Numerical Methods in Thermal Problems

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THERMAL RESISTANCE OF ARBITRARILY SHAPED CONTACTS

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Summary

A novel analytical-numerical surface element method is proposed for computing the centroidal and the average temperature rise of single and multiple contact areas subjected to uniform or distributed heat flux. Certain geometric parameters for an arbitrary surface element are developed using analytical solutions for right triangles. The accuracy and efficiency of the method is demonstrated by the computation of the centroidal temperature rise of single arbitrary areas. Geometric parameters are also developed for two interacting contact areas of arbitrary shape. These parameters are used to determine the constriction resistance of a set of identical contact spots. It is demonstrated that a continuous square contact area can be replaced by a finite set of identical contact spots.

Introduction

In a wide range of fields, from the microelectronics to the nuclear industry, the so-called thermal contact resistance needs to be determined to evaluate the overall thermal performance of systems. This contact resistance is defined as the average temperature rise of the contact area of the thermal source divided by the total heat flow rate from the source.

Specifically one may encounter single or multiple thermal contacts of simple or arbitrary geometries on the surface of a half-space. The flux distribution over the contacts may be uniform in one extreme or give a uniform contact temperature in the other extreme. The proposed paper outlines how all of the above mentioned problems can be resolved conveniently and economically by a common approach.

THERMAL RESISTANCE OF ARBITRARILY SHAPED CONTACTS

Single Contact: Arbitrary Shape

Consider a single planar contact of arbitrary shape located on the surface of a semi-infinite half space as shown in Fig. 1. The governing differential equation is

$$\nabla^2 T = 0 \tag{1}$$

subject to the boundary conditions

$$-k\frac{\partial T}{\partial z} = q \text{ , uniform flux within contact region} \tag{2}$$

$$\frac{\partial T}{\partial z} = 0 \text{ , insulated outside contact region} \tag{3}$$

$$\text{and } T \to 0 \text{ as } \sqrt{(x^2 + y^2 + z^2)} \to \infty \tag{4}$$

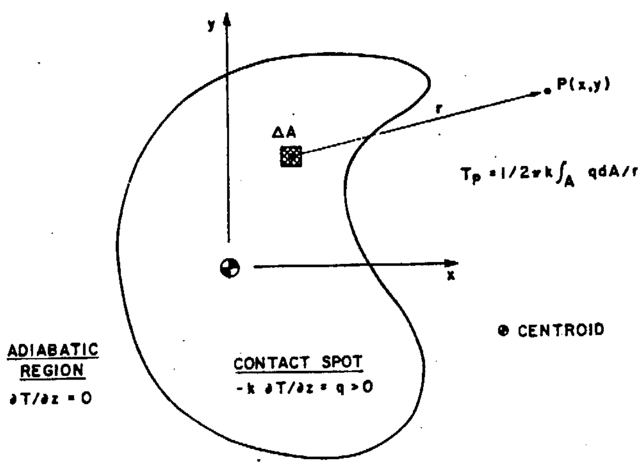

Figure 1 Schematic of Arbitrary Contact Area

Superposition of the solution for a point source subject to (1), (2), (3) and (4) gives the temperature rise at any point P on the plane z = 0 as

$$T = \int_A \left\{\frac{q}{2\pi kr}\right\} dA \tag{5}$$

where r denotes the distance measured from P shown in Fig. 1.

For arbitrarily shaped contact areas, closed form solutions of Eq. (5) generally cannot be obtained. Nevertheless, if P lies outside the contact area, the temperature rise at P may be calculated accurately by a conventional numerical integration scheme. However this is not possible if P is a point within the contact area, since at this point r = o and the integrand is singular. This difficulty may be avoided by considering separately the inner and outer discretized triangular areas shown in Fig. 2. The inner triangular areas have a common apex at P and the temperature rise there due to each of these is given exactly by [1]

