THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS
345 E. 47 St., New York, N.Y. 10017

84-HT-52

TRANSIENT CONSTRICTION RESISTANCE FOR ELEMENTAL FLUX CHANNELS HEATED BY UNIFORM FLUX SOURCES

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ABSTRACT

Series expressions for transient constriction resistances are presented for two elemental flux channel geometries: infinite strip contact bounded by adiabats and a circular spot contact on an insulated cylinder. Both flux channels are semi-infinite in the direction of heat flow and the contacts are heated by a uniform flux source. The resistances were derived from the temperature distributions which were determined by solving the heat equation with Laplace transforms. Results are given in tabular and graphical form, and show that the resistances reach the known steady state values. It is also shown that the resistances approach the half space solutions for small constriction ratios.

NOMENCLATURE

A	= area; channel cross-sectional area
A_c	= contact area
A_m, C, E_m	= intermediate coefficients
a	= contact half width or radius
$B(z,t)$	= expression defined by Eq. (12)
b	= channel half width or radius
$C(z,t)$	= expression defined by Eqs. (12), (18) and (30)
$f(\xi,\theta)$	= expression defined by Eq. (28)
$J_0(\cdot), J_1(\cdot)$	= Bessel functions of the first kind of order 0 and 1, respectively
k	= thermal conductivity
m	= integers 1, 2, 3 . . .
p	= Laplace variable
Q	= total heat flow
q	= $\sqrt{p/\alpha}$
q_a	= $Q/2a$ for flux channels
	= $Q/\pi a^2$ for flux tubes
q_o	= $Q/2b$ for flux channels
	= $Q/\pi b^2$ for flux tubes
R, X, Z	= functions of r,x,z used in the separation of variables method
R_c	= constriction resistance
s	= $p + \alpha\lambda_m^2$
T	= temperature
T_i	= initial temperature
t	= time
$\bar{v}$	= function of s defined in text
x, y, z, r	= space coordinates
α	= thermal diffusivity
δ	= roots of functions $\sin\delta = 0$ or $J_1(\delta) = 0$
ε	= constriction ratio, a/b
η_m	= $\sqrt{\lambda_m^2 + q^2}$
λ_i	= eigenvalues
ξ	= x (or r), the transverse coordinates; or x/a used in Eqs. (28) and (29)
$\theta = \frac{\alpha t}{a^2}$	= Fourier modulus
ψ	= dimensionless resistance

<u>Subscripts</u>

i,m,n	= counters
2D	= flux channel or strip on half space
C	= flux tube or spot on a half space

<u>Superscripts</u>

-	= Laplace transformed variable
ss	= steady state for flux channel or tube
*	= transient solution for half space

INTRODUCTION

The idealization of heat flowing in an elemental flux channel has been shown by Cooper et al. (1), and Yovanovich (2) to be extremely useful in determining the thermal resistance between two rough conforming surfaces. Yovanovich (3,4) has derived expressions for the steady state constriction resistance for two elemental flux channel geometries as shown in Figures 1 and 2: (a) a semi-infinite circular cylinder (henceforth referred to as a flux tube) heated by a concentric circular source, and (b) a semi-infinite region bounded by insulated parallel planes (flux channel) heated by an infinite concentric strip. Exact transient temperature solutions of a surface heated flux tube and a semi-infinite body have been presented by Beck (5-7), but not the transient resistances.

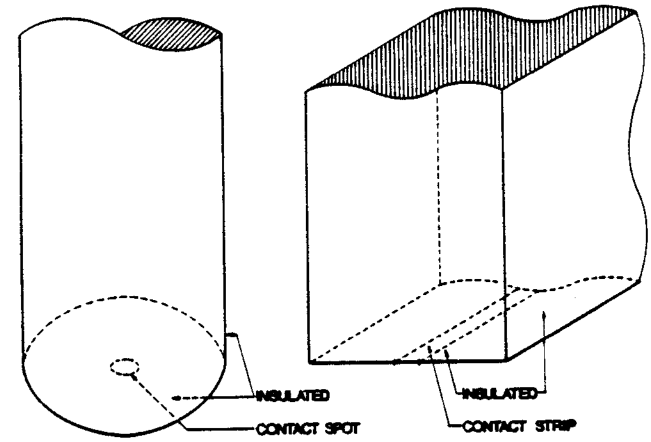

Figure 1 Elemental Flux Channels

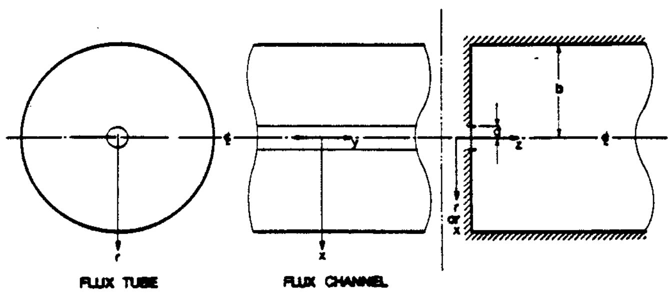

Figure 2 Elemental Flux Channel Geometry

This paper presents expressions for the transient constriction resistance of both elemental flux channel geometries described. The constriction resistance has been defined by Mikic (8) as the difference between the average contact plane and contact temperatures divided by the total heat flow. The temperature solutions are found by the Laplace transform method, and compared to Beck's expressions for the flux tube case. The resistances are deduced from these temperature distributions. Also included are the transient resistances for a circular spot and an infinite strip on a half space as derived from temperature equations in Carslaw and Jaeger (9).

