METHOD OF MOMENTS FORMULATION OF THERMAL CONSTRICTION RESISTANCE OF ARBITRARY CONTACTS

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Abstract

A method of moments formulation is presented for the numerical computation of the steady-state thermal constriction, or spreading, resistance of contacts over which a contact conductance distribution has been specified. The formulation is valid for arbitrarily shaped planar contacts located on isotropic half-spaces. The procedure is applied to the circular and square contacts with uniform contact conductances. The dimensionless results are presented for a practical range of Biot number.

Nomenclature

A_c	contact area (A_o-A_i)
A_i	inner projected area
A_o	outer projected area
a,b	contact or element dimensions
$c_1 \ldots c_6$	correlation coefficients
E	complete elliptic integral of the second kind
G	the geometry matrix
g	an influence coefficient
H	conductance matrix
h	contact conductance
K	complete elliptic integral of the first kind
ℓ	a linear differential or integral operator
l	element of the operator matrix
P	arbitrary field point
Q	total heat flow rate
q	local heat flux
R_c	constriction resistance
R_f	film or contact resistance
R_t	total resistance
r_z	radial coordinate on z=0 plane
r	position vector of a field point
S	arbitrary source point
s	position vector of source point
T	temperature
$T(\infty)$	reference temperature
u	basis functions
w	weighting functions
x,y,z	cartesian coordinates

Superscripts

*	non-dimensional

Subscripts

i,j,m,n	element representation

Greek

$\delta(r)$	Dirac delta function
δ_{mn}	Kronecker delta function
θ	$(T-T(\infty))$ temperature excess
θ_a	applied temperature excess
θ_c	contact temperature excess
λ	thermal conductivity

Introduction

The thermal constriction resistance of a number of contacts with Dirichlet or Neumann boundary conditions have been investigated in the past(1). The purpose of this paper is to develop a method for determining the constriction resistance of arbitrary contacts when the boundary condition of the third kind (Robin condition) is specified.

This type of boundary condition arises in many practical systems; for example when a source is separated from a half-space by a conducting film, such as an oxide, the boundary condition of the third kind describes the situation. The effective conductance of the film is not necessarily uniform over the entire contact, but may in fact be a function of the numerous characteristics and physical parameters which describe the system. In the development of the method described in this paper the contact conductance is assumed to be a function of position. However in the interest of simplicity when interpreting the results the contact conductance has been restricted to the uniform case for the two examples which are presented.

Problem Statement and Resistance Formulation

Steady heat conduction from a planar contact to an isotropic half-space is governed by Laplace's equation

$$\nabla^2\theta = 0 \qquad (1)$$

which is valid within the volume of the conducting medium. The temperature excess, θ, is defined as

$$\theta = T - T(\infty)$$

where $T(\infty)$ is a convenient reference temperature associated with points which are located at distances large relative to some characteristic dimension of the contact area.

With reference to Fig. 1 the boundary conditions which are applied to the solution of equation (1) are

i) z=0 within A_c, $h(r)(\theta_a(r)-\theta_c(r)) + \lambda\frac{\partial\theta(r)}{\partial z} = 0$

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ii) z=0 outside A_c, $\frac{\partial\theta(r)}{\partial z} = 0$

iii) $|r| \to \infty$ $\qquad \theta(r) \to 0$

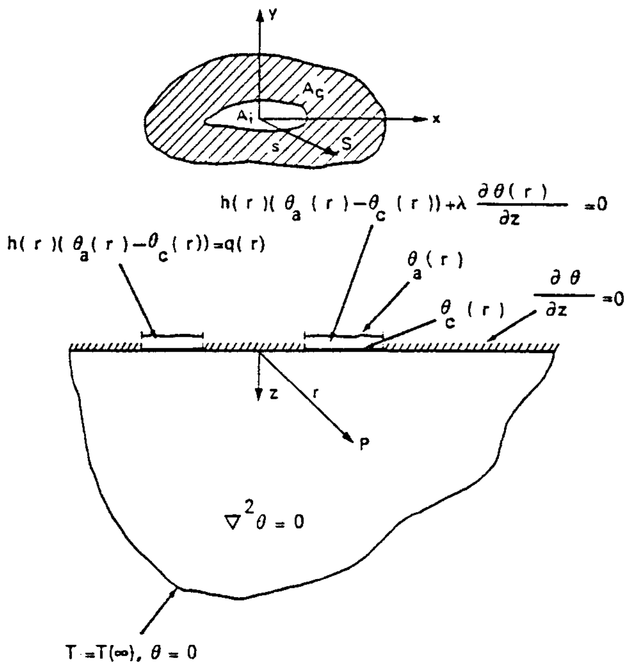

Fig. 1 Boundary Condition of the Third Kind.