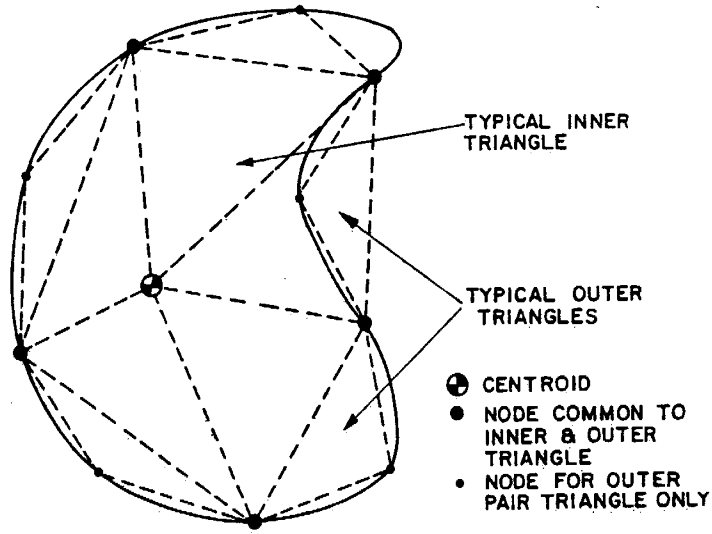

Figure 2 Discretization of Arbitrary Contact Area

$$T_{IN} = \frac{q\delta}{2\pi k} \ell n \left\{ \frac{\tan\left[\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\left(\frac{\overline{AC}}{\delta}\right)\right]}{\tan\left[\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\left(\frac{\overline{BC}}{\delta}\right)\right]} \right\} \tag{6}$$

where the distances $\overline{AC}$, $\overline{BC}$ and δ are shown in Fig. 3.

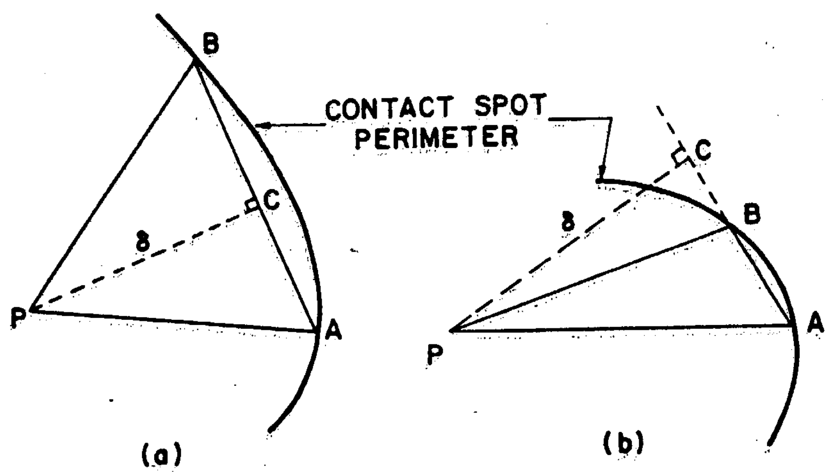

Figure 3 Inner Triangle Geometric Parameters

The outer triangles, each of which has one common side with an inner triangle, approximate the portion of the contact area beteween its perimeter and the inner triangles. For each outer triangle the temperature rise at P is given exactly by the infinite series (see Appendix).

$$T_{OUT} = \frac{q}{2\pi k} \left\{\frac{A}{R} + \frac{2I_o - 3I_{RR}}{2R^3} + \ldots\right\} \tag{7}$$

where A, R, I_o and I_{RR} denote, respectively, the area, the distance from P to the centroid of the outer triangle, the polar second moment of area and the second moment of area with respect to the line of length R, all as shown in Fig. 4.

