	

[a] c_f is taken from Ref. 8 with absolute error bounds of $\pm 3\%$.
[b] c_f is calculated from the formula of ellipsoid in Ref. 9.
[c] c_f is from point matching method with square subareas, with 8 subareas across D.

TABLE IV. Comparison of capacitance of hollow square pipes and solid prism of the same dimensions.

$\gamma=\frac{L}{D}$	Hollow pipe C_h(pF)	Percentage C_h from C_s	Prism	
			C_s(pF)	c_f
1	70.1	−4.4%	73.3	0.953[a]
2	92.8	−2.7%	95.4	0.961
3	112.5	−2.0%	114.8	0.977
4	130.5	−1.7%	132.7	0.996



$D = 1$ m

Hollow pipe (prism with end plates removed)

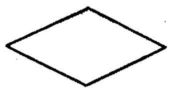
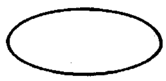
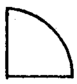
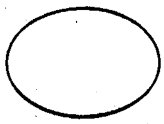
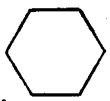
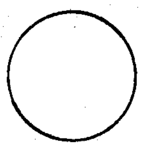
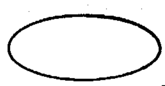
All capacitances are calculated from the point matching method, with 10 square subareas across D.
[a] At $\gamma = 1$ the solid prism is a cube. The c_f obtained by the point matching here happens to agree to the third significant figure with that calculated from Ref. 8. The c_f from Ref. 8 has absolute error bounds of $\pm 0.9\%$.

These two properties are observed not only for the spheroids but also for other structures listed in Table II and III. Table II shows the shape factor of the right cylinders, elliptical disks, and rectangular plates. The chosen aspect ratios of these three structures are defined in Table II.

Table III shows that shape factor of disks of arbitrary shapes and their equivalent elliptical disks (of the same aspect ratio). A comparison of Table I–III shows that since the shape factor is a slowly changing function, the shape factor of an arbitrarily shaped but convex conductor can be approximated by the factor of an ellipsoid of the same aspect ratio.

In an arbitrarily shaped conductor the definition of the aspect ratio is frequently a bit arbitrary. Such arbitrariness is not important, however, since the shape factor is a slowly changing function of the shape.

TABLE III. Comparison of shape factors or arbitrary disks and elliptical disks of similar aspect ratio γ.

Arbitrary Disks	c_f[a]	Elliptical Disks	c_f[b]
 2 x 1 diamond	0.928	 $\gamma = 2$	0.928
 a quadrant	0.918	 $\gamma = \sqrt{2}$	0.912
 a hexagon	0.897	 $\gamma = 1$	0.900
 half circle	0.933	 $\gamma = 2$	0.928

[a] All c_f of arbitrary disks are obtained from Ref. 10.
[b] All c_f of elliptical disks are calculated from the formula in Ref. 9.

V. THE SHAPE FACTOR OF CONDUCTORS WITH CONCAVE SURFACES

We begin this section by comparing a hollow square pipe (a concave conductor) to a solid prism (a convex conductor) of the same dimensions as shown in Table IV.

Let both the prism and the hollow pipe be charged to a potential of V_s. Let Φ_0 be the *exact* potential distribution around the hollow pipe. Now let Φ_0 also be the *approximate* potential distribution around the solid prism. Since the side plates of the prism are the same as those of the hollow pipe there is no boundary potential error on the side plates of the prism due to the approximation of Φ_0. If the length L of the prism is longer than the width D, we observe that (i) on the end plates of the prism Φ_0 still approximates V_s [i.e., $(\Phi_0/V_s - 1)^2$ is a small number], and (ii) the end plate area is a small area out of the surface area of the prism. Based on these two observations we see in Eq. (3) that the ratio $\Delta C/C_1$ can only be a small number. This means that the difference ΔC is small between the capacitances C_1 and C_0, respectively, of the solid prism and the hollow pipe of the same dimensions.

The solid prism can be considered to be the smallest of a convex conductor that can circumscribe the hollow square pipe. Therefore, with a little thought, on other similar cases, we can say in general that the capacitance of a conductor with concave surfaces can be approximated by the capacitance of a smallest convex conductor that can circumscribe the concave conductor.

As an example, Table IV compares the capacitance of a series of hollow square pipes to that of the circumscribing convex conductors of solid prism. One expects that the capacitance of a long pipe, say, the aspect ratio of $\gamma = 4$, is close to that of the circumscribing long prism. However, one sees in Table IV that even for a short pipe of $\gamma = 1$, the capaci-

TABLE Va. Concave circumscribing structures and similar structure with known shape factor.

