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Short Communication

THE CAPACITANCES OF TWO ARBITRARY CONDUCTORS

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The capacitance or charges of two arbitrary conductors are normally computed by numerical methods. There are a number of well known methods, e.g. the point matched boundary integral method [1,2], the charge simulation method [3] and the finite element method [4]. All these numerical methods frequently require substantial computer time and programming effort. It would be desirable to have some simple method in place of these numerical methods and still give acceptable errors of say less than 5 or 10%.

There is, in principle, a simple method to calculate the charges and capacitance of a two conductor system. The method is based on the elastance matrix by Smythe [5], with elastance defined as the reciprocal of capacitance. Through the inversion of a 2× 2 elastance matrix, Smythe obtained the capacitance matrix and, subsequently, simple formulas (i.e. eqn. (2) to (3)) for the charge and capacitance calculations of the two conductors.

The simplicity of these "two conductor" formulas actually depends on the simplicity of obtaining the capacitance of each conductor in isolation (i.e. eqn. (1a) and (1b)). Recently a simple "(square) root of area" formula of such capacitance for arbitrarily shaped conductors has been derived [6]. As a result the simplicity of these formulas has been extended, from a few simple conductor shapes (such as spheres or prolate spheroids) to a vast family of arbitrarily shaped conductors (such as cubes, polygon plates or even short pipes).

In addition to being simple, the accuracy of the formulas is high. In fact the accuracy is so high that the restriction of "distant conductors" from Smythe [5] can be substantially relaxed or even deleted. For example, with two conductors of equal potential, the two conductors can touch each other and frequently the error in capacitance is still less than 5%.

When two capacitances touch, they can be regarded as being fused into one single complex conductor. Therefore the simple "two-conductor of equal potential" formula of Smuthe can also be used for some complex shaped conductors (such as a notched rectangular plate, or a small cube on top of

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a large cube). Comparisons of the single capacitances from the simple "two conductor" formula are made not only with those from the boundary integral method, but also with those from the simple "root of area" formula of [6].

In the conclusions, limitations of the simple formulas are discussed.

1. The elastance matrix and its capacitance formulas

The elastance matrix equation of Smythe [5] for two conductors with potentials V and charges Q is:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{1}$$

where the self elastances are

$$\xi_{11} = 1/C_1 \tag{1a}$$

$$\xi_{22} = 1/C_2 \tag{1b}$$

and the mutual elastances are approximately

$$\xi_{12} = \xi_{21} = \frac{1}{4\pi\epsilon_0 d}\ ; \tag{1c}$$

C_1 is the capacitance of conductor 1 in isolation, similar for C_2, and d is the distance between the centroids of the two conductors.

The inversion of the elastance matrix equation gives the capacitance matrix equation:

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{2}$$

where

$$C_{11} = \frac{C_m^2\, C_1}{C_m^2 - C_1 C_2}$$

$$C_{22} = \frac{C_m^2\, C_2}{C_m^2 - C_1 C_2}$$

$$C_{12} = C_{21} = -\frac{C_1 C_2}{C_m}$$

and

$$C_m = 4\pi\epsilon_0 d\ , \tag{3}$$

the inverse of mutual elastance in (1c).

Equation (2) reduces to a simpler form when the two conductors have the same isolation capacitance, i.e. $C_1 = C_2$. Then if the two conductors are of equal potentials their total capacitance is

$$C_e = \frac{2\,C_1 C_m}{C_m + C_1}\ , \tag{4}$$

if the two conductors are of opposite potentials the capacitance between them is

$$C_0 = \frac{C_1 C_m}{2(C_m - C_1)}\ . \tag{5}$$

2. The capacitance of a conductor in isolation

Recently Chow and Yovanovich [6] have proved analytically and numerically that the capacitance of a convex conductor in isolation is given by the "roots of area" formula

$$\mathrm{C} = c_f\, \epsilon_0 \sqrt{4\pi S} \tag{6}$$

where S is the surface area of the conductor and c_f is called the "shape factor" of the conductor.

Before discussing the significance of c_f, it is to be noted that a plate is considered to be a compressed convex body; therefore its area S is equal to twice the surface area of one side of the plate. Also it is to be noted that if a concave conductor is enveloped by a minimum convex conductor, then the capacitance of the minimum convex conductor approximates that of the concave conductor [6].