The applied temperature excess $\theta_a(r)$ acts on the half-space through a resistance characterized by h(r); both of which are assumed to functions of position. The remaining two boundary conditions specify that outside the contact area the surface of the half-space is impervious to heat transfer, and secondly that far from the contact the temperature excess goes to zero.

A solution to equation (1) which also satisfies boundary conditions ii and iii is [2]

$$\theta_c(r) = \frac{1}{2\pi\lambda}\iint_{A_c}\frac{q(s)dA_c}{|s-r|} \qquad (2)$$

Boundary condition i may be written as

$$h(r)(\theta_a(r)-\theta_c(r)) = -\lambda\frac{\partial\theta}{\partial z} = q(r) \qquad (3)$$

Combining equations (2) and (3) and rearranging yields

$$\theta_a(r) = \frac{q(r)}{h(r)} + \frac{1}{2\pi\lambda}\iint_{A_c}\frac{q(s)dA_c}{|s-r|} \qquad (4)$$

Equation (4) satisfies Laplace's equation and the boundary conditions specified. The unknown quantity in equation (4) which must be determined is the flux distribution q(s). The distribution q(s) of equation (4) is determined by following the general method of moments described by Harrington [3] for obtaining solutions of its type.

Equation (4) is a linear integral equation which may be written as

$$L(q(s)) = \theta_a(r) \qquad (5)$$

where the operator L is given by

$$L = \frac{1}{h(r)} + \frac{1}{2\pi\lambda}\iint_{A_c}\frac{dA_c}{|r-s|} \qquad (6)$$

The assumption is made that q(s) may be represented by a finite series of expansion or basis functions which when summed with appropriate, but as yet undetermined coefficients will approximate q(s). Therefore let

$$q(s) = \sum_{n=1}^{N} q_n u_n \qquad (7)$$

The magnitude of the coefficients q_n are determined by solving the set of equations which arise when the projections of both $\theta_a(r)$ and $L(\Sigma q_n u_n)$ on a set of weighting functions, w_m, are equated.

The projection of one function on another is determined by taking the inner product of the two. The inner product of two functions f and g is given by

$$\langle f,g\rangle = \int f(x)g(x)dx \qquad (8)$$

After performing the required operations and using the fact that the operator is linear, equation (5) may be written as

$$\sum_{n=1}^{N} q_n \langle w_m, L(u_n)\rangle = \langle w_m, \theta_a(r)\rangle \qquad (9)$$

using the notation introduced in equation (8). In matrix notation equation (9) may be written as

$$\left[\ell_{mn}\right]\left\{q_n\right\} = \left\{\theta_{am}\right\} \qquad (10)$$

With

$$\ell_{mn} = \langle w_m, L(u_n)\rangle \qquad (11)$$

and

$$\theta_{am} = \langle w_m, \theta_a(r)\rangle \qquad (12)$$

The contact is divided into N surface elements with the assumption being made that the actual distribution q(s) can be adequately represented by an approximate distribution which has a uniform value on each element. Therefore the basis functions have the form

$$u_n = \begin{cases} 1 \text{ on } A_{cn} \\ 0 \text{ on all other } A_{cn} \end{cases} \qquad (13)$$

The weighting functions, w_m, are chosen to be Dirac delta functions so that

$$w_m = \delta(r-r_m) \qquad (14)$$

where r_m is the location of the centroid of the mth element. The effect of choosing these weighting functions is that the solution procedure becomes a collocation technique satisfying equation (5) exactly at only the points r_m. With the expansion and weighting functions specified equations (11) and (12) may now be evaluated to determine the elements of their corresponding matrices.