PROBLEM STATEMENT

To derive expressions for resistance, the temperature distribution must first be found by solving the heat equation:

$$\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}.$$

For the flux channel, this is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}; \qquad (1a)$$

for the flux tube:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}. \qquad (1b)$$

The initial condition is a uniform temperature. The boundaries of the flux channels are insulated except for the contact at which a constant uniform flux is applied at $t = 0$. At infinity, the temperature remains at the initial value:

Initial condition:

$$T(\xi,z,t) = T_i \qquad t = 0: \qquad (2)$$

Boundary conditions:

$$\frac{\partial T}{\partial \xi} = 0 \qquad \xi = 0 \quad \text{all } z \qquad (3a)$$

$$\frac{\partial T}{\partial \xi} = 0 \qquad \xi = b \quad \text{all } z \qquad (3b)$$

$$\frac{\partial T}{\partial z} = f(\xi) = \begin{cases} -Q/kA_c & 0 \le \xi \le a \quad z = 0 \qquad (3c) \\ 0 & a \le \xi \le b \quad z = 0 \qquad (3d) \end{cases}$$

$$T(\xi,z,t) = T_i \qquad \text{all } \xi \quad z \to \infty. \qquad (3e)$$

Here, ξ is the transverse coordinate, x or r.

Taking the Laplace transforms (Chap. 12 of Ref. (9)) of Eqs. (1)

$$\frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial z^2} = \frac{p}{\alpha}\tilde{T} - \frac{T_i}{\alpha}. \qquad (4a)$$

$$\frac{\partial^2 \tilde{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{T}}{\partial r} + \frac{\partial^2 \tilde{T}}{\partial t^2} = \frac{p}{\alpha}\tilde{T} - \frac{T_i}{\alpha}. \qquad (4b)$$

where the tildes (˜) denote the transformed temperatures and p is the Laplace variable. The initial condition has become part of the differential equation. If T_i is taken as 0, the solution proceeds easily; the modification to the final solution is trivial if $T_i \neq 0$. The boundary conditions (3) remain unchanged except that T becomes $\tilde{T}$, and (3c) becomes

$$\frac{\partial \tilde{T}}{\partial z} = -\frac{Q}{kA_c p} \quad \text{for} \quad 0 \le \xi \le a, \quad z = 0. \qquad (5)$$

Equation (4) may now be solved by the separation of variables.

Flux Channel Solution

Letting $q^2 = p/\alpha$, and assuming a solution of the form $\tilde{T}(x,z) = X(x)\,Z(z)$, Eq. (4a) becomes

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Z}\left(\frac{d^2Z}{dz^2} - q^2 Z\right) = \pm\lambda^2.$$

Taking $-\lambda^2$ as the separation constant

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \qquad (6a)$$

$$\frac{d^2Z}{dz^2} - (q^2+\lambda^2)Z = 0. \qquad (6b)$$

The solutions to Eqs. (6) are easily obtained. Upon application of boundary conditions (3a), (3b) and (3e), and superposing all the solutions (Eq. (6a) is an eigenvalue problem), the general transformed solution is

$$\tilde{T}(x,z,p) = \sum_{m=0}^{\infty} E_m \cos(\lambda_m x)\exp(-\eta_m z), \qquad (7)$$

where $\lambda_m = \frac{m\pi}{b}$, m = 0, 1, 2, 3 ... (8a)

$$\eta_m^2 = \lambda_m^2 + q^2, \quad (8b)$$

and E_m are constant coefficients. To find these coefficients, apply boundary condition (5) at z = 0:

$$\left.\frac{\partial \tilde{T}}{\partial z}\right|_o = f(x) = -\sum_{m=o}^{\infty} E_m \eta_m \cos(\lambda_m x). \quad (9)$$

Multiply both sides by $\cos(\lambda_n x)dx$ and integrate from x = 0 to x = b with respect to x. (The heat flux in (5) is

$$-\frac{Q}{pkA_c} = -\frac{Q}{pk(2a)} \cdot \frac{b}{b} = -\frac{q_o}{pk} \cdot \frac{b}{a},$$

where q_o is the heat flux per unit length of channel).

$$-\int_0^a \frac{q_o}{kp} \frac{b}{a} \cos(\lambda_n x)dx - \int_a^b 0dx$$

$$= -\sum_{m=0}^{\infty} E_m \eta_m \int_0^b \cos(\lambda_m x)\cos(\lambda_n x)dx.$$

For m = n = 0 $\quad E_o = \frac{q_o}{kpq}$, (10a)

and for m = n = 1,2,3 . . . $E_m = \frac{2q_o}{kp\eta_m} \frac{\sin(m\pi\varepsilon)}{m\pi\varepsilon}$. (10b)

Thus the transformed temperature is

$$\tilde{T}(x,y,p) = \frac{q_o}{k} \frac{\exp(-qz)}{pq} + \sum_{m=1}^{\infty} \frac{2q_o}{k} \frac{\sin(m\pi\varepsilon)}{(m\pi\varepsilon)} \cos(\lambda_m x) \frac{\exp(-\eta_m z)}{p\eta_m}. \quad (11)$$

It now remains to find $T = L^{-1}(\tilde{T})$.

