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ELLIPSOIDAL THERMAL CONSTRICTION MODEL FOR CROWNED CYLINDER/FLAT ELASTIC CONTACTS

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ABSTRACT

An ellipsoidal contact model is proposed to account for the very large discrepancies observed between recent experiments with cylinder/flat contacts in vacuum and the corresponding predictions based upon a line contact model. An approximate ellipsoidal model is developed and shown to be in excellent agreement with the complex exact solution requiring numerical computation of incomplete and complete elliptic integrals by the method of the arithmetic-geometric means. A parametric study shows that the proposed exact and approximate ellipsoidal models are in qualitative agreement with the light load experimental results.

NOMENCLATURE

a, b - Contact Dimensions (m)
A, B - Hertz Geometric Parameters (m^{-1})
D - Cylinder Diameter (m)
E - Young's Modulus (Pa)
$E(\chi)$ - Elliptic Integral (complete) of the second kind of modulus χ
$F(\chi,\phi)$ - Incomplete Elliptic Integral of the first kind of modulus χ and amplitude ϕ
k - Thermal Conductivity (W/mK)
$K(\chi)$ - Complete Elliptic Integral of the first kind of modulus χ
m - Surface Asperity Slope (radians)
m, n - Hertz Parameters (-)
N - Total Load (N)
N^* - Dimensionless load $\equiv N\Delta/2wD$
Q - Heat Flow Rate (W)
R_c - Resistance (K/W)
R_c^* - Dimensionless Resistance $\equiv 2wkR_c$
T - Temperature (K)
w - Cylinder Half-length (m)
α, β, γ - Semi-axes of Ellipsoid Along x, y, z - axes
Δ - Elasticity Parameter $\equiv 1/2\left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right)$ (Pa^{-1})
λ - Ellipsoidal Coordinate (Root of $x^2/(a^2+\lambda) + y^2/(b^2+\lambda) + z^2/\lambda = 1$)
ν - Poisson's Ratio (-)
ρ - Radius of Cylinder Due to Crowning (m)
σ - Surface Roughness (μm)
ϕ - Amplitude Angle $\equiv \sin^{-1}\sqrt{(1/(1+\lambda/a^2))}$
χ - Modulus $\equiv \sqrt{1-(b/a)^2}$
χ' - Complementary modulus $\equiv \sqrt{1-\chi^2} = b/a$

Subscript 1 - cylinder surface
2 - flat surface
e - elliptical
s - harmonic mean

INTRODUCTION

One of the assumptions used in modelling the resistance of a cylinder-flat contact is that the contacting bodies are free of any surface roughness, waviness, oxide film or any error in form. Provided these assumptions are valid, the contact formed when these two bodies are pressed together will be a rectangular strip.

Unfortunately, these conditions are not met in many practical situations. The realities of machining processes, material handling, and so on will only allow the experimentalist to approach these ideal conditions. In the case of the cylinder/flat contact, the most troublesome imperfections are from defects.

If, instead of a right circular cylinder, the cylindrical body is slightly barrel shaped, Figure 1, then a small elliptical contact will form instead of the rectangular strip. This crowned geometry will provide much greater constriction to heat conduction than the rectangular geometry at the same applied load. With increasing load the elliptical contact will grow until it becomes the expected strip contact. Experimental data for thermal contact resistance which shows this crowning effect [1] at light loads is seen in Figure 2. Figure 2 also shows a comparison between the experimental and predicted dimensionless contact resistance R_c^* as a function of the dimensionless load parameter N^*.

