LAMINAR NATURAL CONVECTION FROM A VERTICAL PLATE WITH VARIATIONS IN WALL TEMPERATURE

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ABSTRACT

An approximate analytical method extendible to problems involving a class of non-similar thermal conditions at the wall with a wide range of Prandtl numbers in natural convection is developed. An analysis is presented and a derivation is reported herein which details the method specifically applied to a vertical flat plate with a step change in wall temperature. A number of illustrative calculations are made and the resulting local surface heat-fluxes as well as the temperature and velocity distributions are presented for selected cases. Comparisons are made and it is shown that the results are in good agreement with existing data obtained by other investigators.

NOMENCLATURE

a, A	constants in Eq. (4)
b, B	constants in Eq. (4)
C	dimensionless constant, $C = C_u$ or $C = \mathrm{Pr}/(2C_\eta^2)$
C_u, C_η	constants defined by Eqs. (11) and (12)
$D(n,p)$	function defined by Eq. (A.20)
$\mathrm{erfc}\, z$	Complementary Error function, $\dfrac{2}{\sqrt{\pi}}\displaystyle\int_z^\infty e^{-t^2}\,dt$
f_T, f_u	functions defined by Eqs. (A.5) and (A.6)
g	gravitational acceleration
G_e, G_m	functions defined by Eqs. (A.13) and (A.12)
h	local coefficient of heat transfer
h_e, h_m	functions defined by Eqs. (A.28) and (A.27)
H_e, H_m	functions defined by Eqs. (A.26) and (A.25)
$\mathrm{i}^n\mathrm{erfc}\, z$	Complementary Error function integrated n-times, $\displaystyle\int_z^\infty \mathrm{i}^{n-1}\mathrm{erfc}\, t\, dt$, $\mathrm{i}^1\mathrm{erfc}\, z = \mathrm{ierfc}\, z$, $\mathrm{i}^0\mathrm{erfc}\, z = \mathrm{erfc}\, z$
k	thermal conductivity of fluid
m, M	constants in Eq. (7)
n, N	constants in Eq. (7)
p	square root of Prandtl number, $\sqrt{\mathrm{Pr}}$
q	local heat-flux
q^*	non-dimensionalized local heat-flux, q/q_{w1}
t	*pseudo-time* defined in $t - y$ plane
t_o	*pseudo-time* when $x = x_o$
T	temperature excess over ambient fluid temperature
u	local velocity parallel to plate, in x-direction
u_c	characteristic u-velocity across the boundary layer
u^*	non-dimensionalized u-velocity, $u/\sqrt{4g\beta T_{w1}x}$
v	local velocity normal to plate, in y-direction
x	vertical coordinate measured from leading edge
x_o	value of x at which functional discontinuity of wall temperature occurs, see Eq. (4)
y	horizontal coordinate measured from plate surface

Greek Symbols

α	thermal diffusivity of fluid, $k/(\rho c_p)$
β	thermal expansion coefficient, $-(\partial\rho/\partial T)_P/\rho$
γ	parameter, η_1/η_2 or $\sqrt{(1-1/\xi)\phi_2/\phi_1}$
$\Gamma(z)$	Gamma function, $\displaystyle\int_0^\infty t^{z-1}e^{-t}\,dt$
η_1	variable, $y/\sqrt{4\alpha t}$ in $t - y$ plane, or $C_\eta/\sqrt{\phi_1}\cdot(\mathrm{Gr}_x/4)^{1/4}\cdot y/x$ in $x - y$ plane
η_2	variable, $y/\sqrt{4\alpha(t-t_o)}$ in $t - y$ plane, or $C_\eta/\sqrt{(1-1/\xi)\phi_2}\cdot(\mathrm{Gr}_x/4)^{1/4}\cdot y/x$ in $x - y$ plane
θ	non-dimensionalized temperature excess, T/T_{w1}
$\Lambda(n)$	parameter, $\Lambda(n) = 1/\mathrm{i}^n\mathrm{erfc}\, 0 = 2^n\Gamma(\frac{n}{2}+1)$
ν	kinematic viscosity, μ/ρ
ξ	non-dimensionalized x-coordinate, x/x_o
ϕ_1, ϕ_2	parameters defined by Eqs. (16) and (17), $\phi_1 = 1$ when $x \le x_o$
ψ	parameter, $\phi_1^2\xi$

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Subscripts

$w1$	wall conditions for $x \le x_o$
$w2$	wall conditions for $x > x_o$

Dimensionless Groups

Gr_x	local Grashof number, $g\beta T_{w1}x^3/\nu^2$
Nu_x	local Nusselt number, hx/k
Pr	Prandtl number, ν/α
Ra_x	local Rayleigh number, $\mathrm{Gr}_x \cdot \mathrm{Pr}$

INTRODUCTION

Since the first successful experimental and theoretical study of a flat plate with uniform wall temperature by Schmidt and Beckmann [1], natural convection heat transfer from a vertical flat plate in a quiescent medium has been studied by many investigators. Among them, Ostrach [2] solved laminar boundary layer equations through similarity methods for the case of uniform wall temperature for various Prandtl numbers. Similarity solutions to boundary layer equations have been further extended by Sparrow and Gregg [3,4]. They have documented the results for the cases of uniform surface heat-flux and surface temperature variations of the power and exponential form.