Using the transformations in Carslaw and Jaeger (9), Appendix V, and Laplace transform theory, Eq. (11) can be transformed to the real time domain. (Details of this may be found in the Appendix.) The temperature field is thus:

$$T(x,z,t) = \frac{q_o}{k}\left(B(z,t) + \sum_{m=1}^{\infty} a \frac{\sin(m\pi\varepsilon)}{(m\pi\varepsilon)^2} \cdot \cos(\lambda_m x)\, C_m(z,t)\right), \quad (12)$$

$$\lambda_m = \frac{m\pi}{b}, \quad (12a)$$

$$B(z,t) = \sqrt{4\alpha t}\ \text{ierfc}\left(\frac{z}{\sqrt{4\alpha t}}\right). \quad (12b)$$

$$C_m(z,t) = e^{-\lambda_m z}\ \text{erfc}\left(\frac{z}{\sqrt{4\alpha t}} - \sqrt{\alpha\lambda_m^2 t}\right) - e^{\lambda_m z}\ \text{erfc}\left(\frac{z}{\sqrt{4\alpha t}} + \sqrt{\alpha\lambda_m^2 t}\right). \quad (12c)$$

Flux Tube Solution

The flux tube solution proceeds in the same manner as the flux channel solution. It is assumed that $\tilde{T}(r,z) = R(r)\, Z(z)$. Equation (6b) remains unchanged, and Eq. (6a) becomes Bessel's equation

$$\frac{d^2R}{dr^2} + \frac{1}{R}\frac{dR}{dr} + \lambda^2 R = 0. \quad (13)$$

Boundary condition (5) becomes

$$\frac{\partial \tilde{T}}{\partial z} = \frac{-Q}{k\pi a^2 p} \cdot \frac{b^2}{b^2} = \frac{-Q}{\pi b^2} \cdot \frac{1}{kp} \frac{b^2}{a^2} = \frac{-q_o}{kp} \frac{b^2}{a^2}. \quad (14)$$

The general transformed solution is

$$\tilde{T}(r,z,p) = \sum_{i=0}^{\infty} E_i J_o(\lambda_i r)\exp(-\eta_i z), \quad (15)$$

where the eigenvalues λ_i are obtained from $J_1(\lambda_i b) = 0$ and $\eta_i^2 = \lambda_i^2 + q^2$. Taking $\partial\tilde{T}/\partial z$ at z = 0, multiplying both sides by $rJ_o(\lambda_j r)dr$ and integrating from r = 0 to r = b with respect to r:

$$\frac{-q_o}{kp}\frac{b^2}{a^2}\int_0^a rJ_o(\lambda_j r)dr = -\sum_{i=0}^{\infty} E_i \eta_i \int_0^b rJ_o(\lambda_i r)J_o(\lambda_j r)dr.$$

For i = j = 0 $\quad E_o = \frac{q_o}{kpq}$, (16a)

and for i = j = 1, 2, 3 . . .

$$E_i = \frac{2q_o}{kp\eta_i J_o^2(\lambda_i b)} \cdot \frac{J_1(\lambda_i b\varepsilon)}{\lambda_i b\varepsilon}. \quad (16b)$$

The transformed temperature is

$$\tilde{T}(r,z,p) = \frac{q_o}{k}\frac{\exp(-qz)}{pq} + \sum_{i=1}^{\infty} \frac{2q_o J_1(\lambda_i b\varepsilon)J_o(\lambda_i r)}{k\varepsilon(\lambda_i b)J_o^2(\lambda_i b)} \frac{\exp(-\eta_i z)}{p\eta_i}. \quad (17)$$

In finding the inverse Laplace transform of (17), the flux channel results are useful. The first term of (17) is identical to that in (11), and the exponential part of the second term may be transformed as in (A5). Hence

$$T(r,z,t) = \frac{q_o}{k}\left\{B(z,t) + \frac{b}{\varepsilon}\sum_{i=1}^{\infty} \frac{J_1(\lambda_i b\varepsilon)J_o(\lambda_i r)C_i(z,t)}{(\lambda_i b)^2 J_o^2(\lambda_i b)}\right\}, \quad (18)$$

$\lambda_i b$ are roots of $J_1(\lambda_i b) = 0$, i = 1, 2, 3 . . , (18a)

$$B(z,t) = \sqrt{4\alpha t}\ \text{ierfc}\left(\frac{z}{\sqrt{4\alpha t}}\right), \quad (18b)$$

$$C_i(z,t) = e^{-\lambda_i z}\ \text{erfc}\left(\frac{z}{\sqrt{4\alpha t}} - \sqrt{\alpha\lambda_i^2 t}\right) - e^{\lambda_i z}\ \text{erfc}\left(\frac{z}{\sqrt{4\alpha t}} + \sqrt{\alpha\lambda_i^2 t}\right). \quad (18c)$$

Beck (5) obtained the same result as in Eq. (18) by integrating the Green's function solution for an instantaneous point source in an infinite insulated cylinder in Carslaw and Jaeger (9) §14.13 (7). It should be pointed out here that Beck's heat flux $q = Q/\pi a^2$ (call this q_a so as not to confuse it with $q^2 = p/\alpha$), and so $q_a = q_o \frac{b^2}{a^2}$. Otherwise Beck's solution and Eq. (18)

are identical. It is interesting that both (12) and (18) share common terms viz. $B(z,t)$ and $C_i(z,t)$, where in $C_i(z,t)$, the λ_i's are the eigenfunctions. For both geometries, if the constriction parameter is set $\varepsilon = 1$, the explicit sums in (12) and (18) drop out leaving

$$T = \frac{q_o}{k} \sqrt{4\alpha t}\ \text{ierfc}\left(\frac{z}{\sqrt{4\alpha t}}\right),$$

which is the solution to the problem of uniform heating over the end of the elemental flux channel, or a semi-infinite body (see Ref. (9) §2.9).