For a line contact in vacuum [1,2'

$$R_c^* = \frac{1}{2\pi}\frac{k_s}{k_1}\ \ln\left(\frac{\pi}{N^*}\right) - \frac{k_s}{2k_1} + \frac{1}{2\pi}\frac{k_s}{k_2}\ \ln\left(\frac{1}{4\pi N^*}\right) \qquad (1)$$

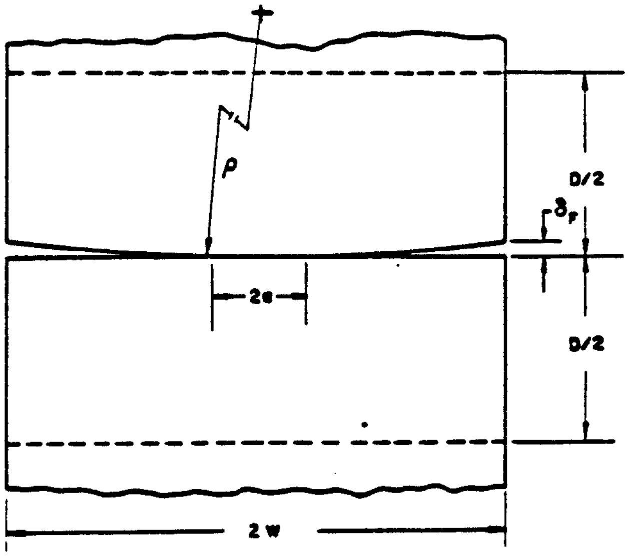

Fig. 1 Form Error Analysis Geometry

Table 1 contains the specimen thermophysical properties, dimensions, and the surface characteristics, for which experimental data were obtained.

It can be seen in Figure 2 that the experimental and theoretical values are in very good agreement at the higher loads ($N^* > 5\times10^{-6}$). At the lighter loads the difference between the experimental and theoretical values is very large. This discrepancy is attributed to the error in form or crowning of the cylindrical body.

The purpose of this paper is to investigate a model which relaxes the 'perfect cylinder' constraint imposed in the ideal case and to perform a parametric study on the ellipsoidal model solution to compare with the light load experimental contact resistance values.

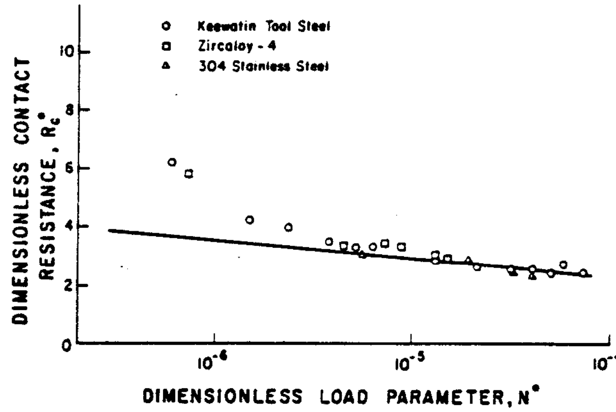

Fig. 2 Variation of Resistance with Load in Vacuum

Ellipsoidal Constriction Model

As with most other elasto-constriction resistance models, the analysis is divided into mechanical and thermal solutions. Hertz' theory is used to predict the contact shape and dimensions. The thermal resistance of the contact is then approximated by considering two isothermal semi-ellipsoids separated by a small isothermal elliptic contact area centred on their half-plane. The semi-ellipsoids are then chosen to fit inside the cylinder/flat geometries as shown in Figures 3 and 4.

For this geometry, Figure 3, the Hertz geometric parameters become [2]:

$$2(A+B) = (2/D + 1/\rho)$$
$$2(B-A) = (2/D - 1/\rho) \qquad (2)$$

These are related to the contact shape through the transcendental equation:

$$B/A = 2\rho/D = \frac{(1/\chi')E(\chi)-K(\chi)}{K(\chi)-E(\chi)} \qquad (3)$$

where K and E are the complete elliptic integrals of the

Table 1 Specimen Properties and Geometry

Property	Keewatin T.S.	304 S.S.	Zircaloy-4
k(W/mK)	34.13-1.7993E-3T	10.67+1.59E-2T	7.51+2.09E-2T-1.45E-5T
E(GPa)	228.8-7.6E-2T	207.5-7.6E-2T	117.11-6.7E-2T
ν	0.3	0.3	0.333-1.261E-4T
Hardness	R_c60	R_b75	R_b94
Roughness			
Cyl.: σ(μm)	0.464	0.339	0.607
m(rad)	0.110	0.070	0.230
Flat: σ(μm)	0.111	0.117	1.372
m(rad)	0.027	0.022	0.090
Dimensions			
D (mm)	25.4	20.0	25.4
2w (mm)	25.4	40.0	25.4