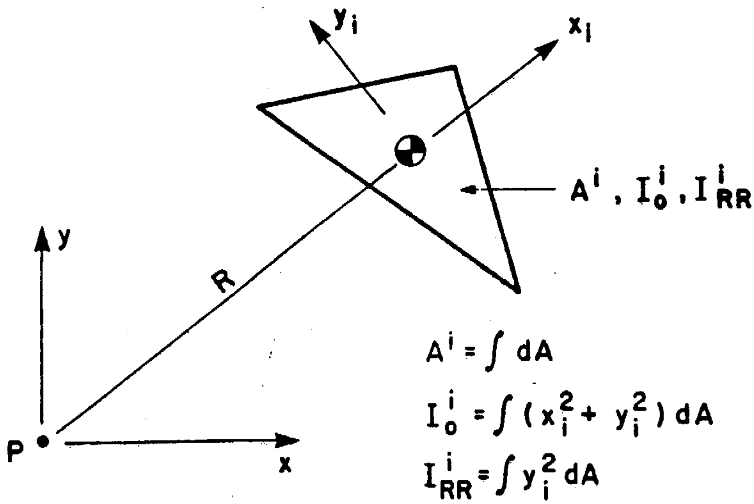

Figure 4   Outer Triangle Geometric Parameters

Summing the contributions from each triangle gives the required temperature rise at P. Thus

$$T_P = \Sigma\, T_{IN}^i + \Sigma\, T_{OUT}^i \tag{8}$$

the superscript denoting the contribution from the ith inner and outer regions.

The convenience, efficiency and accuracy of this technique for computations of the temperature rises and thermal constriction resistances will now be examined for the family of areas within the hyperellipses

$$(\frac{x}{a})^n + (\frac{y}{b})^n = 1$$

and for a semi-circle.

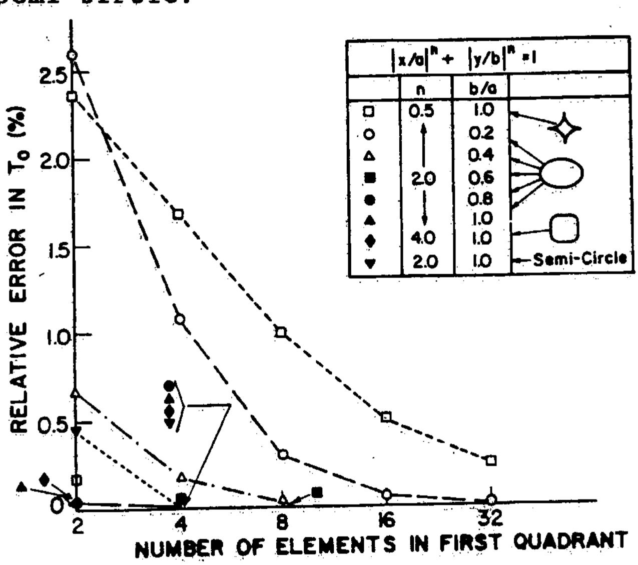

Figure 5   Effect of Discretization on Relative Error of Centroidal Temperature

The boundary points by which the contact area was discretized into inner and outer triangles were positioned such that each inner triangle had the same angle at the centroid. Symmetry permitted consideration of just the first quadrant for each shape. Fig. 5 shows the relative error of the centroidal temperature (or centroidal constriction resistance) versus the number of discretized inner-outer triangular elements.

For all shapes considered, the error decreases rapidly as the number of elements increases giving a more accurate approximation of the true contact area. With only four elements the maximum error was generally much less than 0.2%, except for the hyperellipse with n = 1/2 and the elongated ellipse, (shapes for which the contact area is poorly approximated with less than 8 elements).

In summary, with this method it is very simple to compute precisely the centroidal temperature rise for virtually any arbitrarily shaped contact area subjected to uniform flux.

Multiple Contacts: Arbitrary Shapes

Thermal contact resistance problems frequently involve multiple contacts located close to one another on the surface of a half-space. The temperature rise at the centroid of any one of the contacts and/or the total constriction resistance of the set are often needed.

Centroidal Temperatures

For convenience, the centroidal temperature rise T_i of the ith contact is written as the sum of the contribution from the flux on the ith contact, plus the contributions from all other contacts,

$$T_i = T_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{N} T_{ij} \tag{9}$$

The first of these, T_{ii}, can be calculated by the technique given in the previous section. Each T_{ij} is given by

$$T_{ij} = \frac{q_j}{2\pi k} \left\{ \frac{A^j}{R_{ij}} + \frac{2I_o^j - 3I_{RR}^j}{R_{ij}^3} + \ldots \right\} \tag{10}$$

where R_{ij} is the distance between the centroids of the ith and jth contacts, while A^j, I^j_o and I^j_{RR} denote, respectively, the area, second polar moment of area and second linear moment of areas for the jth contact.