The above cited literature and many others deal with similarity problems with continuous and well-behaved boundary conditions at the wall. However, most practical problems in application often involve wall conditions that differ from those allowed for in a similarity method. The study of non-similar natural convection heat transfer from a vertical flat plate with arbitrarily varying thermal conditions at the wall is a complex investigation. The problem has attracted a great deal of interest over the past years in several technological applications, particularly in the thermal design of microelectronic circuit boards and in the consideration of system behaviour under conditions where the usual mode of heat transfer fails. However, to the authors' present knowledge, practical solutions that can assign general surface conditions do not exist.

To understand and solve problems involving general non-similar conditions at the wall, it is useful to investigate problems subjected to a step change in wall temperature. The problems impose a mathematical singularity and severe non-similar conditions at the wall.

An early attempt was made by Schetz [5,6], to develop an approximate analytical model for the problems of discontinuous constant wall temperature types. In his study, the temperature and velocity profiles were assumed essentially the same as those due to unsteady one-dimensional conduction alone. The model employed a collocation method, and resulted in overestimating the local heat transfer coefficient, as compared to that of Ostrach [2], by approximately 14% when it was applied to a uniform surface temperature case. Subsequently, his approximate method as applied to the step change in wall temperature was concluded to be unsuccessful.

Numerous investigations on the same problem were continued by using an experimental technique [7], numerical methods [8,9], or by using series expansions [10-12].

In this paper, a new approximate model is presented. The method reported herein has been developed in view of extending it to problems having general non-similar surface conditions and it is applicable to a family of non-similar problems. The method utilizes a linearization of convective terms in transforming the non-linear steady-state equations to linear *pseudo-transient* equations, from which the temperature and velocity profiles are derived. A comparison is made between the similarity groups of the transient solutions and those of the steady-state solutions of Sparrow and Gregg [4], yielding a preliminary transformation function. The von Kármán-Pohlhausen integral method is then employed in order to complete the reverse-transformation of the *pseudo-transient* terms, which results in a set of approximate but *simple analytical* solutions to the original equations. Although the present analysis is applicable to a class of functionally discontinuous wall temperature variations, the illustration is focused on problems with a step change in constant wall temperature, mainly due to the availability of other data for comparison. The validity of the method is demonstrated by comparisons with the results of the aforementioned investigators. The general agreement is good and the results obtained with the solution of this work are remarkably close to existing numerical data of Hayday et al. [8] and the perturbation series solution of Kao [12]. The method is equally applicable to problems with a step change in wall heat-flux.

ANALYSIS

Governing Equations. The general conservation equations written for a full description of flow and energy within the fluid, have been reduced to a set of boundary layer equations, governing two-dimensional, steady-state, laminar natural convection heat transfer from a vertical plate. The plate considered is semi-infinite and suspended in a quiescent fluid which is maintained at uniform temperature. The equations may be expressed as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta T \qquad (2)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (3)$$

where x, y are coordinates parallel and normal to the plate, u, v are corresponding components of the velocity, and T is the local temperature excess over the ambient fluid temperature. In addition to the boundary layer approximations, the assumption of constant fluid properties, except the density in the use of the Boussinesq approximation, is included, and the dynamic pressure work and viscous dissipation terms are neglected in the above equations. The boundary conditions associated with the foregoing equations are

$$\begin{aligned} \text{at } y = 0\,, \quad & u = v = 0\,, \\ & T = T_{w1} = Ax^a && \text{for } x \le x_o\,, \\ & T = T_{w2} = Ax^a + B(x - x_o)^b && \text{for } x > x_o\,, \\ \text{as } y \to \infty\,, \quad & u \to 0\,, \quad T \to 0\,, \end{aligned} \qquad (4)$$

where a, b, A, and B are constants. The conditions at $x = 0$ are trivial and are not stated for brevity.

Although the wall condition for T is continuous at $x = x_o$, when $b > 0$, a general form of wall temperatures given by Eq. (4) will be referred to as functionally discontinuous at $x = x_o$. For a heated plate, the wall temperatures are such that $T_{w1} > 0$ and $T_{w2} \geq 0$. Notice that T_{w1} is expressed in a form of power law for which the similarity solution is available when $x \leq x_o$.

Pseudo-Transient Equations. The reduced equations previously shown are non-linear and mutually coupled, representing an extremely complicated mathematical problem. Similarity solutions do not exist for a family of the given boundary conditions unless the coefficient B vanishes, in which case, of course, the functional discontinuity of the wall temperature disappears.

An approximate method is sought in which a *pseudo-time*, t is introduced. The downstream location x is viewed as $u_c \times t$, where u_c is defined as a characteristic streamwise velocity over the boundary layer, as a function of x. The original $x - y$ plane is now transformed into the $t - y$ plane. An assumption that diffusion is dominant across the boundary layer in the y-direction at given time implies that the temperature and velocity profiles would take forms of those due to a transient conduction heat transfer into a half space. Subsequently, the convective terms are replaced by transient terms resulting in a set of *pseudo-transient* equations written as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + g\beta T \tag{5}$$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} . \tag{6}$$

The associated boundary conditions that are compatible with the conditions prescribed by Eq. (4) are

$$\begin{aligned} &\text{at } y = 0 , \quad u = 0 , \\ &\qquad T = T_{w1} = N t^{n/2} && \text{for } t \leq t_o, \\ &\qquad T = T_{w2} = N t^{n/2} + M (t - t_o)^{m/2} && \text{for } t > t_o, \\ &\text{as } y \to \infty , \quad u \to 0 , \quad T \to 0 , \end{aligned} \tag{7}$$

where n, m, N, M are constants, t is currently an unknown function of x, and t_o is the corresponding time of the discontinuity at x_o.