Note: Temperatures (T) in Kelvin
R_c is Rockwell hardness number
R_b is Brinell hardness number

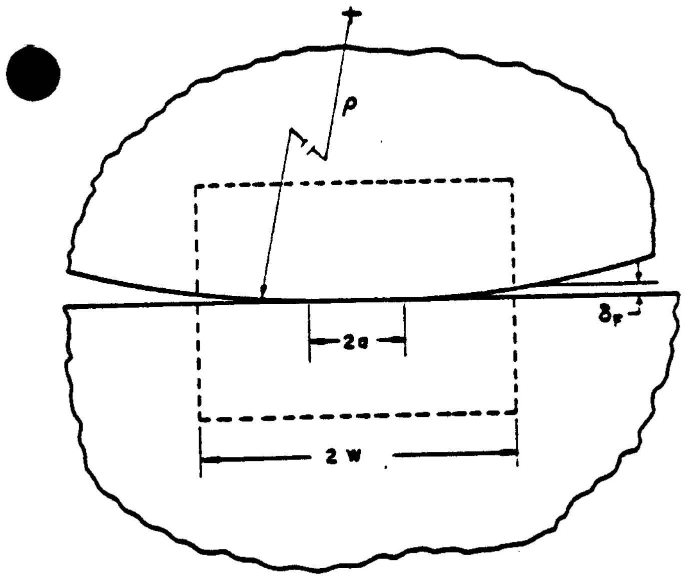

Fig. 3 Form Error Geometry for Elastic Model

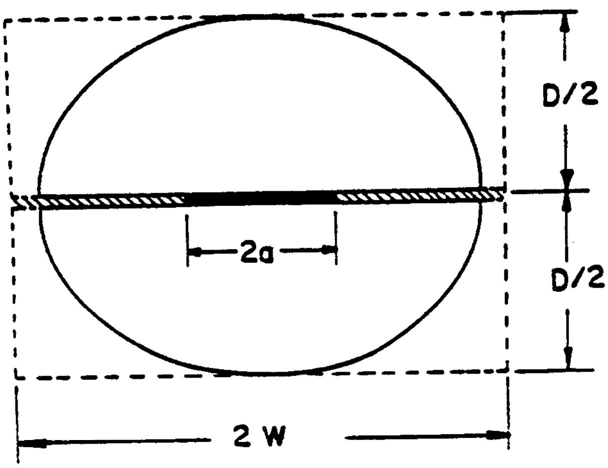

Fig. 4 Form Error Geometry for Thermal Model

first and second kind, respectively. The single, positive root of Eq. (3) is called the modulus χ and it is defined by:

$$\chi \equiv \sqrt{1-(b/a)^2} \qquad (4)$$

The semi-major and semi-minor axes of the elliptical contact are denoted by a and b respectively.

Using the process of the arithmetic-geometric mean [3] to evaluate the elliptic integrals, and a Newton-Raphson root-finding technique, Eq. (3) may be easily solved for the modulus χ. Blahey [4] has correlated the Hertz data of Walowit and Anno [5]:

$$\chi = [1.0 - (0.9446(D/2\rho)^{0.6135})^2]^{1/2} \qquad (5)$$

This expression may be used to compute a first guess to start the root-finding algorithm.