Contact Resistance

In order to compute the total thermal contact resistance, it is necessary to deal with average temperatures. As before the self and mutual effects are separated giving

$$\bar{T}_i = \bar{T}_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \bar{T}_{ij} \tag{11}$$

where the (-) denotes an average value.

For thermal contact resistance problems, individual contact spots would usually be modelled as circles or ellipses. For these shapes the $\bar{T}_{ii}$ have been determined [2] and no further computation is necessary. For future computations, it will be convenient to have $\bar{T}_{ii}$ expressed in the form

$$\bar{T}_{ii} = \frac{q_i}{k} g_{ii} \tag{12}$$

As shown in Appendix I, $\bar{T}_{ij}$ is given exactly by

$$\bar{T}_{ij} = \frac{q_i}{k} g_{ij} \tag{13}$$

$$\text{where } g_{ij} = \frac{1}{2\pi} \left\{ \frac{A^j}{R_{ij}} + \frac{2I_o^j - 3I_{RR}^j}{2R_{ij}^3} + \left(\frac{A^j}{A^i}\right)\left(\frac{2I_o^i - 3I_{RR}^i}{2R_{ij}^3}\right) \right.$$

$$\left. + \left(\frac{12I_o^i - 15I_{RR}^i}{2R_{ij}^5}\right)\left(\frac{2I_o^j - 3I_{RR}^j}{2A^i}\right) + \ldots \right\} \tag{14}$$

with A^j, A^i, I_o^j, I_o^i, I_{RR}^j, I_{RR}^i and R_{ij} defined as previously and shown in Fig. 6. Note that g_{ij} depends only on geometry.

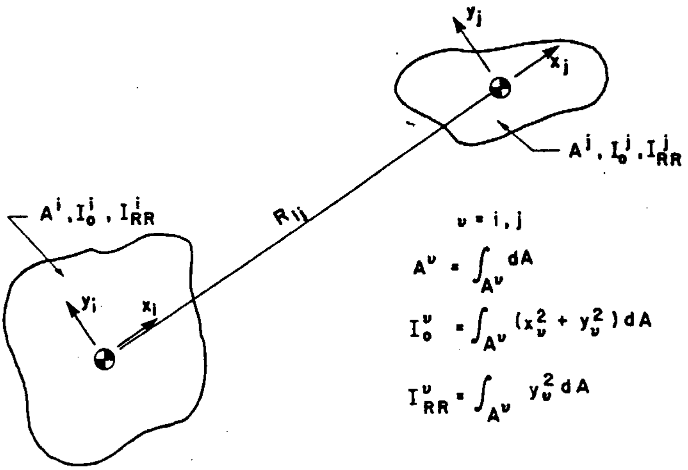

Figure 6 Geometric Parameters for Two Arbitrary Contact Areas

For the full set of contacts, Eqs. (11), (12) and (13) lead to a system of linear algebraic equations of the form

$$\frac{1}{k} [G_{ij}] \{q_j\} = \{\bar{T}_i\} \tag{15}$$

where $[G_{ij}]$ represents a matrix composed of the geometry dependent g_{ii} and g_{ij} terms defined above.

Note that Eq. (15), which was derived for the case of uniform flux on each contact spot, may also be used for isothermal or mixed boundary conditions if q_j is simply interpreted as the average flux on the jth spot.

For any of the above boundary conditions, the total contact resistance of the system of contact spots is given by

$$\bar{R} = \frac{\bar{T}}{Q} = \frac{\dfrac{\Sigma \bar{T}_i A^i}{\Sigma A^i}}{\Sigma q_i A^i} \qquad (16)$$

the summation taken over all spots.

Constriction Resistance of an Arbitrarily Shaped Contact

Consider a singly connected contact area such as shown in Fig. 7(a), and the system of discrete circular contacts shown in Fig. 7(b) which cover the same apparent area as the single contact. For a single arbitrary shape with arbitrary boundary conditions, the constriction resistance would be difficult to compute by conventional techniques. However, it is reasonable to anticipate that the difference in constriction resistance between the single contact and the discrete system will tend to zero as the area covered by the latter system increases towards that of the single contact. For the system of discrete contacts, calculation of the constriction resistance involves only a straightforward systematic computation by the technique described in the previous section.