Expanding equation (12) gives

$$\langle w_m, \theta_a(r)\rangle = \iint_{A_c} \delta(r-r_m)\, \theta_a(r)\, dA_c \qquad (15)$$

which when integrated results in

$$\theta_{am} = \theta_a(r_m) \qquad (16)$$

Similarly expanding equation (11) yields

$$\ell_{mn} = \iint_{A_c} \delta(r-r_m) \left[\frac{u_n}{h(r)} + \frac{1}{2\pi\lambda} \iint_{A_c} \frac{u_n dA_c}{|r-s|} \right] dA_c \qquad (17)$$

or equivalently

$$\ell_{mn} = \iint_{A_c} \delta(r-r_m) \frac{u_n}{h(r)}\, dA_c + \frac{1}{2\pi\lambda} \iint_{A_c} \left[\delta(r-r_m) \iint_{A_c} \frac{u_n dA_c}{|r-s|} \right] dA_c \qquad (18)$$

The integrands containing u_n are equal to zero except when the integration is being carried out over A_{cn} in which case u_n is equal to one, from equation (13). By using this fact along with the property of the Dirac delta function the first integral of equation (18) may be evaluated giving

$$\iint_{A_c} \delta(r-r_m) \frac{u_n}{h(r)}\, dA_c = \delta_{mn} \frac{1}{h(r_m)} \qquad (19)$$

where δ_{mn} is the Kronecker delta function. Consider the second term of equation (18)

$$\frac{1}{2\pi\lambda} \iint_{A_c} \left[\delta(r-r_m) \iint_{A_c} \frac{u_n dA_c}{|r-s|} \right] dA_c \qquad (20)$$

Since u_n is equal to one on A_{cn} only and zero elsewhere, expression (20) may be written as

$$\frac{1}{2\pi\lambda} \iint_{A_c} \delta(r-r_m) \iint_{A_{cn}} \frac{dA_{cn}}{|r-s|}\, dA_c \qquad (21)$$

Completing the outer integration equation (21) reduces to

$$\frac{1}{2\pi\lambda} \iint_{A_{cn}} \frac{dA_{cn}}{|r_m-s|} \qquad (22)$$

The expression for determining the elements of the L matrix is therefore

$$\ell_{mn} = \delta_{mn} \frac{1}{h(r_m)} + \frac{1}{2\pi\lambda} \iint_{A_{cn}} \frac{dA_{cn}}{|r_m-s|} \qquad (23)$$

If

$$g_{mn} = \frac{1}{2\pi} \iint \frac{dA_{cn}}{|r_m-s|} \qquad (24)$$

and the g_{mn} are elements of the G matrix, and the terms

$$\delta_{mn} \frac{1}{h(r_m)}$$

are elements of the diagonal conductance matrix, H, then equation (9) may be written as

$$\frac{1}{\lambda} [G] + [H] \quad \{q\} = \{\theta_a\} \qquad (25)$$

The elements of the G matrix are determined from the contact geometry and the discretization scheme used for the contact. The elements of the H and θ matrices are obtained from the h(r) and $\theta_a(r)$ distributions specified for the problem. Therefore it is now possible to solve for the q matrix, which when combined with equations (7) and (13) will yield the approximate flux distribution over the contact.

The elements of the G matrix, the g_{mn}, when multiplied by the quotient of the flux divided by the thermal conductivity, represent the temperature rise at the centroid of element m due to a unit flux applied over element n. The major difficulty in analysing complex geometries is the evaluation of the g_{mn}, or influence coefficients. The evaluation of g_{mn} depends on the shape of the surface element and the location of the point with respect to the element. Expressions are available for a large number of geometries[4]. As an example the influence coefficient of a rectangular surface element and a point on the same plane internal or external to the element, with reference to Fig. 2 and equation (24), is given by

$$g = \frac{1}{2\pi} \int_{x_2-a/2}^{x_2+a/2} \int_{y_2-b/2}^{y_2+b/2} \frac{dxdy}{\sqrt{(x-x_1)^2+(y-y_1)^2}} \qquad (26)$$

When integrated equation (26) becomes (27)

$$2\pi g = X_2\, \ell n \left[\frac{\sqrt{X_2^2+Y_2^2} + Y_2}{\sqrt{X_2^2+Y_1^2} + Y_1} \right] + Y_2\, \ell n \left[\frac{\sqrt{X_2^2+Y_2^2} + X_2}{\sqrt{X_1^2+Y_2^2} + X_1} \right]$$

$$-X_1\, \ell n \left[\frac{\sqrt{X_1^2+Y_2^2} + Y_2}{\sqrt{X_1^2+Y_1^2} + Y_1} \right] - Y_1\, \ell n \left[\frac{\sqrt{X_2^2+Y_1^2} + X_2}{\sqrt{X_1^2+Y_1^2} + X_1} \right]$$