Once the modulus χ has been determined, the contact dimensions are computed from [2]:

$$a = \left[\frac{3\ E(\chi)\ N\Delta}{\pi\chi'^2\ (2/D+1/\rho)}\right]^{1/3} \qquad (6)$$

or

$$a = \left[\frac{3\ E(\chi)\ D^2\ 2w\ N^*}{\pi\ \chi'^2\ (1+D/2\rho)}\right]^{1/3} \qquad (7)$$

then

$$b = \chi' \cdot a \qquad (8)$$

The thermal resistance of a semi-ellipsoid whose outer surface is isothermal, whose half contact plane contains a centred isothermal elliptic contact, and is adiabatic elsewhere is given by [2]:

$$R = \frac{1}{2\pi\ k\ a}\ (K(\chi)-F(\chi,\phi)) \qquad (9)$$

where $F(\chi,\phi)$ is the incomplete elliptic integral of the first kind of modulus χ and amplitude ϕ. The modulus is given by

$$\chi = \sqrt{1-(b/a)^2}$$

as before, while the amplitude is defined by

$$\phi \equiv \sin^{-1} \sqrt{\frac{1}{1+\lambda/a^2}} \qquad (10)$$

The outer surface of the semi-ellipsoid is described by the ellipsoidal coordinate λ which is chosen to fit the desired semi-ellipsoid into the cylinder-flat contact, Figure 4. One of these semi-ellipsoids can be used to represent each of the contacting bodies. Since they are thermally connected in series, therefore the total thermal constriction resistance of the two contacting semi-ellipsoids is [2]:

$$R_e = \frac{1}{2\pi\ a}\ (K(\chi)-F(\chi,\phi))\ [\frac{1}{k_1} + \frac{1}{k_2}] \qquad (11)$$

$$\text{where } \phi = \sin^{-1} \sqrt{\frac{1}{1+(D/2a)^2}} \qquad (12)$$

having put $z = \gamma = D/2$. This particular choice fits the semi-ellipsoid so its semi-minor axis γ is along the z axis in the Cartesian coordinate system, and is of length D/2. Other choices for λ are possible, but they are less realistic and yield less accurate results.

Then, using the definition of the nondimensional resistance:

$$R^* = 2w\ k_s\ R \qquad (13)$$

the dimensionless resistance of the ellipsoid may be expressed as:

$$R_e^* = \frac{2w}{\pi\ a}\ (K(\chi)-F(\chi,\phi)) \qquad (14)$$

Sample Calculation

As an example of the evaluation of the point contact model described above, the case of $D = 2w = 25.4$ mm., $\rho = 8.6$ m., $N^* = 4 \times 10^{-7}$ will be computed.

We begin the computations by determining the ratio $(D/2\rho)$:

$$D/2\rho = (25.4 \times 10^{-3})/(2.0)(8.6)$$
$$= 1.477 \times 10^{-3} \qquad (15)$$

Then, using Eq. (5), we obtain a first guess for the modulus:

$$\chi = (1.0 - (0.9446(1.477 \times 10^{-3})^{0.6135})^2)^{1/2} \qquad (16)$$
$$= 0.99985$$

Then, the complementary modulus is

$$\chi' = \sqrt{1 - (0.99985)^2} = 17.323 \times 10^{-3} \qquad (17)$$

The method of the arithmetic-geometric means involves repeatedly taking the arithmetic and geometric means of a pair of numbers, until the numbers compare to within a tolerance, i.e.:

$$\begin{aligned} A_i &= 1/2(A_{i-1} + B_{i-1}) \\ B_i &= \sqrt{A_{i-1} \cdot B_{i-1}} \\ C_i &= 1/2(A_{i-1} - B_{i-1}) \end{aligned} \qquad (18)$$

Starting values are $A_0 = 1$, $B_0 = \chi'$, $C_0 = \chi$. Iteration is continued until C_i is less than the required tolerance. Then

$$\begin{aligned} K(\chi) &= \frac{\pi}{2 A_n} \\ E(\chi) &= K(\chi)(1 - 1/2 \sum 2^i C_i^2) \end{aligned} \qquad (19)$$

where n is the number of the last iteration. Typical results are given in Table 2.