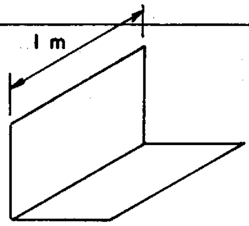
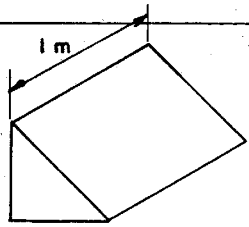
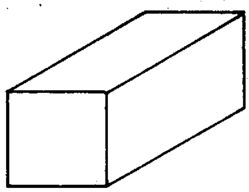
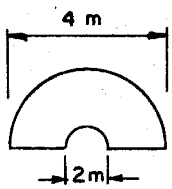
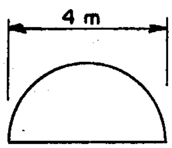
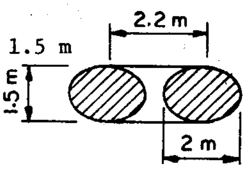
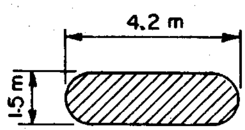
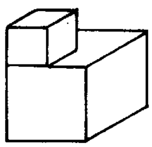
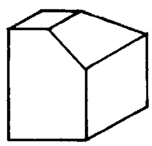
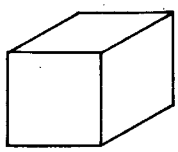
	(I)	(II)	(III)
	Concave structure	Circumscribing structure to (I) with known surface area and unknown γ	Structure with similar γ as (II) but known c_f
A	 folded sq. plate	 right Δ prism	 $\gamma=\sqrt{2}$ square prism
B	 half annulus	 half circle	same half circle
C	 Toroid with elliptic x-section (x-section shown)	 thick circular disk (x-section shown)	 ob. spheroid $\gamma \doteqdot 0.36$
D	 0.5 m cube on 1 m cube	 a trapezohedron on cube	 cube

TABLE Vb. Parameters taken from the structures in Table Va to calculate and compare the approximated capacitances C_s of the concave conducting structures of column (I).

	(I) Capacitance C_h of (I) (pF)	(II) Surface area S of (II) (sgm)	(III) Shape c_f of (III)	Approximate capcitance $C_a = c_f \epsilon_0 (4\pi S)^{1/2}$ (pF)	Percentage C_h from C_a
a	42.1[a)]	1.96	0.957[c)]	42.1	0.0%
b	95.0[b)]	12.57	0.933[d)]	103.8	− 8.5%
c	189.5[a)]	38.7	0.986[e)]	192.5	− 1.6%
d	76.2[g)]	7.06	0.953[f)]	79.5	− 4.2%

[a)] C_h is calculated by point matching method using approximately 10 matching points per side of plate.
[b)] C_h is taken from Ref. 10.
[c)] c_f is taken from Table IV.
[d)] c_f is taken from Table III.
[e)] c_f is calculated from the formula for ellipsoid in Ref. 9.
[f)] c_f is taken from Table IV.
[g)] C_h is taken from Ref. 11.

tance is only 5% from that of the circumscribing prism, which is a cube.

Tables V(a) and V(b) compare the capacitance of other concave conductors to that of the circumscribing convex conductors. Column I gives [in Table V(a)] the shape of the conductor and [in Table V(b)] the numerically calculated capacitance. Similarly, column II gives the circumscribing conductors and their surface areas.

Most of the circumscribing convex conductors listed in column II do not have known shape factor, therefore column III gives the closet convex conductors with the known shape factor. The approximated capacitances C_a are then calculated from Eq. (7) using the areas in column II and the shape factor c_f for column III. One sees that the approximate capacitances resulted are generally accurate.

VI. DISCUSSION

Both the theoretical proof and the large number of examples from Tables I–V show that the shape factor of capacitance is indeed a slowly changing value with respect to the change in shape. With the shape factor of the sphere normalized to unity, the examples show that the shape factor ranges from a minimum of 0.9 of a flat circular disk to 1.15 of a long elliptical plate of aspect ratio of 8. As the aspect ratio gets still bigger, the shape factor does approach infinity. However, one can show, based on the approximation to the capacitance of an ellipsoid,[7] that the shape factor is very slowly changing with γ, the aspect ratio. That is, $c_f = [(2\gamma)^{1/2}/\ln(4\gamma)]$ for $\gamma > 5$ with error $< 1\%$.

For convex conductors the minimum of the shape factor seems to be at 0.9 for a circular disk. Table III gives the shape factor of the hexagonal disk at 0.897. As this number is only 0.3% from 0.9, this may be only a numerical error. As pointed out at the end of Sec. III, the numerical errors may be up to 3%.

The shape factor is not defined for a conductor with concave surfaces. However, one can always circumscribe the concave conductor with a convex conductor. The shape factor of the convex conductor is defined. Using the shape factor and the surface area of the convex conductor, a capacitance can be calculated from Eq. (7). The capacitance of the concave conductor is normally slightly lower than this calculated value. Exceptions to this rule are concave bodies, of which the smallest circumscribing body still encounters large potential error on integrating over the circumscribing surface in Eq. (3). Some of these bodies are the half annulus of Tables V(a) and V(b), a square pipe of Table IV but with $\gamma < 1$ or, similarly, a toroid with large center hole.

The shape factor of a convex conductor, either the circumscribing one or the original one, may not be known. However, since the shape factor is a slowly changing value with the shape, the known shape factor of a similar convex conductor can be used. Tables III and V show that the errors involved are usually less than 5%.

Since the shape factor is a slowly changing value, Tables I–V indicate that even if the shape factor is taken to be unity (i.e., that of a simple sphere) instead of the values tabulated, the capacitance calculated from Eq. (7) normally has an error not larger than 10%.

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