with $X_1 = x_2 - x_1 - (a/2)$ $\quad Y_1 = y_2 - y_1 - (b/2)$

$X_2 = x_2 - x_1 + (a/2)$ $\quad Y_2 = y_2 - y_1 + (b/2)$

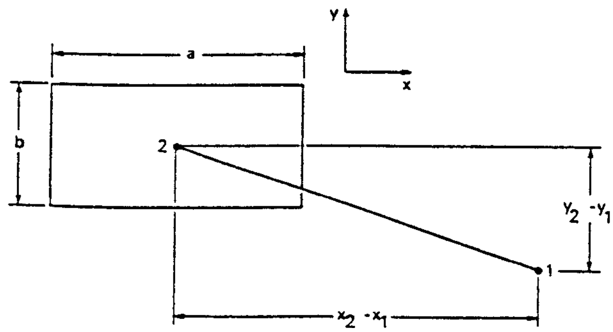

Fig. 2 Square Element Geometry and Coordinates.

The influence coefficients for circular elements and points on the same plane were required to determine the resistance of the circular contacts. With reference to Fig. 3a and b the influence coefficient for a circular element and a point internal or external are respectively [5]

$$g = (2/\pi) a E(r_z/a) \tag{28}$$

and

$$g = \frac{2}{\pi} r_z [E(a/r_z) - (1-(a/r_z)^2) K(a/r_z)] \tag{29}$$

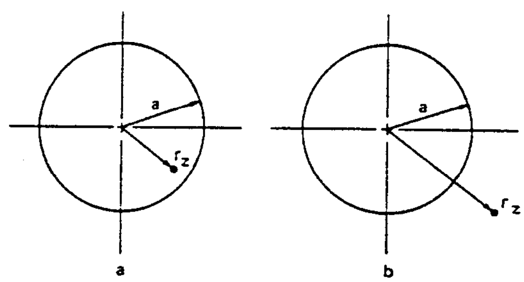

Fig. 3 Circular Element Geometry and Coordinates.

Expressions 28 and 29 may be combined to give the influence coefficient for an annular element.

The elements of the H matrix multiplied by the flux of a corresponding surface element give the difference between the applied temperature excess and the temperature excess at the centroid of the element. This difference is due to the contact resistance, or equivalently due to a film resistance.

Therefore the approximate distribution is obtained by matching the applied temperature excess of each element at its centroid, to the temperature drop through the local contact resistance plus the temperature rise at its centroid due to the flux distribution of the entire contact.

Once the flux distribution has been determined the temperature excess anywhere on the contact surface (or anywhere within the half-space) may be calculated by use of equation (2). The average contact temperature excess may therefore be calculated and used to determine the constriction resistance.

The resistance of a medium to heat flow is defined as the average temperature drop across the medium divided by the total heat flow through it. When the boundary condition of the third kind is imposed on a contact the total resistance of the system has two components: the contact resistance and the constriction resistance.

An area average is used to determine the average contact temperature. The total heat flow rate is determined from the relationship

$$Q = \iint_{A_c} q(s)\, dA_c \tag{30}$$

or

$$Q = \iint_{A_c} h(s)\, (\theta_a(s) - \theta_c(s))\, dA_c \tag{31}$$

The total resistance is therefore

$$R_t = R_c + R_f = \frac{(1/A_c) \iint_{A_c} \theta_a(s) dA_c}{\iint_{A_c} q(s) dA_c} \tag{32}$$

and the contact resistance is given by

$$R_f = \frac{(1/A_c) \iint_{A_c} (\theta_a(s)-\theta_c(s)) dA_c}{\iint_{A_c} h(s)(\theta_a(s)-\theta_c(s)) dA_c} \tag{33}$$

and the constriction resistance is given by

$$R_c = \frac{(1/A_c) \iint_{A_c} \theta_c(s) dA_c}{\iint_{A_c} q(s) dA_c} \tag{34}$$