For the example under consideration we have

$$K(\chi) = \frac{\pi/2}{0.28862} = 5.4424 \qquad (20)$$

and

$$E(\chi) = 5.4424\ (1-1/2(1.6247)) = 1.00074 \qquad (21)$$

Differentiation of Eq. (3), and some manipulation allows use of the Newton-Raphson technique to improve the estimate of χ:

$$\chi_{j+1} = \chi_j - \frac{NUM(\chi_j)}{DEN(\chi_j)} \qquad (22)$$

with

$$NUM(\chi_j) = E(\chi_j) \cdot (1/\chi_j'^2 + 2\rho/D) - K(\chi_j) \cdot (1+2\rho/D) \qquad (23)$$

and

$$DEN(\chi_j) = E(\chi_j) \frac{2\chi_j - \chi_j \cdot \chi_j'^2 \cdot (2\rho/D)}{\chi_j'^4} - K(\chi_j) \cdot \chi_j/(\chi_j'^2) \qquad (24)$$

Equation (22) is repeatedly evaluated, using updated values of K_j, E_j determined using the arithmetic-geometric mean procedure. Typical results are shown in Table 3.

These results (j = 4) are then used to determine the contact dimensions, amplitude angle, and resistance.

From Eq. (7), the semi-major axis of the elliptical contact is

$$a = \frac{3(1.00082)(25.4\text{x}10^{-3})^2(25.4\text{x}10^{-3})(4\text{x}10^{-7})}{(18.343\text{x}10^{-3})^2(1+1.477\text{x}10^{-3})} = 2.649\ \text{mm}. \qquad (25)$$

From Eq. (8), the semi-minor axis of the elliptical contact is

$$b = (18.344\text{x}10^{-3})(2.649\text{x}10^{-3}) = 48.59\ \mu\text{m}. \qquad (26)$$

From Eq. (10), the amplitude is

$$\phi = \sin^{-1}\left[\frac{1}{1 + (12.7/2.649)^2}\right]^{1/2} = 0.2056 \qquad (27)$$

All that remains to determine the constriction resistance is to evaluate the incomplete elliptic integral $F(\chi,\phi)$. Once again, the method of the arithmetic-geometric means is used. In this case, A, B, C are defined as in Eq. (18), with the additional parameters:

$$\begin{aligned} \theta_{i+1} &= \theta_i + \text{Tan}^{-1}(\xi_{i+1}) + I_{i+1} \cdot \pi \\ I_{i+1} &= \text{INT}(1/2(2\theta_i/\pi + 1)) \\ \xi_{i+1} &= (B_i/A_i)\tan(\theta_i) \end{aligned} \qquad (28)$$

where 'INT(χ)' means the integer part of χ. Then, when C_i is less than the required tolerance,

Table 2 Application of arithmetic-geometric means

i	A_i	B_i	C_i	$2^i C_i$
0	1	17.323E-3	0.99985	0.99970
1	0.50866	0.13162	0.49134	0.48283
2	0.32014	0.25874	0.18852	0.14216
3	0.28944	0.28781	3.0698E-2	7.5389E-3
4	0.28862	0.28862	8.1625E-4	1.0660E-5
5	0.28862	0.28862	5.7715E-7	1.0659E-11
6	0.28862	0.28862	$< 10^{-10}$	$< 10^{-18}$

Table 3 Results of Newton-Raphson root-finding method

j	χ_j	χ_j'	$K(\chi_j)$	$E(\chi_j)$	NUM	DEN
1	0.99985	17.323E-3	5.4424	1.00074	321.79	19.948E6
2	0.99983	18.230E-3	5.3913	1.00081	32.947	16.065E6
3	0.99983	18.342E-3	5.3852	1.00082	0.4266	15.651E6
4	0.99983	18.344E-3	5.3851	1.00082	----	----

$$F(\chi,\phi) = \theta_{i+1}/(2^n \cdot A_{i+1}) \tag{29}$$

where n is the number of iterations. To start, A, B, are as before, while $\theta_o = \phi$. Typical results are shown in Table 4.