AVERAGE TEMPERATURES AND RESISTANCE

The constriction resistance arises due to the fact that the contact width is smaller than the width of the channel. The heat flow lines must spread out into the channel rather than take the easier, straight path (hence the alternative term spreading resistance). The constriction may be visualized as shown in Figure 3. $R_c(t)$ is the spreading or constriction resistance and $R_m(t)$ is the resistance to heat flow due to the material. The constriction resistance is defined by

$$QR_c = \bar{T}_c(t) - \bar{T}(z=0,t), \tag{19}$$

where the over bars denote average temperatures. The average plane temperature is

$$\bar{T}(z,t) = \frac{1}{A}\int_0^A T(\xi,z,t)dA, \tag{20a}$$

and the average contact temperature is

$$\bar{T}_c(t) = \frac{1}{A_c}\int_0^{A_c} T(\xi,0,t)dA. \tag{20b}$$

For the flux channel, the contact plane temperature is

$$\bar{T}(0,t) = \frac{q_o}{k}\left[\frac{B(0,t)}{b}\int_0^b dx + \sum_{m=1}^{\infty} \frac{a\sin(m\pi\varepsilon)}{(m\pi\varepsilon)^2}\,\frac{C_m(0,t)}{b}\int_0^b \cos\left(\frac{m\pi x}{b}\right)dx\right].$$

The integral within the summation is zero. This leaves

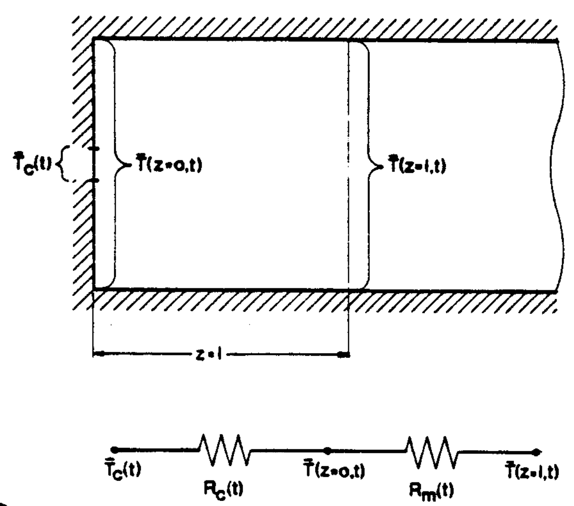

Figure 3 Resistance Circuit in an Elemental Flux Channel

$$\bar{T}(0,t) = \frac{q_o}{k} B(0,t) = \frac{2q_o}{k}\sqrt{\frac{\alpha t}{\pi}},$$

which is the surface temperature of a semi-infinite body heated by a uniform flux (cf. (9) §2.9. Recall that $q_o = Q/2b$ which, in effect, smears the heat flow across the entire channel. To include the effect of the constriction $\bar{T}(0,t) = \frac{2q_a\varepsilon}{k}\sqrt{\frac{\alpha t}{\pi}}$, where q_a is $Q/2a$.)

Similarly, the average contact temperature is

$$\bar{T}_c(t) = \frac{q_o}{k}\left[\frac{B(0,t)}{a}\int_0^a dx + \sum_{m=1}^{\infty}\frac{a\sin(m\pi\varepsilon)}{(m\pi\varepsilon)^2}\,\frac{C_m(0,t)}{a}\int_0^a \cos\left(\frac{m\pi x}{b}\right)dx\right],$$

$$\bar{T}_c(t) = \frac{q_o}{k}\left[B(0,t) + \sum_{m=1}^{\infty}\frac{b\sin^2(m\pi\varepsilon)}{\varepsilon^2(m\pi)^3}C_m(0,t)\right]. \tag{22}$$

Now, $C_m(0,t) = \text{erfc}\left(-\frac{m\pi}{b}\sqrt{\alpha t}\right) - \text{erfc}\left(\frac{m\pi}{b}\sqrt{\alpha t}\right)$. Substituting $\theta = \frac{\alpha t}{a^2}$, and using $\text{erfc}(x) = 1 - \text{erf}(x)$ and $\text{erf}(-x) = -\text{erf}(x)$:

$$C(0,t) = 2\text{erf}(m\pi\varepsilon\sqrt{\theta}). \tag{23}$$

The resistance is

$$QR_c = 2q_o bR_c = \frac{2q_o b}{k\pi^3\varepsilon^2}\sum_{m=1}^{\infty}\frac{\sin^2(m\pi\varepsilon)}{m^3}\text{erf}(m\pi\varepsilon\sqrt{\theta}),$$

$$\psi_{2D}(\theta) = R_c k = \frac{1}{\pi^3\varepsilon^2}\sum_{m=1}^{\infty}\frac{\sin^2(m\pi\varepsilon)}{m^3}\text{erf}(m\pi\varepsilon\sqrt{\theta}). \tag{24}$$

For the flux tube

$$\bar{T}_c(t) - \bar{T}(0,t) = \frac{q_o b}{k\varepsilon}\sum_{i=1}^{\infty}\frac{J_1(\lambda_i b\varepsilon)C_i(0,t)}{(\lambda_i b)^2 J_o^2(\lambda_i b)} \cdot \frac{1}{\pi a^2}\int_0^a 2\pi r J_o(\lambda_i r)dr.$$

Again $C_i(0,t) = 2\text{erf}(\lambda_i b\varepsilon\sqrt{\theta})$, and the integral is $\frac{2a}{\lambda_i}J_1(\lambda_i a)$, from which

$$QR_c = \pi q_o b^2 R_c = \frac{4q_o b^2}{ka\varepsilon}\sum_{i=1}^{\infty}\frac{J_1^2(\lambda_i b\varepsilon)\text{erf}(\lambda_i b\varepsilon\sqrt{\theta})}{(\lambda_i b)^3 J_o^2(\lambda_i b)},$$

$$\psi_c(\theta) = 4kaR_c = \frac{16}{\pi\varepsilon}\sum_{i=1}^{\infty}\frac{J_1^2(\lambda_i b\varepsilon)\text{erf}(\lambda_i b\varepsilon\sqrt{\theta})}{(\lambda_i b)^3 J_o^2(\lambda_i b)}. \tag{25}$$

Equations (24) and (25) give the constriction resistance as a function of time after a uniform flux is applied to the end of a semi-infinite elemental flux channel. As mentioned above, R_c is the resistance of the heat flow paths to spread out into the channel. When the contact width equals the channel width ($\varepsilon = 1$), there is no spreading, and from (24) and (25) $\psi(\theta) = 0$.