The discretized equivalents for the average temperature and total heat flow rates are

$$\theta = \frac{1}{A_c} \sum_{i=1}^{N} \theta_i A_{ci} \tag{35}$$

where θ_{ci} is the value of θ_c at the centroid of the ith element; and

$$Q = \sum_{i=1}^{N} q_i A_{ci} \tag{36}$$

Expressions (35) and (36) can be used in equations (30) through (34) to determine the resistance of the system. In order to use equations (30) through (34) to determine the contact and constriction resistance it is necessary to know the surface temperature of the elements at their centroids. The surface temperature at the centroid of the elements can be determined from the relationship

$$[G] \left\{\frac{q}{\lambda}\right\} = \left\{\theta_c\right\} \tag{37}$$

Numerical Results and Correlations

The numerical procedure described above is applied to two contact geometries: the circle and the square. For both geometries the contact conductance is assumed to be uniform. These examples were chosen to illustrate the method because solutions for both contacts at the limits of the boundary condition are known. When the case of uniform contact conductance is considered, the procedure simplifies very little except that all elements in the H matrix have the same value and equation (33) reduces to

$$R_f = \frac{1}{hA_c} \tag{38}$$

The relative magnitude of the contact conductance is characterized by a Biot number defined as

$$Bi = \frac{h\sqrt{A_c}}{\lambda} \tag{39}$$

where the characteristic dimension is chosen to be the square root of the contact area. As the Biot number becomes small (<0.1) the contact boundary condition approaches the uniform Neumann condition. Conversely as the Biot number becomes large (>1000) the contact boundary condition approaches the uniform Dirichlet condition.

The computed values of the constriction reresistance obtained have been nondimensionalized by means of the thermal conductivity of the half-space and the square root of the contact area

$$R_c^* = \lambda R_c \sqrt{A_c} \tag{40}$$

The constriction resistance for three limiting combinations of the geometries and boundary conditions considered can be obtained analytically. Analytic solutions are available for both the circular and the square contacts with the Neumann boundary condition. An analytic solution is also available for the circular contact with the Dirichlet condition. Upper and lower bounds have been established for the constriction resistance of the isothermal square contact. As well as these limits numerical solutions are available for this contact. The values of the constriction resistance for these four limiting cases are given in Table 1.

Table 1 Resistance Limits

Geometry	Boundary Condition	$R^*=\lambda R\sqrt{A_c}$
Circular	Neumann	0.4789
	Dirichlet	0.4431
Square	Neumann	0.4732
	Dirichlet	
	Investigator	
	Schneider	0.438
	Stepleman	0.4339
	Pólya and Szegö	
	Upper Limit	0.4434
	Lower Limit	0.4238
	Present Work	0.434

The constriction resistance values obtained for both contact geometries at various Biot numbers ranging from 0.001 to 100,000 are listed in Table 2. These results are also displayed graphically in Fig. 4.

Table 2 Resistance vs. Biot

Biot	Circle	Square
0.001	0.4790	0.4738
0.0025	0.4790	0.4737
0.005	0.4790	0.4737
0.0075	0.4790	0.4737
0.01	0.4790	0.4737
0.025	0.4789	0.4737
0.05	0.4788	0.4736
0.075	0.4788	0.4735
0.1	0.4787	0.4734
0.25	0.4782	0.4729
0.5	0.4775	0.4722
0.75	0.4768	0.4715
1.0	0.4762	0.4708
2.5	0.4729	0.4673
5.0	0.4690	0.4632
7.5	0.4662	0.4603
10	0.4640	0.4581
25	0.4572	0.4514
50	0.4529	0.4474
75	0.4508	0.4456
100	0.4496	0.4446
250	0.4468	0.4425
500	0.4456	0.4417
750	0.4451	0.4415
1000	0.4449	0.4413
2500	0.4444	0.4411
5000	0.4443	0.4410
7500	0.4442	0.4409
10,000	0.4442	0.4409
25,000	0.4441	0.4409
50,000	0.4441	0.4409
75,000	0.4441	0.4409
100,000	0.4441	0.4409

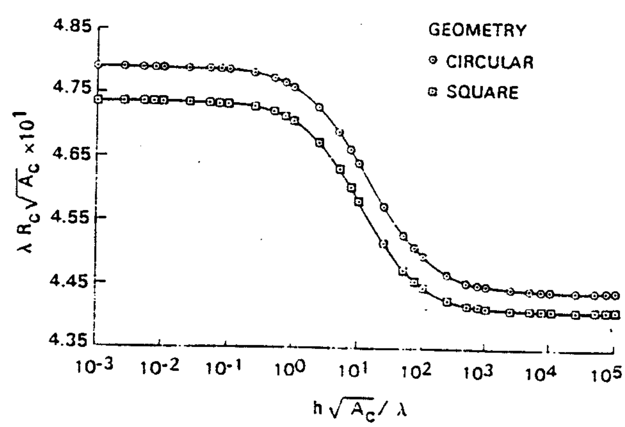

Fig. 4 Resistance vs. Biot Number, Circular and Square Contacts; Uniform Conductance.