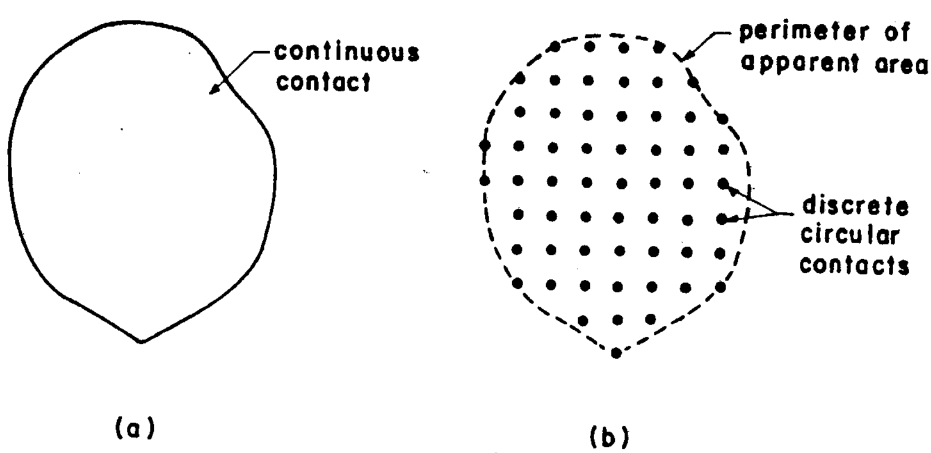

Figure 7 Approximation of Continuous Contact By Discrete Contact Spots

For an example of this method, consider a square contact area as shown in Fig. 8. The apparent area has been discretized by circular contacts and the total area of the circular contacts is increased by uniformly increasing the number or density of the contacts. Table 1 contains the ratio of the resistance of the discrete system, $\bar{R}^*_{CALC}$, calculated by (15) and (16), to the theoretical resistance, $\bar{R}^*_{THEO} = 0.4732$ [2] as the number of contacts, N, and their percentage of the total apparent area, A%, increase. In addition, a simple attempt to extrapolate the calculated $\bar{R}^*_{CALC}$ was made by fitting the latest 3 values of $\bar{R}^*_{CALC}$ to the model

$$C_0 + C_1 N^{-1} + C_2 N^{-2} = \bar{R}^*_{CALC}(N) \qquad (17)$$

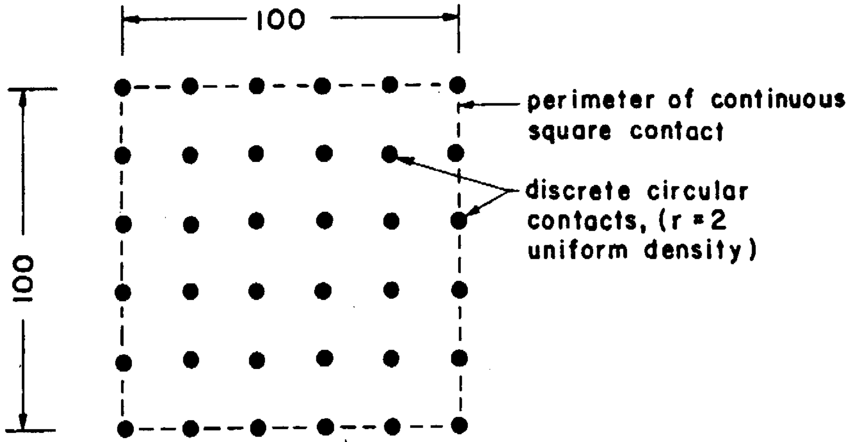

Figure 8 Discretization of Square Contact Area

After determining the coefficients, an evaluation of Eq. (17) at N = 625 (where the contact density is such that the circles touch each other tangentially) produces the value $\bar{R}^*_{PRED}$ which is also reported in Table 1 as a ratio of $\bar{R}^*_{THEO}$.