Some values of c_f for different convex body have been tabulated in [6]. Nevertheless the significance of c_f lies in the fact that it is a very slowly changing value with shape. For most commonly encountered bodies, c_f is found to range from 0.9 to 1.1 with $c_f = 1$ for a sphere [6]. Therefore the unknown c_f of a conductor can be approximated by that of a similar conductor. For example, a square and a circular disc have the same shape factor of $c_f = 0.90$. Also with the same length to width ratio of 4.0, a finite cylinder and a prolate spheroid have similar shape factors of $c_f = 1.01$ to 1.04 respectively.

3. The example of two identical spheres

We shall begin with a simple example of the capacitance of two identical conducting spheres of radius r_0 and a centre to centre separation of d as shown in Fig. 1. In this case

$$C_1 = C_2 = 4\pi\epsilon_0 r_0 \tag{7}$$

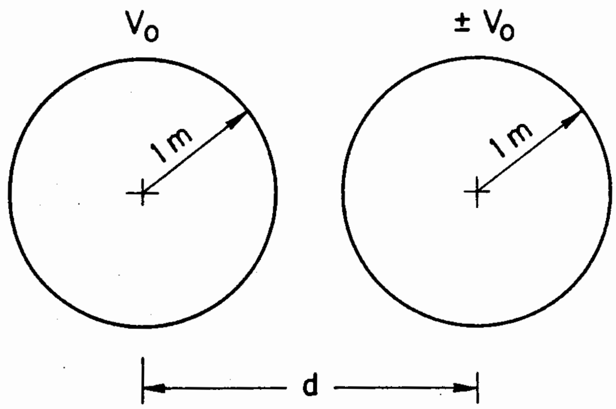

Fig. 1. Two identical conducting spheres.

and

$$C_m = 4\pi\epsilon_0 d \tag{3}$$

with $r_0 = 1$ m. The capacitance C_e and C_0 of equal and opposite potentials are calculated from (4) and (5) and plotted in Fig. 2.

Figure 2 shows these results agree very well with those from the exact solutions with 100 images from Smythe [7]. In fact, for the equal potential case the agreement is still within 3% when the two spheres touch.