The velocity normal to the wall no longer plays a part in determining the profiles as its effect has been neglected by the assumption. The qualitative discrepancies on heat transfer results due to this assumption are expected to be less when the magnitude of the exponents, a and b, are relatively small so that a change in the streamwise velocity, $\partial u/\partial x$, near the wall is kept moderate. It is easy to visualize a proportional relationship between the exponents and the velocity gradient. When the magnitude of the exponents are large, acceleration or deceleration induced by the corresponding change in buoyant force along the flow near the wall will result in a rapid change in the streamwise velocity. On the other hand, when the exponents are small, $\partial u/\partial x$ will be small. It is then clear from the continuity equation, Eq. (1), that when $\partial u/\partial x$ is small, $\partial v/\partial y$ is also small in favour of the assumption. This becomes especially true near the wall by noting that $v = 0$ and $\partial v/\partial y = 0$ at the wall. The omission of the normal velocity component is thus partially justified under limited conditions implied on the exponents.

Pseudo-Transient Equations (5) and (6) are linear and unilaterally decoupled, conditions that respectively allow the solutions to be superposed and the temperature equation to be solved independently prior to solving the velocity equation. The identical equations may be obtained by simply replacing the convective derivatives with $\bar{u}\frac{\partial}{\partial x}$, where $\bar{u} = u_c/(1 - d\ln u_c/d\ln x)$. These have been called *linearized approximations* by some investigators. The above transient equations can be solved by means of either Laplace transforms [13,14], or similarity methods. The resulting solutions for $t > t_o$ are

$$T = T_{w1} \Lambda(n)\, \mathrm{i}^n \mathrm{erfc}\, \eta_1 + (T_{w2} - T_{w1})\, \Lambda(m)\, \mathrm{i}^m \mathrm{erfc}\, \eta_2 \tag{8}$$

$$u = \begin{cases} 2g\beta \left[T_{w1}\, t\, \Lambda(n)\, \eta_1\, \mathrm{i}^{n+1}\mathrm{erfc}\, \eta_1 \right. \\ \left. + (T_{w2} - T_{w1})(t - t_o)\, \Lambda(m)\, \eta_2\, \mathrm{i}^{m+1}\mathrm{erfc}\, \eta_2 \right] & \text{for Pr} = 1 , \\ 2g\beta \dfrac{2}{1 - \mathrm{Pr}} \left[T_{w1}\, t\, \Lambda(n)\, (\mathrm{i}^{n+2}\mathrm{erfc}\, \eta_1 - \mathrm{i}^{n+2}\mathrm{erfc}\, \dfrac{\eta_1}{\sqrt{\mathrm{Pr}}}) \right. \\ + (T_{w2} - T_{w1})(t - t_o) \\ \left. \quad \times \Lambda(m)\, (\mathrm{i}^{m+2}\mathrm{erfc}\, \eta_2 - \mathrm{i}^{m+2}\mathrm{erfc}\, \dfrac{\eta_2}{\sqrt{\mathrm{Pr}}}) \right] & \text{for Pr} \neq 1 , \end{cases} \tag{9}$$

where n, m are apparently constrained to be integers that are greater than or equal to -1, η_1, η_2 are similarity variables given as $\eta_1 = y/\sqrt{4\alpha t}$, $\eta_2 = y/\sqrt{4\alpha(t - t_o)}$, and $\Lambda(n)$ is defined as $2^n \Gamma(\frac{n}{2} + 1)$. These solutions are exact for transient natural convection heat transfer from an infinitely long plate experiencing wall temperature variations given by Eq. (7). The solutions for $t \leq t_o$ can also be obtained by simply discarding the second term of each equation. Note that while $n = 0$ represents uniform T_{w1}, it is easy to verify from Eq. (8) that the heat-flux at the wall is uniform for $t \leq t_o$ when $n = 1$. The problem is now reduced to finding a proper transformation function or functions for $u_c(x)$ to relate t and x, and the values of n and m.

t–x Transformations for $x \leq x_o$. Since the above similarity solutions for $t \leq t_o$ are approximate representations of solutions of the original steady-state equations for $x \leq x_o$, it is reasonable to compare the similarity groups from each solution domain as follows:

$$\text{Wall Temperature Excess}; \qquad A x^a = N t^{n/2} \tag{10}$$

$$\text{Reference Velocity Group}; \quad C_u \sqrt{4 g\beta T_{w1} x} = 2 g\beta T_{w1} t \tag{11}$$

$$\text{Similarity Variables} \quad ; \quad C_\eta \left(\frac{\mathrm{Gr}_x}{4}\right)^{1/4} \frac{y}{x} = \frac{y}{2\sqrt{\alpha t}} \tag{12}$$

where C_u and C_η are inserted as dimensionless proportionality constants for the given problem. The parameters shown on the left hand side of Eqs. (11) and (12) are from the similarity analysis of Sparrow and Gregg [4]. A simple manipulation of either Eq. (11) or Eq. (12) upon substitution of Eq. (10) for T_{w1} yields

$$n = \frac{4a}{1-a} \,. \tag{13}$$

From Eq. (11) , $$t = C_u \frac{x}{\sqrt{g\beta T_{w1} x}} \tag{14}$$

and from Eq. (12) , $$t = \frac{\mathrm{Pr}}{2C_\eta^2} \frac{x}{\sqrt{g\beta T_{w1} x}} \,. \tag{15}$$

Equation (13) defines the value of n, and it also asserts a mathematical constraint on a value of the exponent such that $-1/3 \leq a < 1$. This is merely a consistent consequence imposed by the assumption discussed earlier. When $|a|$ is relatively large, an effect of normal advection by v-velocity becomes significant, and the profiles derived solely from the basis of transient diffusion, would no longer be able to simulate the behaviour of the actual profiles for $a \geq 1$.