Note from these results that with a discrete system covering approximately 1/4 of the total apparent area, the resistance of the system is only 2.45% above that of the continuous shape, and by using the simple extrapolation discussed, the resistance of the continuous shape is predicted to within 0.9%.

Table 1

N	Area, %	$\bar{R}^*_{CALC}/\bar{R}^*_{THEO}$	$\bar{R}^*_{PRED}/\bar{R}^*_{THEO}$
9	1.1	3.730	-
25	2.9	1.823	-
49	5.7	1.344	0.903
81	9.4	1.166	0.951
121	14.0	1.086	0.971
169	19.6	1.046	0.982
225	26.1	1.025	0.991
289	33.6	1.013	0.996
361	41.9	1.006	0.999
441	51.2	1.003	1.000

Conclusions

An original analytical-numerical method has been developed for accurate and efficient computation of centroidal and average contact area temperature rise or resistance of arbitrary single or multiple areas. The method was used to determine the resistance of numerous geometries. It was also shown that the method can be used to determine the resistance of a single arbitrary area by means of a set of identical contact spots.

References

1. Yovanovich, M.M., Thermal Constriction of Contacts on a Half-Space: Integral Formulation. Prog. in Astronautics and Aeronautics, Vol. 49, Edited by Allie M. Smith, AIAA, New York, 1976, pp. 397-418.

2. Yovanovich, M.M., Burde, S.S. and Thompson, J.C., Thermal Constriction Resistance of Arbitrary Planar Contacts with Constant Flux. Prog. in Astronautics and Aeronautics, Vol. 56, Edited by Allie M. Smith, AIAA, New York, 1977, pp. 127-139.

APPENDIX I

For the two planar contacts shown in Fig. I-1, the temperature rise at any point P on the ith contact due to a uniform flux q_j on the jth contact is

$$T_r = \left(\frac{q_j}{2\pi k}\right) \int_{A^j} \frac{dA^j}{\rho} \qquad \text{(I-1)}$$

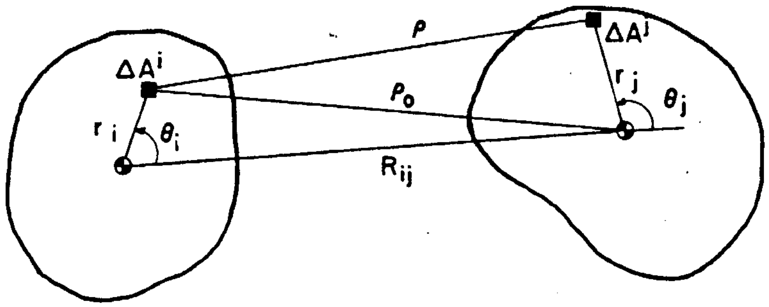

Figure I-1 Local Polar Coordinates on ith and jth Contact Areas

By the cosine law

$$\rho^2 = \rho_o^2 + r_j^2 - 2\, r_j \rho_o \cos\theta_j \qquad \text{(I-2)}$$

Hence

$$T_P = \left(\frac{q_j}{2\pi k \rho_o}\right) \int_{A^j} \left(1 + \frac{r_j^2}{\rho_o^2} - \frac{2 r_j}{\rho_o} \cos\theta_j\right)^{-1/2} dA^j \qquad \text{(I-3)}$$

Since $r < \rho_o$, the binomial theorem may be used to expand the integrand, giving

$$T_p = \left(\frac{q_j}{2\pi k\rho_o}\right) \int_{A^j} \left(1 - \frac{r_j^2}{2\rho_o^2} + \frac{r_j}{\rho_o}\cos\theta_j + \frac{3r_j^2\cos^2\theta_j}{2\rho_o^2} + \ldots\right) dA^j \quad \text{(I-4)}$$

Eq. (I-4) becomes

$$T_r = \left(\frac{q_j}{2\pi k\rho_o}\right) \left\{\int_{A^j} dA^j + \left(\frac{1}{\rho_o}\right)\int_{A^j} r_j \cos\theta_j \, dA_j + \int_{A^j}\left(\frac{2r_j^2 - 3r_j^2\sin^2\theta_j}{2\rho_o^2}\right) dA^j + \ldots\right\} \quad \text{(I-5)}$$