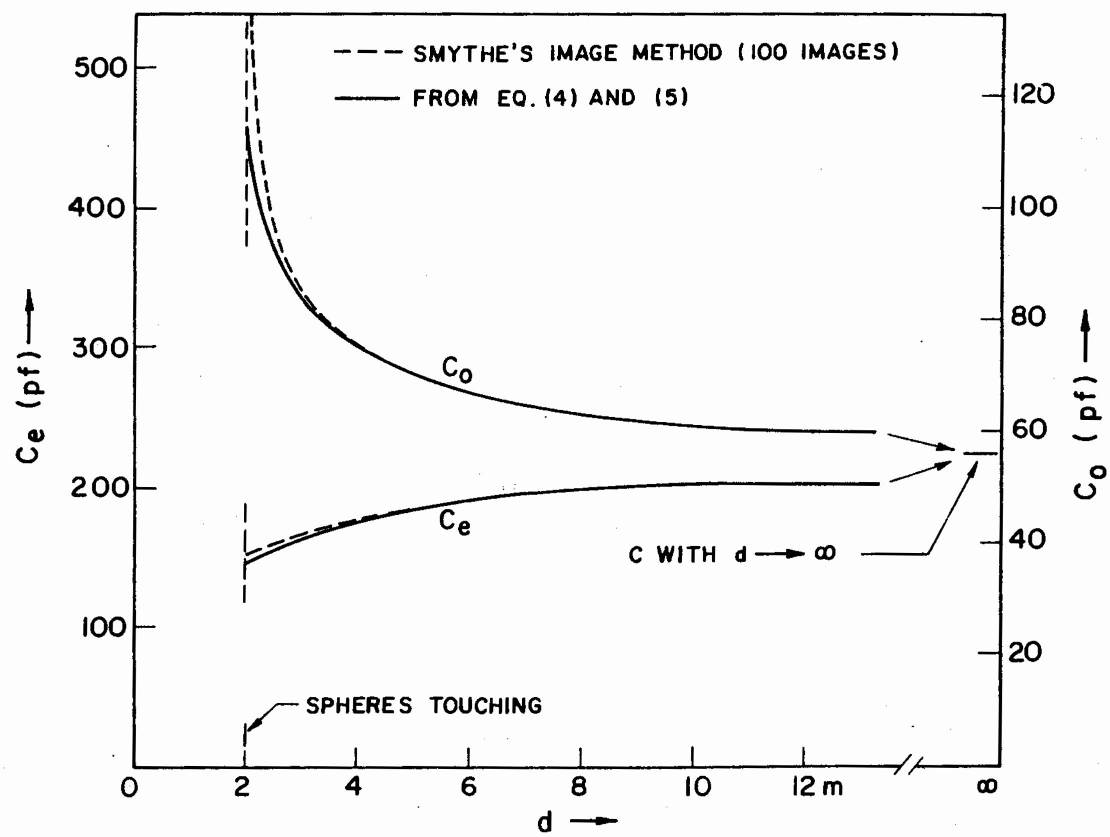

Fig. 2. The capacitances of equal and opposite potentials on the two spheres in Fig. 1, vs. separation.

4. Example of two identical conducting square plates

The approximation in (1c) gives the mutual elastance ξ_{12} as a function of centre separation and not a function of orientations of general conducting bodies with respect to each other. This implies that the true mutual elastance ξ_{12} (and therefore the capacitances of the two bodies with equal and opposite potentials) is only weakly dependent on the orientations of the conducting bodies.

An example of two identical conducting square plates is to demonstrate this observation. As shown in Fig. 3, one conducting square is made to rotate around but parallel to the other conducting square. The capacitances of equal and opposite potentials are then computed by the point-matching moment method [1] and are plotted in Fig. 4. To ensure a two significant digit accuracy, 64 match points in each square plate are used in the method.

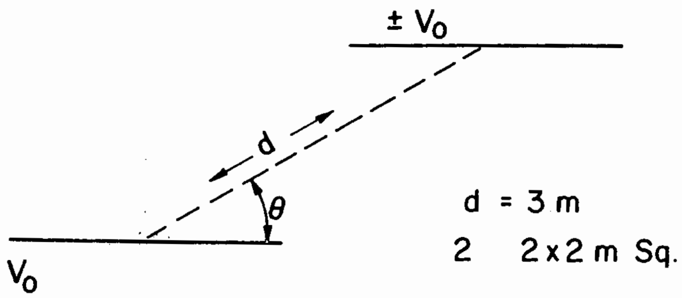

Fig. 3. Two identical conducting plates.

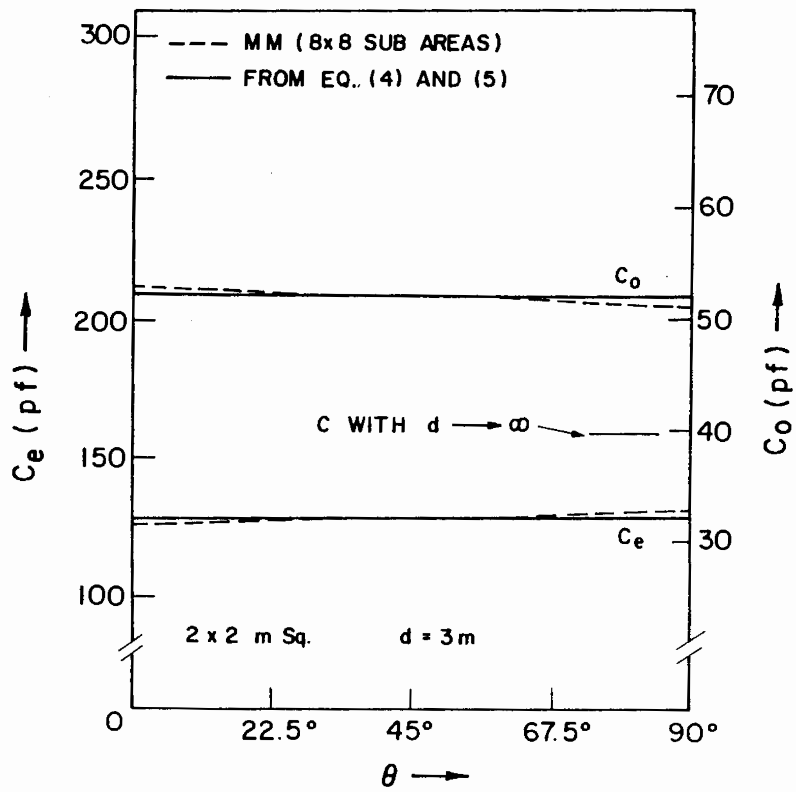

Fig. 4. The capacitances of equal and opposite potentials on the two square plates in Fig. 3, vs. orientation.