Then, by means of Eq. (29):

$$F(0.99985, 0.2056) = \frac{3.866}{(64)\cdot(0.2917)} = 0.2071 \tag{30}$$

Finally, from Eq. (14), the dimensionless constriction resistance is

$$R_e{}^* = \frac{(25.4 \times 10^{-3})}{\pi(2.649 \times 10^{-3})}(5.3851 - 0.2071) = 15.804 \tag{31}$$

For the case of D = 2w = 25.4 mm., ρ = 8.6 m., and $N^* = 4 \times 10^{-7}$, the point contact model gives a value of 15.804 for the dimensionless contact resistance. The line contact model predicts a value of 3.6 and the experimental values are approximately 6.

Approximate Analytic Method

The method described above is complex, requiring the use of micro-computers or main-frame computers. It is desirable to have a simple, approximate method of computing the various parameters and the ellipsoidal constriction resistance. The method to be considered here is restricted to the elliptical contact formed by a crowned cylinder for values of the parameter χ close to unity and values of the complementary parameter χ' very small (< 0.02), the complete and incomplete elliptic integrals appearing in the ellipsoidal model can be approximated by the following expressions:

$$K(\chi) = \ln(4/\chi') \tag{32}$$

$$E(\chi) = 1 + (1/2)(\ln(4/\chi') - 1/2)\,(\chi')^2 \tag{33}$$

$$F(\chi,\phi) = \ln\tan\left[\frac{\pi}{4} + \frac{\phi}{2}\right] \tag{34}$$

For the case considered above, Eq. (5) gives χ' = 0.0173 and χ = 0.99983. By means of Eq. (7) we obtain

$$\frac{a}{2w} = 14.718\,(N^*)^{1/3} \tag{35}$$

With D = 2w and Eq. (36) we can express the amplitude as

$$\phi = \sin^{-1}\left[\frac{1}{1 + 0.001154/(N^*)^{2/3}}\right]^{1/2} \tag{36}$$

Finally, the dimensionless constriction resistance, Eq. (14), can be approximated by

$$R_e^* = \frac{2w}{\pi a}\left[\ln(4/\chi') - \ln\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right] \tag{37}$$

The results of the exact and approximate ellipsoidal models are compared in Table 5 for an interesting load range.

An examination of Table 5 shows that the simple, approximate ellipsoidal results are in good agreement for the amplitude ϕ and the incomplete elliptic integral F(χ,ϕ). The approximate and exact values of the dimensionless ellipsoidal resistance are also in

Table 4 Application of arithmetic-geometric means

i	A_i	B_i	C_i	ε_i	I_i	θ_i
0	1	18.344E-3	0.99983	---	--	0.2056
1	0.5092	0.1354	0.4908	3.827E-3	0	0.2095
2	0.3223	0.2626	0.1869	56.55 E-3	0	0.2660
3	0.2925	0.2909	2.985E-2	2.220E-1	0	0.4844
4	0.2917	0.2917	7.636E-4	5.234E-1	0	0.9666
5	0.2917	0.2917	5.00 E-7	1.499E 0	0	1.933
6	0.2917	0.2917	$<10^{-10}$	-2.638E 0	1	3.866