Since

$$\int_{A^j} dA^j = A^j \qquad \int_{A^j} r_j \cos\theta_j \, dA^j = 0 \quad \text{(I-6\&7)}$$

$$\int_{A^j} r_j^2 \, dA^j = I_o^j \qquad \int_{A^j} r_j^2 \sin^2\theta \, dA^j = I_{RR}^j \quad \text{(I-8\&9)}$$

the temperature rise at P may conveniently be expressed in terms of easily computable geometric terms,

$$T_p = \left(\frac{q_j}{2\pi k}\right)\left[\frac{A^j}{\rho_o} + \frac{2I_o^j - 3I_{RR}^j}{2\rho_o^3} + \ldots\right] \quad \text{(I-10)}$$

The average temperature rise on the ith contact is

$$\bar{T}_{ij} = \frac{1}{A^i}\int_{A^i} T_p \, dA^i \quad \text{(I-11)}$$

Thus

$$\bar{T}_{ij} = \left(\frac{1}{A^i}\right)\left(\frac{q_j}{2\pi k}\right)\int_{A^i}\left[\frac{A_j}{\rho_o} + \frac{2I_o^j - 3I_{RR}^j}{2\rho_o^3} + \ldots\right] dA^i \quad \text{(I-12)}$$

The A^j and I_o^j are constant. In this derivation it is assumed that $I^j{}_{RR}$ can be treated as a constant evaluated about the axis joining the centroids of the ith and jth contacts. Then for the terms of the infinite series as shown in (I-12), evaluation of $\bar{T}_{ij}$ requires the integrals,

$$\int_{A^i}\frac{dA^i}{\rho_o} \quad \text{and} \quad \int_{A^i}\frac{dA^i}{\rho_o^3}$$

These integrals are evaluated by making the substitution

$$\rho_o^2 = R_{ij}^2 + r_i^2 - 2r_i R_{ij}\cos\theta_i \quad \text{(I-13)}$$

expanding by the binomial theorem, and integrating as for (I-6) through (I-9), giving

$$\int_{A^i}\frac{dA^i}{\rho_o} = \frac{A^i}{R_{ij}} + \frac{2I_o^i - 3I_{RR}^i}{2R_{ij}^3} \qquad \int_{A^i}\frac{dA^i}{\rho_o^3} = \frac{A^i}{R_{ij}^3} + \frac{12I_o^i - 15I_{RR}^i}{2R_{ij}^5} \quad \text{(I-14\& 15)}$$

Substitution of (I-14) and (I-15) into (I-12) gives the average temperature in terms of readily computable quantities,

$$\bar{T}_{ij} = \left(\frac{q_j}{2\pi k}\right) \left\{\frac{A^j}{R_{ij}} + \frac{2I_o^j - 3I_{RR}^j}{2R_{ij}^3} + \left(\frac{A^j}{A^i}\right) \left(\frac{2I_o^i - 3I_{RR}^i}{2R_{ij}^3}\right) + \left(\frac{2I_o^i - 15I_{RR}^i}{2R_{ij}^5}\right) \left(\frac{2I_o^j - 3I_{RR}^j}{2A^i}\right) + \ldots\right\} \qquad \text{(I-16)}$$

Nomenclature

A	= area
G_{ij}	= matrix of geometric coefficients
g_{ij}	= geometric coefficient
I_o	= polar second moment of area
I_{RR}	= radial second moment of area
k	= thermal conductivity
N	= number of contact spots
P	= field point location
Q	= total heat flow
q	= heat flow per unit area
$\bar{R}^*$	= non-dimensional constriction resistance
R_{ij}	= distance between centroids of ith and jth contacts
r	= radial coordinate
T	= temperature rise
T_o	= centroidal temperature rise
$\bar{T}$	= average temperature rise
x,y,z	= Cartesian coordinates

Greek Symbols

θ	= polar coordinate
π	= pi
ρ	= polar coordinate
ρ_o	= distance from P to centroid of jth contact
∇^2	= Laplacian operator

Subscripts and Superscripts

i,j = reference to ith and jth contacts