In addition to the moment method calculation, the reduced formulas (4) and (5) of C_e and C_0 are also calculated using the mutual elastance in (1c) and the self elastance in (1a), equalling the reciprocal of the capacitance C_1 of an isolated square plate. Based on Chow and Yovanovich [6], the capacitance of an isolated square plate of sides $2b$ may be taken as

$$C_1 = \epsilon_0 c_f \sqrt{4\pi(2b)^2 2} \quad (8)$$

with $c_f = 0.904$.

When the point-match results and these from (4) and (5) are computed in Fig. 4, one sees that the capacitances are indeed only weakly dependent on the orientation of the square plates, and the constant capacitances C_e and C_0 given by (4) and (5) do closely approximate the point-matched values of capacitances for all orientations of square plate separations.

Orientations can also mean the rotations of the bodies with respect to their own centroids. Based on the same reason as the above example, it is expected that the capacitances are also weakly dependent on such type of orientations.

5. Example of a square and a rectangular conducting plate

The two examples before all have two identical conducting bodies. In this example as shown in Fig. 5 the two bodies are different, that is: a conducting square plate and a conducting rectangular plate. Therefore, instead of (4) and (5) the general formula (1) has to be used. When the two plates have different areas the capacitance of opposite potentials is difficult to define. Therefore, only the equal potential case is studied.

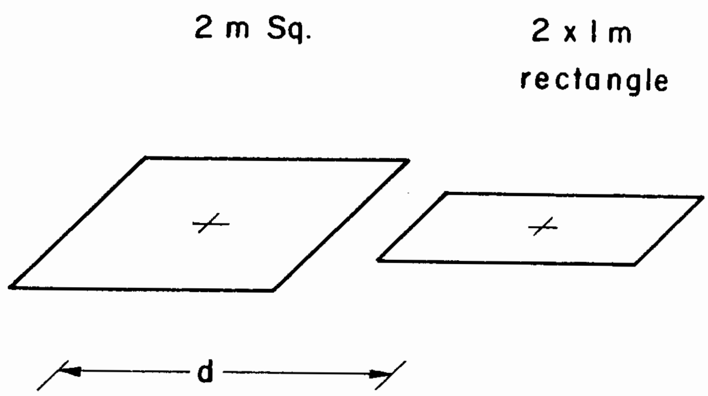

Fig. 5. A square and a rectangular conducting plate.

The capacitance of the isolated square plate has been given by (8). Based on Chow and Yovanovich [6], the capacitance of the isolated rectangular plate of side ratio of 2 to 1 in Fig. 5 can be taken to be

$$C_2 = \epsilon_0 c_f \sqrt{4\pi 2b^2 2} \quad (9)$$

where $c_f = 0.931$ and $2b$ is the long side dimension. Now we approximate the self elastance $\xi_{11} = 1/C_1$, and $\xi_{22} = 1/C_2$ and the mutual elastance ξ_{12} by (1c). When the elastances are substituted in (1), the capacitance (vs. separation) is calculated.

The capacitance from the calculation is plotted in Fig. 6. When compared with that from the point-matching method, the agreement is still very good. In fact it is observed when the rectangular plate touches the square plate, the difference is still only 3% between the two capacitances.

On the other hand, the difference in complexity between the two methods is striking. While (1) is quite simple, to ensure a two significant digit accuracy the point-matching method has required 96 equally spaced matching-points, 64 on the square and 32 on the rectangle.