It is interesting to compare these results with the steady state results of Yovanovich (3,4):

Flux channel:

$$\psi_{2D}^{ss} = \sum_{m=1}^{\infty}\frac{\sin^2(m\pi\varepsilon)}{\pi^3\varepsilon^2 m^3} = \sum_{m=1}^{\infty}\psi_m^{ss}; \tag{26a}$$

Flux tube:

$$\psi^{ss} = \sum_{i=1}^{\infty} \frac{16}{\pi\varepsilon} \frac{J_1^2(\lambda_i b\varepsilon)}{(\lambda_i b)^3 J_o^2(\lambda_i b)} = \sum_{i=1}^{\infty} \psi_i^{ss}. \qquad (26b)$$

It is easy to see that in general

$$\psi(\theta) = \sum_{n=1}^{\infty} \psi_n^{ss} \operatorname{erf}(\delta_n \varepsilon\sqrt{\theta}), \qquad (27)$$

where δ_n, n = 1, 2, 3 . . . are the roots of either $\sin \delta_n = 0$ or $J_1(\delta_n) = 0$ ($n\pi$ or $\lambda_n b$ respectively).

The modifier $\operatorname{erf}(\delta_n\varepsilon\sqrt{\theta}) \cong 1$ for $\delta_n\varepsilon\sqrt{\theta} \geq 3.8$ approximately. If the first term in the summation has $\delta_1\varepsilon\sqrt{\theta} \doteq 3.8$, then all the terms in the summation will have $\operatorname{erf}(\delta_n\varepsilon\sqrt{\theta}) = 1$ because δ_n increases with n. The resistance is thus said to have reached the steady state. From this, the time at which steady state is reached, θ_{ss}, may be approximated: for the flux channel, $\delta_1 = \pi$, $\pi\varepsilon\sqrt{\theta_{ss}} = 3.8$, or $\theta_{ss} = (3.8/(\pi\varepsilon))^2$; for the flux tube, $\delta_1 \doteq 3.8$ and $\theta_{ss} = 1/\varepsilon^2$. For example, if $\varepsilon = 0.1$ in the flux channel, the steady state is reached at around $\theta = 150$. As ε becomes larger, the steady state is achieved at earlier θs. This will be shown in the section on numerical results.

It should also be pointed out that although the temperature levels at the contact plane do not reach a steady state, the constriction resistance does reach the steady state asymptotically. This indicates that the resistance at large times does not depend on the temperature level, but on the temperature distribution, and further implies that heat flow paths have settled into time invariant positions.

NTACT ON A HALF-SPACE

Elemental flux channel analysis is useful when dealing with heat transfer between contacting rough surfaces. In most applications, the ratio of contact spot to flux channel width is very small, usually $\varepsilon << 0.1$. In such cases, one surmises that the constriction resistance approaches that of a contact on a half space. Hence it is also instructive to investigate the transient behaviour of a contact strip and spot on a half space. Here, results from Carlaw and Jaeger (9) §10.5 are used for strip and spot contacts with a uniform flux applied to the contact surfaces.

Strip Contact

The surface temperature for an infinite strip contact of half width a is (strip runs parallel to the y axis)

$$T(x,t) = \frac{q_a a}{k} \sqrt{\frac{\theta}{\pi}}\; f(\xi,\theta), \qquad (28a)$$

$$f(\varepsilon,\theta) = \operatorname{erf}\left(\frac{1+\xi}{\sqrt{4\theta}}\right) + \operatorname{erf}\left(\frac{1-\xi}{\sqrt{4\theta}}\right) + \frac{1+\xi}{\sqrt{4\pi\theta}} E_1\left(\frac{(1+\xi)^2}{4\theta}\right) + \frac{1-\xi}{\sqrt{4\pi\theta}} E_1\left(\frac{(1-\xi)^2}{4\theta}\right). \qquad (28b)$$

where $\xi = x/a$, $q_a = Q/2a$, and E_1 is the exponential integral

$$E_1(x) = \int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt.$$

The resistance of a contact on a half space is

$$QR_c = \bar{T}_c(t) - T_\infty,$$

where T_∞ is the temperature far from the strip and is assumed to be zero as before. The average contact temperature is

$$\bar{T}_c(t) = \frac{Q}{2k}\sqrt{\frac{\theta}{\pi}} \int_0^1 f(\xi,t)\,d\xi$$

$$\text{and} \quad \psi^*_{2D}(\theta) = R_c k = \frac{1}{2}\sqrt{\frac{\theta}{\pi}} \int_0^1 f(\xi,t)\,d\xi. \qquad (29)$$

Circular Contact Spot

The temperature in a half space due to a uniform flux applied at t = 0 to a contact spot of radius a is

$$T(r,z,t) = \frac{q_a a}{2k} \int_0^{\infty} J_o(\lambda r) J_1(\lambda a) C(z,t) \frac{d\lambda}{\lambda}, \qquad (30a)$$

$$C(z,t) = e^{-\lambda z}\left(\operatorname{erfc}\frac{z}{\sqrt{4\alpha t}} - \lambda\sqrt{\alpha t}\right) - e^{\lambda z}\left(\operatorname{erfc}\frac{z}{\sqrt{4\alpha t}} + \lambda\sqrt{\alpha t}\right). \qquad (30b)$$