In order to establish the accuracy of the procedure the resistance values obtained at very high and very low Biot numbers are compared to the numerical values listed in Table 1 for the isothermal and uniform flux cases. It can be seen from Fig. 4 and Table 2 that using a range of Biot numbers from 0.001 to 100,000 to span the difference between the uniform flux and isothermal contact is sufficient. Since extending the range past these extremes will not cause the numerical solution to approach any closer to its limiting values.

The numerical values of the constriction resistance obtained for both geometries at a Biot number of 0.001 compare well with the corresponding analytic solutions. The difference between the analytic and numerical solution is larger for the square contact and is about 0.15%. The resistance value predicted for a Biot of 100,000 for the circular geometry also compares well with the analytic solution for the isothermal case with the difference being approximately 0.2%. Examination of the values listed for the isothermal square contact in Table 1 shows that the numerical value obtained is well within the limits established by Pólya and Sezgö[6]. The value obtained by this method also agrees well with the values obtained by Stepleman[7] and Schneider[8].

The convergence of the solution for the high Biot number square contact is shown in Fig. 5, where the dimensionless resistance versus the number of elements used is plotted. The solution converges monotonically from above. The solution has these two characteristics for both geometries at any Biot number. Extrapolation of the numerical data used to plot Fig. 5 gives an asymptotic value of 0.434 for the resistance of the isothermal square contact.

The dimensionless constriction resistance as a function of Biot number may be correlated for each geometry using

$$R_c^* = c_1 - c_2 \tanh\left(c_3 \ln(Bi) - c_4\right) \qquad (41)$$

The form correlation equation was determined by inspection of Fig. 4. The constants c_1 through c_4 used in Equation (41) were determined using a least squares analysis and are listed in Table 3. By using the exponential definition of the hyperbolic tangent function Equation (41) may be written as

$$R_c^* = c_1 - c_2 \frac{Bi^{c_5} - c_6}{Bi^{c_5} - c_6} \qquad (42)$$

with

$$c_5 = 2c_3 \qquad (43)$$

and

$$c_6 = e^{2c_4} \qquad (44)$$

The correlation equation is very accurate with a maximum difference between its prediction and the numerical data of 0.1%.

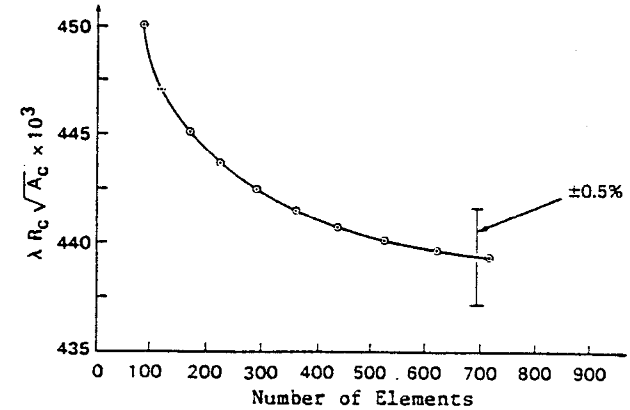

Fig. 5 Convergence of the Solution for a Square Contact with a Biot Number of 100,000.

Table 3 Correlation Coefficients

	Circle	Square
c_1	0.46159	0.45733
c_2	0.017499	0.016463
c_3	0.43900	0.47035
c_4	1.1624	1.1311
c_5	0.87800	0.94070
c_6	10.225	9.6042

Conclusions and Discussion

The moment method has been introduced to the solution of thermal constriction resistance problems with the boundary condition of the third kind imposed. The method has been demonstrated for planar contacts located on a half-space. The solution technique is not limited to contacts of regular geometries. The applied temperature excess and conductance distributions are assumed to be functions of position.

The method was applied to square and circular contacts with uniform conductance and applied temperature excess distributions. The constriction resistance of these two contact geometries was determined for a practical range of the Biot number.

The overall accuracy and the convergence of the solution were established by comparison of the resistance values obtained at the extremes of the boundary condition to the results available for the isothermal and uniform flux contacts. The method proved to be extremely accurate agreeing closely with these results.

The resistance values obtained were nondimentionalized using the square root of the contact area as the characteristic dimension. The nondimensional resistance was correlated as a function of the Biot number. The correlation function chosen was very accurate differing from the numerical data by a maximum of 0.1%.

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