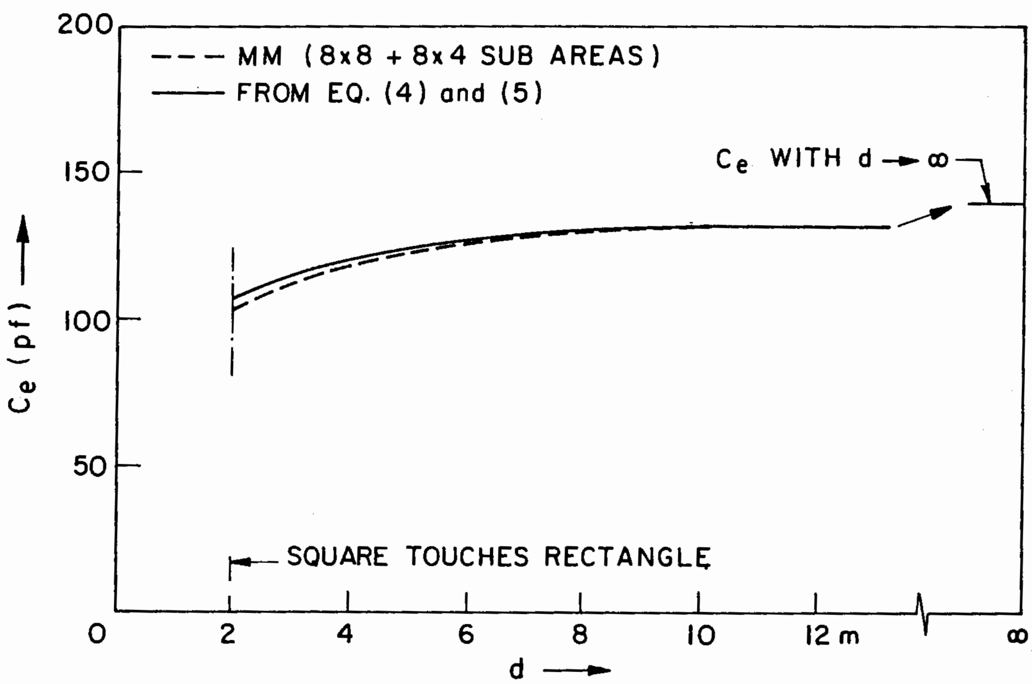

Fig. 6. The capacitance of equal potentials on the square and rectangular conducting plates in Fig. 5, vs. separation.

6. Complex simple conductors by fusing two conductors

When two conductors of the same potential touch they can be considered to be fused into one complex conductor. However, despite such close separation, the capacitances calculated from the two conductor formula from (4) or from its more general form (1) are still very accurate when compared with those from the boundary integral method, in Table 1. Details of the geometries in Table 1 are given in Fig. 7.

It may be noted that when the conductors fuse, the fused surface areas of the two conductors are not considered to disappear in the calculation of the

TABLE 1

Examples of conductors formed by fusing two parts. Capacitance in pF (with errors from II in parenthesis)

	I Two conductor formula from eqns. (1), (3) and (6) (pF)	II Numerical method results* (pF)	III $C = c_f \epsilon_0 \sqrt{4\pi S}$ from eqn. (6) (pF)	 Approx. c_f value and its source
Two equal spheres (Figs. 7a and 1)	148.4 (−4%)	154.8 [7]	158.3 (+2%)	$c_f = 1.006$ (from a 2 :1 prolate spheroid [6])
A square and a rectangle plate (Figs. 7b and 5)	117.3 (+1%)	116.5	109.0 (−4%)	$c_f = 0.928$ (from a 2 :1 elliptic plate [6, 8])
Half annulus (fusing of two quarter annulus) (Fig. 7c)	101.1 (+6%)	95.0 [1]	103.8 (+9%)	$c_f = 0.933$ (from a half circular disc [6, 8])
A small cube on a large cube (Fig. 7d)	75.6 (−1%)	76.2 [9]	79.5 (+4%)	$c_f = 0.953$ (from a cube [6])
Two squares fused at right angle (Fig. 7e)	53.1 (−4%)	55.2 [9]	62.8 (+14%)	$c_f = 0.953$ (from a cube [6])
Four square plates forming a hollow pipe (Fig. 7f)	67.4 (−4%)	70.1 [9]	73.3 (+5%)	(from a cube [6])

*Over 50 evenly spaced match-points are used in each numerical computation.

self elastance of each conductor. An example of this fusing is in Fig. 7d.

It may also be pointed out that the last complex conductor in Table 1 is a hollow pipe formed by four square plates. This means that the 2×2 matrix in (1) has to be expanded into a 4×4 matrix. Nevertheless, because of the four-fold symmetry, the 4×4 matrix is reducible to 2×2. The details of the 4×4 matrix and its reduction are very simple. Therefore they are not derived here.