Equations (14) and (15) are the $t-x$ transformations sought for $x \leq x_o$. Although they express different relationships between t and x, they both exhibit an identical functional form such that $u_c = \sqrt{g\beta T_{w1} x}/C$, where $C = C_u$ or $C = \mathrm{Pr}/(2C_\eta^2)$. The former equation defines the time anticipated in the reference velocity group and the latter defines the time anticipated in the similarity variable. A practice which equates these two to obtain $C_u = \mathrm{Pr}/(2C_\eta^2)$ satisfies the u-momentum equation at the wall. It should be emphasized, however, that this practice is only useful in satisfying the transient equations, as it would not conserve the momentum over the boundary layer for the steady-state problems. The constants, C_u and C_η are dependent only on the power a and the Prandtl number, both of which are fixed for the given problem. Their values are determined by employing the von Kármán-Pohlhausen integral method and the derivation is provided in the Appendix.

t–x Transformations for x > x$_o$. Owing to the boundary layer approximations, the similarity solutions at $x \leq x_o$ remain unchanged. However, as a *secondary thermal boundary layer* establishes from the wall at $x = x_o$ due to the induction of an abrupt potential change at the wall, the similarity characteristic of the existing boundary layer, which initiated at the leading edge, will no longer be maintained at the downstream location at $x > x_o$. This secondary boundary layer grows quickly into and will eventually engulf the existing one as x approaches infinity, where, if $b = a$, the solutions become similar once again based on T_{w2}. Not only the characteristic streamwise velocity over the existing boundary layer would hence be altered, but another distinct characteristic velocity would also evolve over the secondary boundary layer. In the transient solutions, Eqs. (8) and (9), the first terms are related to the original boundary layer whereas the second terms are related to the secondary boundary layer.

The $t-x$ relationships are formulated as before which may be viewed as

$$\text{The } \textit{pseudo-time} \text{ lapse} \equiv \frac{\text{streamwise displacement}}{u_c \text{ over the boundary layer}} \,.$$

The characteristic streamwise velocity of each boundary layer experiences a functional transition from the existing one in order to incorporate the non-similarity effect. They may thus be defined by

$$t = \frac{x}{u_c} = C\phi_1 \frac{x}{\sqrt{g\beta T_{w1} x}} \tag{16}$$

$$t - t_o = \frac{x - x_o}{u_c} = C\phi_2 \frac{x - x_o}{\sqrt{g\beta T_{w1} x}} \,. \tag{17}$$

Notice that the time t in Eq. (17) is not the same *pseudo-time* t as in Eq. (16). The term $t - t_o$ in Eq. (17) is a unique variable, representing the *pseudo-time* lapse within the secondary boundary layer. The functions ϕ_1 and ϕ_2 modify the characteristic velocities in the corresponding domain of the boundary layers, and they are dependent on the exponents, Pr, θ_{w2} , and ξ. Since C itself still denotes two different coefficients, namely C_u and $\mathrm{Pr}/(2C_\eta^2)$, it should be recognized that, even when the exponents are fixed so that C_u and C_η are functions only of the Prandtl number, it is necessary to distinguish C from ϕ_1 and ϕ_2. The von Kármán-Pohlhausen method is again employed in evaluating ϕ_1 and ϕ_2. Without loss of extended applicability of the method, the derivation of these functions is outlined, in particular, for a step change in constant wall temperature in the Appendix.

The same relationship and constraints held between a and n may be assumed between b and m. This is suggested by recalling the aforementioned similarity solution written for T_{w2}, to which the solution must approach at large x when $b = a$. Also, note that C_u and C_η needed for C in Eq. (17) are functions of b and Pr, so that if $b \neq a$, they have to be evaluated separately from the Appendix using b and m in place of a and n.

Results and Discussion

Although the present analysis is applicable to cases other than a step change in uniform wall temperature, results provided in the following, report only the case in which both T_{w1} and T_{w2} are uniform. Summarizing the preceding section, the resulting non-dimensionalized approximate solutions are :

$$\theta = \frac{T}{T_{w1}} = \mathrm{erfc}\,\eta_1 + (\theta_{w2} - 1)\mathrm{erfc}\,\frac{\eta_1}{\gamma} \tag{18}$$