The average contact temperature is

$$\bar{T}_c(t) = \frac{1}{\pi a^2}\int_0^a 2\pi r\, T(r,0,t)\,dr$$

$$= \frac{2q_a}{ka}\int_0^{\infty} \frac{J_1(\lambda a) C(0,t)}{\lambda} \int_0^a r J_o(\lambda r)\,dr\,d\lambda.$$

$$\bar{T}_c(t) = \frac{2Q}{k\pi a}\int_0^{\infty} \frac{J_1^2(\lambda a)\operatorname{erf}(\lambda a\sqrt{\theta})\,d(\lambda a)}{(\lambda a)^2}.$$

Recalling that $R_c = \bar{T}_c/Q$

$$\psi^*_c(\theta) = 4kaR_c = \frac{8}{\pi}\int_0^{\infty} \frac{J_1^2(\lambda a)\operatorname{erf}(\lambda a\sqrt{\theta})\,d(\lambda a)}{(\lambda a)^2}. \qquad (31)$$

Note that, again, the time is brought in by the modifier $\operatorname{erf}(\lambda a\sqrt{\theta})$. The steady state results in Ref. (9), §8.2 (15)

$$(4kaR_c)^{ss} = \frac{8}{\pi}\int_0^{\infty} \frac{J_1^2(\lambda a)\,d(\lambda a)}{(\lambda a)^2}.$$

As pointed out by Beck (7), the integral in (31) is difficult analytically and is not suitable to numerical integration due to the infinite domain and the sinusoidal character of the integrand. However, Jaeger (10) gives the approximation (see also Beck (7))

$$\psi^*_c(\theta) = \frac{8}{\pi}\left(\frac{4}{3\pi} - \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2)!}{2^{2m+2}(m+2)!\,[(m+1)!]^2 (2m+1)\sqrt{\pi}\,\theta^{2m+1}}\right) \qquad (32a)$$

which, for small θ ($\theta \leq 1$), is

$$\psi^*_c(\theta) = \frac{8}{\pi}\left(\sqrt{\frac{\theta}{\pi}} - \frac{\theta}{\pi} + \frac{\theta^2}{8\pi} \ . \ . \ .\right). \qquad (32b)$$

Beck (6) also gives expressions for the average temperature of a circular region directly below the source. His expressions may be simplified to obtain the average contact temperature from which the resistance is

$$\psi^*_c(\theta) = \frac{4}{\pi}\left(\sqrt{4\theta}\left(\frac{1}{\sqrt{\pi}}\,(1-e^{-1/4\theta}) + \frac{1}{\sqrt{4\theta}}\operatorname{erfc}\left(\frac{1}{\sqrt{4\theta}}\right)\right)\right.$$

$$+ \frac{8}{3\pi} - 1 + \frac{1}{48\theta\sqrt{\pi\theta}}\left(1 - \frac{1}{5\theta} + \frac{1}{64\theta^2} \cdot \cdot \cdot\right)\Bigg], \quad \theta \geq 0.6 \tag{33a}$$

and for short times

$$\psi_c^*(\theta) = \frac{8}{\pi}\left[\sqrt{\frac{\theta}{\pi}} - \frac{\theta}{\pi} + \frac{\theta^2}{8\pi} + \frac{\theta^3}{32\pi} + \frac{15\theta^4}{512\pi} \cdot \cdot \cdot\right]. \tag{33b}$$

Equation (33b) is the same as (32b) with a few extra terms in the series. Equation (33a) is analytical, and thus has a distinct computational advantage over the series solution (32a). It will be shown that these expressions yield identical results.

COMPUTED RESULTS

The steady state solutions (26a) and (26b) were computed for various ε's by Yovanovich (3,4) and are displayed in Table 1. The transient solutions (24) and (25) were computed here for a range of time and constriction ratio. The results are listed in tabular form in Tables 2 and 3 and plotted in Figures 4 and 5 for the flux channel and flux tube, respectively. The transient solution reaches the steady state quickly, $\theta \leq 10^2$ for constriction ratios $\varepsilon \geq 0.1$. Also, as ε increases, the time to steady state decreases as described above.

Table 1 Steady State Constriction Resistance

Constriction Ratio ε	Constriction Resistance ψ_{ss}	
	Flux Channel	Flux Tube
Half Space	--	1.0808
0.01	1.3583	1.0661
0.1	0.6263	0.9401
0.2	0.4083	0.8010
0.3	0.2836	0.6649
0.4	0.1984	0.5337
0.5	0.1357	0.4092
0.6	0.0882	0.2936
0.7	0.0521	0.1895
0.8	0.0255	0.1008