A complex conductor resulted from the fusing of two parts is usually a concave conductor as shown in the examples of Fig. 7. However, as discussed in Section 2.3, with a modification in the area S the "root of area" formula of (6) still applies. As parts of (1) and (4) come from (6), they cannot be as simple as (6). Therefore results from (6) are also included in Table I for comparison. For eqn. (6) in Table I, the c_f are approximated from similar conductors. The names of the similar conductors are listed.

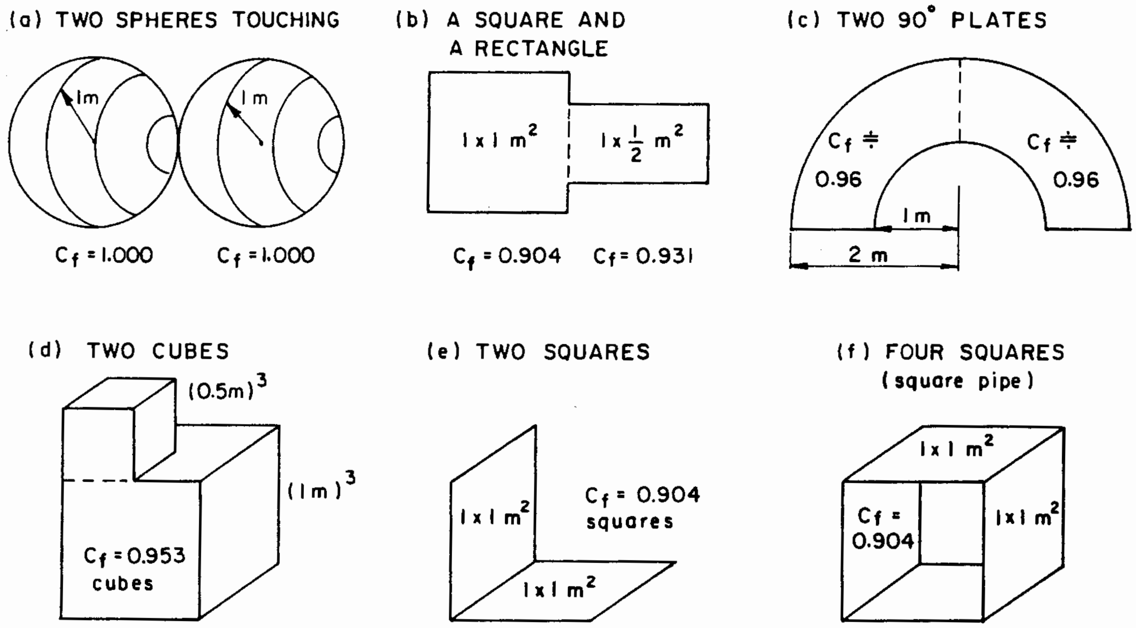

Fig. 7. Complex single conductors by fusing two parts. The c_f value of each part is marked.

Table 1 shows that, with only slightly more complexity, (1) and (4) generally give less error than that from (6).

The two conductors in Table I are fused and the error is still quite small. It is evident therefore if the two conductors were separated, (1) and (4) would give even more accurate results.

7. Conclusions

The first three examples in Figs. 2, 4 and 6 show that the "two conductor" formulas of (1), (4) and (5) are accurate from the infinite separation to very short separations. In the equal potential case of (1) or (4), the two conductors can actually touch and the error is only less than 5%.

When two conductors of the same potential touch, they can be considered to be fused into one complex conductor. Table 1 shows that in many cases the errors are indeed less than 5%.

Table 1 also shows that the application of the "two conductors" formula of (1) to one complex conductor usually gives a better capacitance than the "root of area" equation of (6). The "two conductor" formula (1) however makes use of (6), therefore is slightly more complex than (6) although (1) is still very simple.

There is a situation nevertheless where the "two conductor" formula does not give good results, regardless whether or not the two conductors are of equal or opposite potentials. This is the situation in which the mutual elastance of (1c) is larger than a half of the self elastance of (1a) and (1b).

This large mutual elastance invariably means that the separation between the centroids of the two conductors is smaller than the largest dimension of the conductors. Examples of this situation are the parallel plate capacitor and a sharply folded plate with the angle between the two parts of the plate at or less than the 90° in Fig. 7e. For this situation, a different approximate formula would have to be developed.

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