$$u^* = \frac{u}{2\sqrt{g\beta T_{w1} x}} = \begin{cases} C_u\phi_1 \left[\eta_1\,\mathrm{ierfc}\,\eta_1 + \gamma\,(\theta_{w2}-1)\eta_1\,\mathrm{ierfc}\,\frac{\eta_1}{\gamma}\right] & \text{for Pr} = 1\,, \\ C_u\phi_1 \frac{2}{1-\mathrm{Pr}} \Big[\mathrm{i}^2\mathrm{erfc}\,\eta_1 - \mathrm{i}^2\mathrm{erfc}\,\frac{\eta_1}{\sqrt{\mathrm{Pr}}} & \\ \quad + \gamma^2\,(\theta_{w2}-1)(\mathrm{i}^2\mathrm{erfc}\,\frac{\eta_1}{\gamma} - \mathrm{i}^2\mathrm{erfc}\,\frac{\eta_1}{\gamma\sqrt{\mathrm{Pr}}})\Big] & \text{for Pr} \neq 1\,, \end{cases} \tag{19}$$

where $\theta_{w2} = T_{w2}/T_{w1}$, $\eta_1 = C_\eta/\sqrt{\phi_1} \cdot (\mathrm{Gr}_x/4)^{1/4} y/x$, $\gamma = \sqrt{(1-1/\xi)\phi_2/\phi_1}$, $\xi = x/x_o$, and C_u, C_η, ϕ_1 and ϕ_2 can be obtained from the Appendix. Clearly, as before, the solution to a problem in which the entire wall is at uniform temperature T_{w1} can be found from the above by discarding the last terms and setting $\phi_1 = 1$.

The solutions given in Eqs. (18) and (19) are evaluated for a wide range of Prandtl numbers with various wall temperature

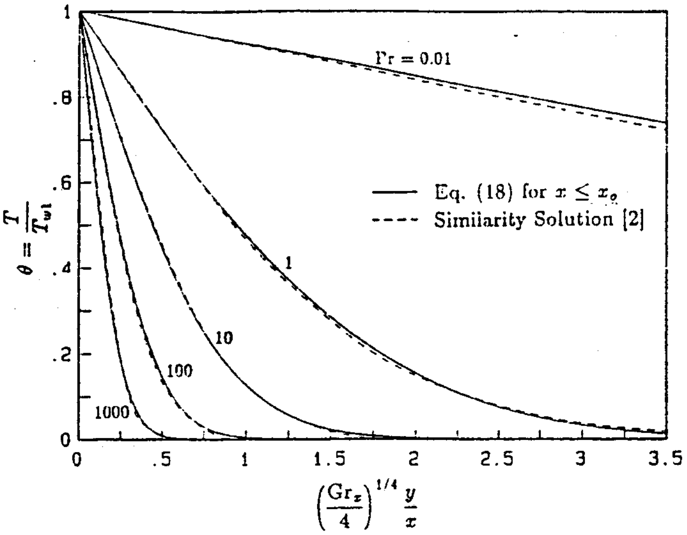

FIG. 1 COMPARISON OF DIMENSIONLESS TEMPERATURE DISTRIBUTIONS FOR VARIOUS PRANDTL NUMBERS WITH A UNIFORM WALL TEMPERATURE.

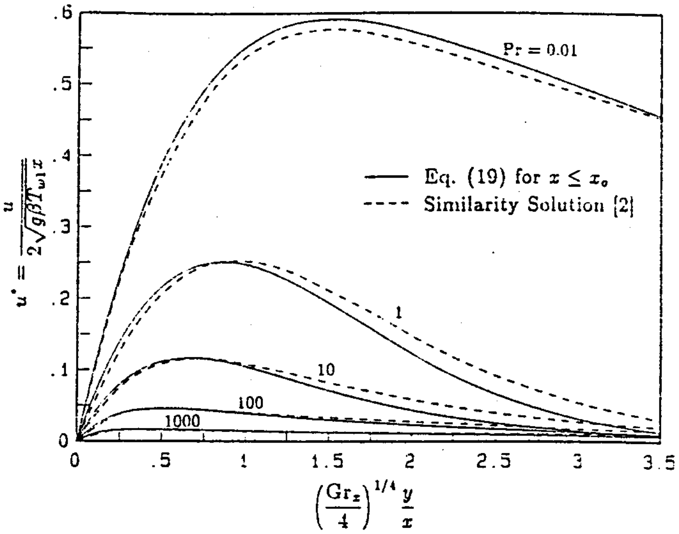

FIG. 2 COMPARISON OF DIMENSIONLESS VELOCITY DISTRIBUTIONS FOR VARIOUS PRANDTL NUMBERS WITH A UNIFORM WALL TEMPERATURE.

ratios θ_{w2}. The resulting temperature and velocity distributions are shown for $x \le x_o$ in Figs. 1 and 2, respectively. And for $x > x_o$, selected evaluations are plotted in Figs. 3, 4, and 5 with those of other investigators for comparison.

Figures 1 and 2 exhibit an excellent agreement between the results obtained by the present method and the results of Ostrach [2], who solved boundary layer equations using similarity methods. This good agreement has partially been observed by Goldstein in his discussion of Siegel's [15] investigation, where he documented the results for Pr = 1 by essentially using the same technique as the present method, applied for $x \le x_o$. Figures 3, 4, and 5 depict the temperature and velocity distributions responding to step changes in wall temperature, and they also exhibit good agreement with the data of others.