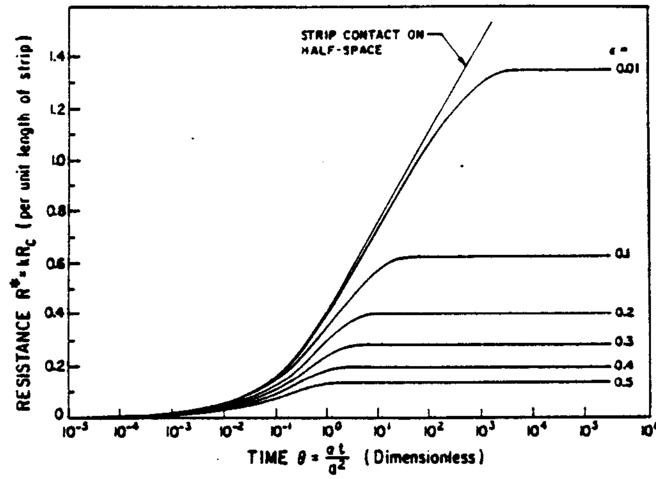

Figure 4 Dimensionless Resistance in the Flux Channel

Table 2 Transient Resistance in the Flux Channel

Time θ	Constriction Resistance (Eq. (24)) $\psi_{2D}(\theta)$					
	ε=0.01	0.1	0.2	0.3	0.4	0.5
10^{-7}	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001
10^{-6}	0.0006	0.0005	0.0005	0.0004	0.0003	0.0003
10^{-5}	0.0018	0.0016	0.0014	0.0012	0.0011	0.0009
10^{-4}	0.0056	0.0051	0.0045	0.0039	0.0034	0.0028
10^{-3}	0.0175	0.0159	0.0141	0.0123	0.0105	0.0088
10^{-2}	0.0542	0.0492	0.0435	0.0379	0.0323	0.0266
10^{-1}	0.1607	0.1447	0.1268	0.1090	0.0911	0.0733
10^{0}	0.4041	0.3533	0.2969	0.2405	0.1846	0.1323
10^{1}	0.7368	0.5763	0.4069	0.2836	0.1984	0.1357
10^{2}	1.0624	0.6263	0.4083	0.2836	0.1984	0.1357
10^{3}	1.3066	0.6263	0.4083	0.2836	0.1984	0.1357
10^{4}	1.3583	0.6263	0.4083	0.2836	0.1984	0.1357

Table 3 Transient Resistance in the Flux Tube

Time θ	Constriction Resistance (Eq. (25)) $\psi_c(\theta)$								
	ε=0.01	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
10^{-7}	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003	0.0003	0.0002	0.0002
10^{-6}	0.0014	0.0014	0.0014	0.0013	0.0012	0.0011	0.0009	0.0007	0.0005
10^{-5}	0.0044	0.0045	0.0044	0.0041	0.0038	0.0034	0.0029	0.0023	0.0016
10^{-4}	0.0140	0.0141	0.0137	0.0130	0.0120	0.0107	0.0091	0.0072	0.0051
10^{-3}	0.0441	0.0442	0.0428	0.0405	0.0374	0.0333	0.0283	0.0224	0.0155
10^{-2}	0.1349	0.1341	0.1298	0.1226	0.1126	0.0997	0.0839	0.0652	0.0436
10^{-1}	0.3737	0.3698	0.3561	0.3334	0.3016	0.2607	0.2108	0.1522	0.0888
10^{0}	0.7474	0.7338	0.6907	0.6189	0.5204	0.4067	0.2933	0.1895	0.1008
10^{1}	0.9670	0.9227	0.8007	0.6649	0.5337	0.4092	0.2936	0.1895	0.1008
10^{2}	1.0428	0.9401	0.8010	0.6649	0.5337	0.4092	0.2936	0.1895	0.1008
10^{3}	1.0642	0.9401	0.8010	0.6449	0.5337	0.4092	0.2936	0.1895	0.1008
10^{4}	1.0661	0.9401	0.8010	0.6649	0.5337	0.4092	0.2936	0.1895	0.1008

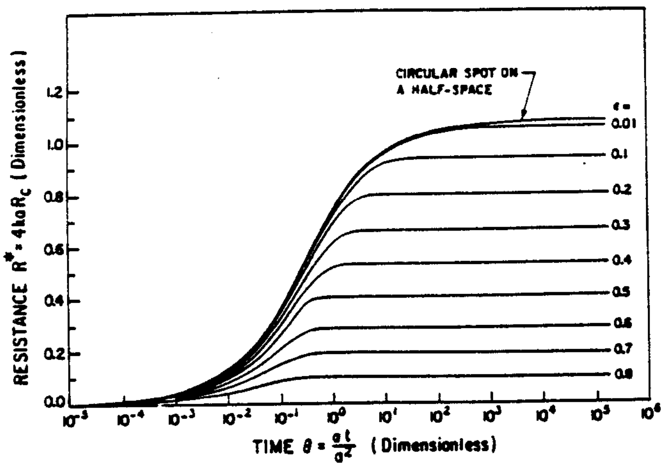

Figure 5 Dimensionless Resistance in the Flux Tube

The half space solutions Eqs. (29) and (32) or (33) are also plotted in Figures 4 and 5, and numerical values are listed in Table 4 and 5. The transient resistance of a strip on a half space does not reach steady state, but increases linearly with log(θ) for $\theta > 1$. This corresponds with the fact that there is no known steady state solution. However, for small $\varepsilon(\varepsilon \leq 0.01)$ and $\theta \leq 10^2$, there is less than 5.5% difference between Eq. (29), the half space solution and Eq. (24), the flux channel solution.

In the case of the circular spot on a half space, the solution approaches the steady state values $32/3\pi^2$ at a time $\theta > 10^7$. As anticipated, there is very little difference, ≤ 1.4%, between the flux tube (ε=0.01) and the half space solutions for all time. The maximum error occurs at steady state where $\psi_c^{ss}(\varepsilon=0.01)=1.0661$. In addition, for times $\theta \leq 10^{-1}$, there is only a 5% difference between resistances of the half space and a flux tube with a relatively large constriction ratio of ε=0.2. And, as shown in Figure 5, all the curves come together at extremely short times, $\theta \leq 10^{-5}$.

The values listed in Table 5 for $\theta > 1$ were identically produced by both Eqs. (32a) and (33a). Equation (32a) was more costly due to the summation. However, both these equations require much less computing time (of the order 40 times less) than Eq. (25) for ε=0.01, and yet they yield the same result to within 1.4%. Thus, for small constriction ratios (ε<0.01), which are usually the case in conforming, rough surface contact analysis, it is recommended that Eq. (33a) be used to calculate the constriction resistance. Also Eq. (33a) may be used for short times and moderately large constriction ratios: $\theta \leq 0.1$ and $\varepsilon \leq 0.2$.