Table 5 Comparison of Approximate and Exact Ellipsoidal Models

Load N*	a (mm)	b (μm)	ϕ (Exact)	ϕ (Approx.)	F(χ,ϕ) (Exact)	F(χ,ϕ) (Approx.)	R_e^* (Exact)	R_e^* (Approx.)
4.0E-7	2.65	48.6	0.206	0.214	0.207	0.215	15.8	15.3
7.0E-7	3.19	58.6	0.246	0.256	0.249	0.258	13.0	12.6
1.2E-6	3.82	70.1	0.292	0.303	0.296	0.308	10.8	10.5
2.1E-6	4.60	84.5	0.348	0.360	0.355	0.369	8.83	8.57
3.7E-6	5.66	102	0.413	0.427	0.425	0.441	7.21	6.99
6.4E-6	6.68	122	0.484	0.500	0.504	0.522	5.91	5.73
1.1E-5	8.00	147	0.562	0.580	0.594	0.615	4.84	4.69
2.0E-5	9.76	179	0.655	0.674	0.708	0.732	3.87	3.75
.5E-5	11.8	216	0.749	0.767	0.828	0.855	3.13	3.03

For ρ = 8.6 m, χ = 0.99983, K(χ) = 5.383, E(χ) = 1.0008 very good agreement over the entire range of the dimensionless load N*.

Discussion of Ellipsoidal Results and Comparison with Experiment

The amount of crowning or radius of curvature of the cylinders is unknown and it cannot be determined by metrology. A parameter study was therefore undertaken to examine the effect of curvature upon the ellipsoidal constriction resistance. The dimensionless resistance R_e^* was computed by means of the exact method for selected values of ρ ranging from 5 to 100 meters for various values of the dimensionless load N*. These results are presented in Figure 5.

The intersections of the ellipsoidal results and the line contact results occur when the semi-axis a is approximately equal to the half-length of the flat. The ellipsoidal model should not be used beyond this point. One also observes that the point of intersection moves towards decreasing values of N* as ρ increases. Obviously, when ρ is infinitely large, the crowning effect disappears and the line contact model is valid for all values of N*.

A precise comparison of the theoretical and experimental results cannot be made because the actual radius of curvature for each test cylinder is unknown. One can, however, compare the experimental results shown in Figure 2 with the theoretical values given in Figure 4. It appears that the radius of curvature due to crowning is of the order of 100 meters. The experimental values are in very good agreement with the line contact model when $N^* > 5\times10^{-6}$ and the ellipsoidal model intersects the line contact model at approximately 4×10^{-6}.

Additional controlled experimental results for well-defined crowned cylinders are required to verify the theoretical ellipsoidal model developed here.

Summary

A thermo-mechanical ellipsoidal model has been presented which attempts to predict the resistance of a joint consisting of a crowned cylinder in mechanical contact with a rectangular flat. The motivation for this analysis lies in the difficulty of manufacturing a perfect right circular cylinder. The model developed in this paper demonstrates clearly that form defects can significantly increase the contact resistance of a joint at light contact loads. The predicted values are in qualitative agreement with some experimental results corresponding to crowned cylinders. An approximate ellipsoidal model is also presented. It is in very good agreement with the complex exact solution which requires the numerical computation of incomplete and complete elliptic integrals.

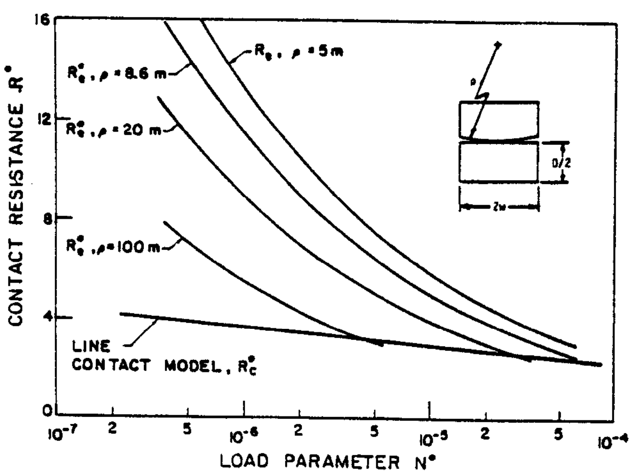

Fig. 5 Results of Form Error Analysis (for selected values of ρ)

Acknowledgments

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