<u>Heat Transfer.</u> The local wall heat-flux at the location $x \le x_o$ may be obtained by using Fourier's law :

$$q_{w1} = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = C_\eta\sqrt{\frac{2}{\pi}}\,\frac{kT_{w1}}{x}\mathrm{Gr}_x^{1/4}\,. \qquad (20)$$

After equating Eq. (20) with Newton's law of cooling, given as

$$q_{w1} = h\,T_{w1} \qquad (21)$$

a dimensionless expression for the local heat transfer coefficient for $x \le x_o$ can be obtained as

$$\frac{\mathrm{Nu}_x}{\mathrm{Ra}_x^{1/4}} = \frac{C_\eta}{\mathrm{Pr}^{1/4}}\sqrt{\frac{2}{\pi}}\,. \qquad (22)$$

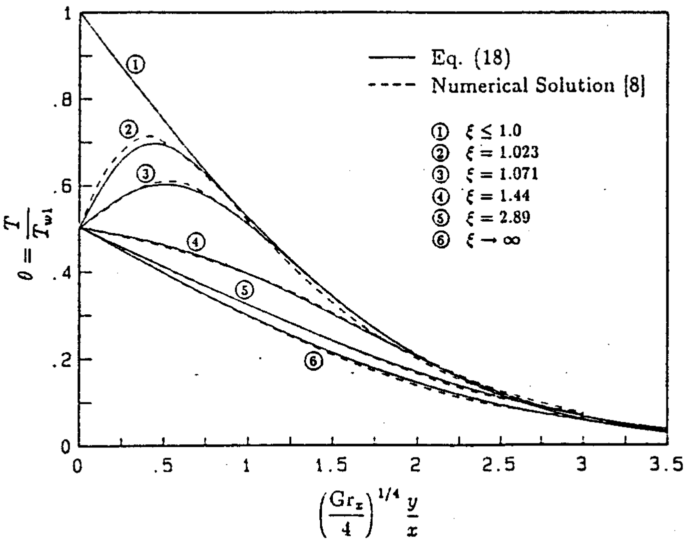

FIG. 3 COMPARISON OF DIMENSIONLESS TEMPERATURE FIELD DEVELOPMENT WITH A STEP CHANGE IN WALL TEMPERATURE : $\theta_{w2} = 0.50251$, Pr = 0.72 .

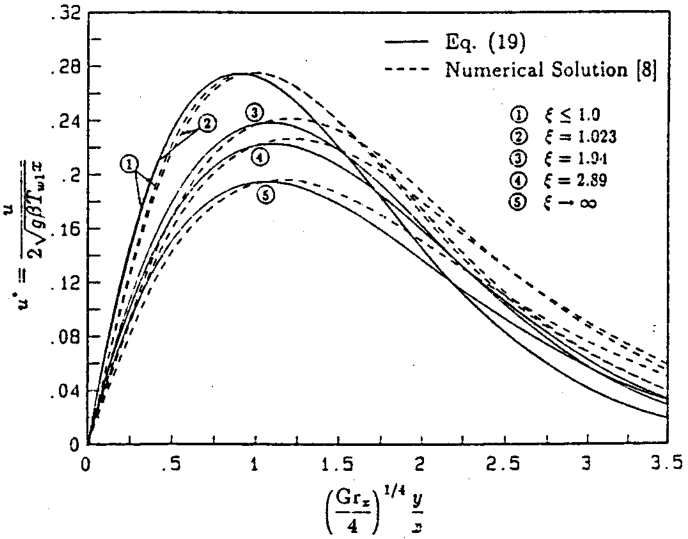

FIG. 4 COMPARISON OF DIMENSIONLESS VELOCITY FIELD DEVELOPMENT WITH A STEP CHANGE IN WALL TEMPERATURE : $\theta_{w2} = 0.50251$, Pr = 0.72 .

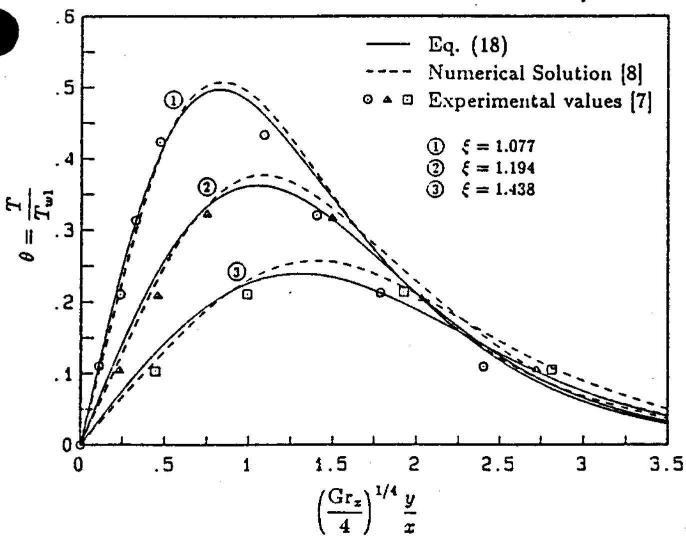

FIG. 5 COMPARISON OF DIMENSIONLESS TEMPERATURE FIELD DEVELOPMENT WITH A STEP CHANGE IN WALL TEMPERATURE : $\theta_{w2} = 0$, $\Pr = 0.72$.