It is also of interest to determine the actual time in which the steady state is reached. For example, two flat, rough surfaces in contact characteristically form contact spots with a radius of about 1 micron (10^{-6} m), with a constriction ratio of ε=0.01 (nominal loads). If the two surfaces are stainless steel ($\alpha \approx 10^{-5}$ m²/s), using the analysis shown above:

$$\theta_{ss} = \frac{\alpha t_{ss}}{a^2} = \frac{1}{\varepsilon^2}$$

$$t_{ss} = \frac{a^2}{\varepsilon^2 \alpha} = \frac{(10^{-6})^2}{(0.01)^2 (10^{-5})} \frac{m^2}{\frac{m^2}{s}} = 10^{-3}\,s.$$

Hence, the real time span in Figure 5 is of the order 1 millisecond (t_{ss} decreases for larger ε). In a practical sense, this means that even though experimental results show that the temperature takes a long time to reach equilibrium values, the maximum contact resistance is reached almost instantaneously.

CONCLUSION

Series expressions have been presented for both transient temperature distributions and constriction resistances for two elemental flux channels heated by uniform sources. It was shown that the expressions for resistance are related to the steady state solutions--the terms in the series modified by an error function multiplier. The resistances for the flux tube can be approximated by the resistance of a spot on a half space for sufficiently small constriction ratios, or very short times. The temperature distributions for the cylindrical geometries (tube and half space) are the same as those reported by Beck.

APPENDIX--THE INVERSE LAPLACE TRANSFORM OF EQ. (11)

From Carslaw and Jaeger (9), Appendix V, (9)

$$L^{-1}\left(\frac{e^{-qz}}{pq}\right) = \sqrt{4\alpha t}\ \text{ierfc}\left(\frac{z}{\sqrt{4\alpha t}}\right) = B(z,t). \quad \text{(A1)}$$

Table 4 Transient Resistance for an Infinite Strip on a Half Space

Time θ	Constriction Resistance $\psi^*_{2D}(\theta)$
10^{-7}	0.0002
10^{-6}	0.0006
10^{-5}	0.0018
10^{-4}	0.0056
10^{-3}	0.0177
10^{-2}	0.0548
10^{-1}	0.1625
10^{0}	0.4096
10^{1}	0.7547
10^{2}	1.1188

Table 5 Transient Resistance for a Circular Spot on a Half Space

Time θ	Constriction Resistance $\psi^*_c(\theta)$
10^{-6}	0.0014[a] Eq. (33b)
10^{-5}	0.0045
10^{-4}	0.0143
10^{-3}	0.0446
10^{-2}	0.1356
10^{-1}	0.3743
10^{0}	0.7480[b] Eq. (32a) or (33a)
10^{1}	0.9681
10^{2}	1.0449
10^{3}	1.0694
10^{4}	1.0772
10^{5}	1.0796
10^{6}	1.0804
10^{7}	1.0806

From the second term of (11)

$$\frac{\exp(-\eta_m z)}{p\eta_m} = \frac{\exp(-z\sqrt{q^2+\alpha\lambda_m^2})}{p\sqrt{q^2+\lambda_m^2}}$$

$$= \frac{\exp\left(-z\sqrt{\frac{p+\alpha\lambda_m^2}{\alpha}}\right)}{(p+\alpha\lambda_m^2)\sqrt{\frac{p+\alpha\lambda_m^2}{\alpha}} - \alpha\lambda_m^2\sqrt{\frac{p+\alpha\lambda_m^2}{\alpha}}} \tag{A2}$$

$$= \bar{v}(p+\alpha\lambda_m^2).$$

Let $s = p+\alpha\lambda_m^2$: $\bar{v}(s) = \dfrac{e^{-z\sqrt{s/\alpha}}}{\sqrt{s/\alpha}(s-\alpha\lambda_m^2)}$. (A3)

From Laplace transform theory (a is some constant)

$$L^{-1}(\bar{v}(p+a)) = e^{-at} L^{-1}(\bar{v}(p)). \tag{A4}$$

Using this ($a = \alpha\lambda_m^2$) and Carslaw and Jaeger (9), Appendix V, (30)

$$L^{-1}(\bar{v}(s)) = e^{-\alpha\lambda_m^2 t} \cdot \frac{e^{\alpha\lambda_m^2 t}}{2} \cdot \sqrt{\frac{\alpha}{\alpha\lambda_m^2}}$$

$$[e^{-z\sqrt{\alpha\lambda_m^2/\alpha}} \operatorname{erfc}(\frac{z}{\sqrt{4\alpha t}} - \sqrt{\alpha\lambda_m^2 t})$$

$$-e^{z\sqrt{\alpha\lambda_m^2/\alpha}} \operatorname{erfc}(\frac{z}{\sqrt{4\alpha t}} + \sqrt{\alpha\lambda_m^2 t})].$$

$$L^{-1}(\frac{e^{-\eta_m z}}{p\eta_m}) = \frac{1}{2\lambda_m}[e^{-\lambda_m z} \operatorname{erfc}(\frac{z}{\sqrt{4\alpha t}} - \sqrt{\alpha\lambda_m^2 t})$$

$$-e^{\lambda_m z} \operatorname{erfc}(\frac{z}{\sqrt{4\alpha t}} + \sqrt{\alpha\lambda_m^2 t})], \tag{A5}$$

$$= \frac{1}{2\lambda_m} C_m(z,t). \tag{A5a}$$

ACKNOWLEDGEMENT

One of the authors (PJT) acknowledges the support of the Natural Sciences and Engineering Research Council of Canada in the form of a Postgraduate Scholarship.

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