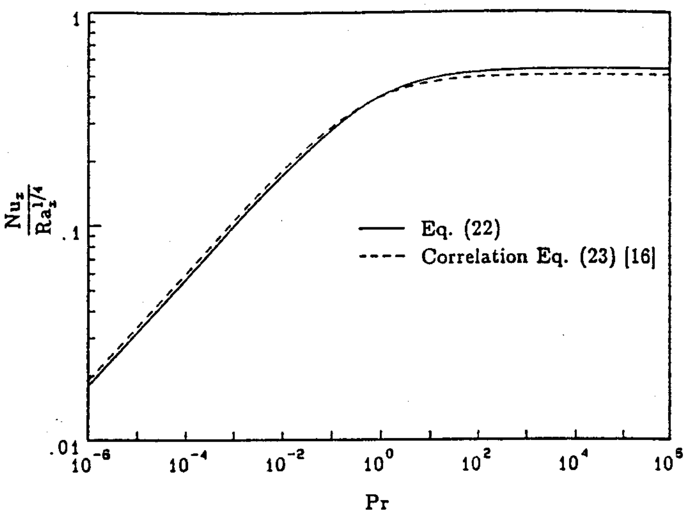

FIG. 6 COMPARISON OF $Nu_x/Ra_x^{1/4}$ AS A FUNCTION OF PRANDTL NUMBER : UNIFORM WALL TEMPERATURE CASE.

The coefficient is evaluated for uniform wall temperature cases over a wide range of Prandtl numbers. Results are compared to an existing correlation equation [16] in Fig. 6, and a maximum difference of 7% is observed as the Prandtl number approaches infinity. This correlation equation is rewritten here as

$$\frac{Nu_x}{Ra_x^{1/4}} = \frac{0.503}{[1 + (0.492/\Pr)^{9/16}]^{4/9}} . \tag{23}$$

The non-dimensionalized local wall heat-flux at the location $x > x_o$ may also be obtained as

$$q_{w2}^{*} = \frac{q_{w2}}{q_{w1}} = \frac{1}{\sqrt{\phi_1}} \left[1 + \frac{\theta_{w2} - 1}{\gamma} \right] \tag{24}$$

where q_{w1} is the local heat-flux at the location of interest with the wall maintained at temperature T_{w1}. Notice that while the

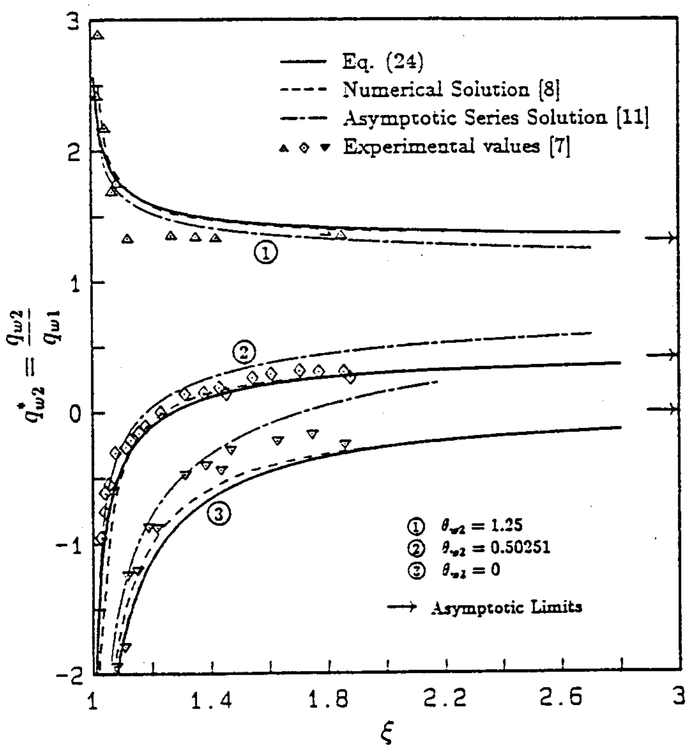

FIG. 7 COMPARISON OF LOCAL WALL HEAT-FLUX VARIATION : $\Pr = 0.72$.

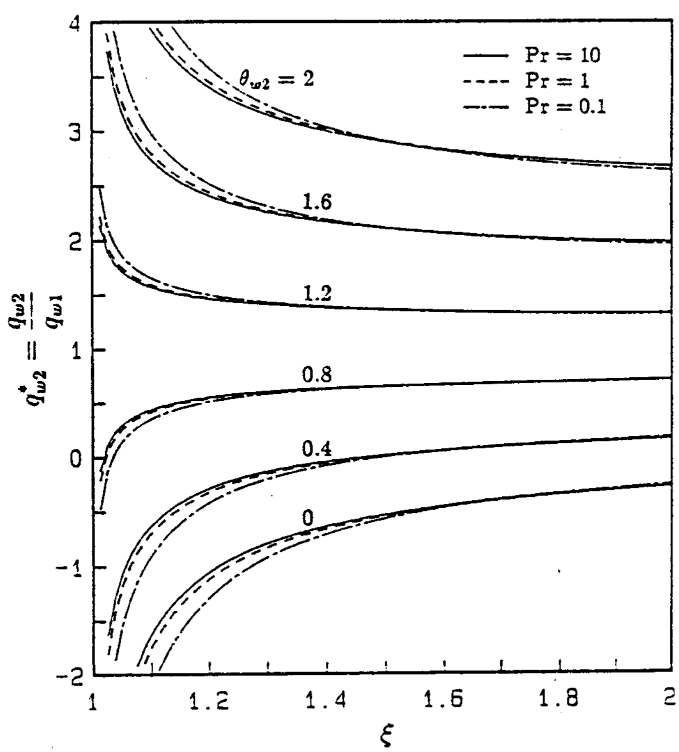

FIG. 8 WALL HEAT-FLUX VARIATION.

resulting magnitude of q_{w1} is directly altered by the differences in the heat transfer coefficient between the calculated and correlated values, q^*_{w2} given by Eq. (24) is free from the choice of expressions for the coefficient.

The result obtained by using Eq. (24) is plotted and compared in Fig. 7 with other data. The values indicated by arrows are the asymptotic values at large x, and they are determined by

$$\lim_{x\to\infty} q^*_{w2} = \theta_{w2}^{5/4} . \tag{25}$$

This equation can be readily obtained by inspecting any valid expression, which is based on the boundary layer theory for natural convection heat transfer from a plate of uniform temperature, and noting that $q_w \propto T_w^{5/4}$ at fixed x.

The laminar regime is not likely to be maintained at far downstream locations in practice. However, it is worthwhile stressing that the resulting solutions satisfy all the limiting conditions at large ξ, as both ϕ_1 and ϕ_2 can be shown to become independent of the Prandtl number and approach $1/\sqrt{\theta_{w2}}$, and thus $\gamma \to 1$. Figure 7 exhibits a good agreement of the result with others, and furthermore, it illustrates a valid correspondence of the method in the limits. Considering the limiting conditions at large ξ, inconsistent deviations are observed in the asymptotic series solution presented by Kelleher [11]. Figure 8 reveals effects of the Prandtl number on the surface heat transfer using Eq. (24). It shows, as observed by Kelleher, the faster thermal response of a higher Prandtl number fluid to the wall temperature change at the initial portion of the downstream location.

Owing to the absence of other data, neither the accuracy nor the range of validity of the present method, as it is formulated for the temperature prescribed wall conditions, could have been explicitly tested for other values of exponents, except for $a = 0.2$ with $B = 0$. This case, as discussed by Sparrow and Gregg [4], corresponds to a plate with uniform surface heat-flux, and the results were found to be satisfactory when compared to those obtained by using the similarity method [3]. The relationship between a and n given by Eq. (13) holds mutual correspondence between the $x-y$ plane and the $t-y$ plane for both wall conditions of uniform temperature ($a = n = 0$) and uniform heat-flux ($a = 0.2$, $n = 1$). In view of this discussion, it may be stated that the present approximate method can also be applied with greater confidence to problems involving a step change in uniform wall heat-flux.

Finally, the present method is basically the von Kármán-Pohlhausen integral method except the temperature and velocity profiles respond to the variations in the thermal conditions imposed at the wall. The combined use of the Rayleigh transformation, $t = x/u_c$, and the linearization technique on the steady state equations not only makes it possible to obtain approximate analytical solutions to the equations, but also allows superposition of the profiles in the $t-y$ plane. The v-velocity, hence the $v\frac{\partial}{\partial y}$ terms are, in essence, neglected by the linearization, but only in determining the forms of the profiles. The effect of these terms and the non-linearity of the original problem are retained after the $t-x$ transformations through the balancing of the energy and the momentum in the $x-y$ plane.

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APPENDIX

Derivation of C_u and C_η. The dimensionless constants, C_u and C_η, involved in the comparison between the similarity groups from the steady-state and transient solution domains are evaluated by using the von Kármán-Pohlhausen integral method. Integrating u-momentum and temperature equations given by Eqs. (2) and (3), across the boundary layer yields

$$\frac{d}{dx}\left[\int_0^\infty u^2\,dy\right] + \nu\left(\frac{\partial u}{\partial y}\right)_{y=0} = g\beta\int_0^\infty T\,dy \qquad \text{(A.1)}$$

$$\frac{d}{dx}\left[\int_0^\infty Tu\,dy\right] + \alpha\left(\frac{\partial T}{\partial y}\right)_{y=0} = 0\,. \qquad \text{(A.2)}$$

Substitute the following solutions obtained for $x \le x_o$;

$$T = T_{w1} f_T(\eta_1) \qquad \text{(A.3)}$$

$$u = C_u\sqrt{4g\beta T_{w1}x}\, f_u(\eta_1) \qquad \text{(A.4)}$$

where

$$f_T(\eta) = \Lambda(n)\mathrm{i}^n\mathrm{erfc}\,\eta \qquad \text{(A.5)}$$

$$f_u(\eta) = \begin{cases} \Lambda(n)\eta\,\mathrm{i}^{n+1}\mathrm{erfc}\,\eta & \text{for Pr}=1 \\ \dfrac{2\Lambda(n)}{1-p^2}(\mathrm{i}^{n+2}\mathrm{erfc}\,\eta - \mathrm{i}^{n+2}\mathrm{erfc}\,\dfrac{\eta}{p}) & \text{for Pr}\neq 1 \end{cases} \qquad \text{(A.6)}$$

$$T_{w1} = Ax^a \qquad \text{(A.7)}$$

$$\eta_1 = C_\eta\left(\frac{\mathrm{Gr}_x}{4}\right)^{1/4}\frac{y}{x} \qquad \text{(A.8)}$$

$$p = \sqrt{\mathrm{Pr}} \qquad \text{(A.9)}$$

into Eqs. (A.1) and (A.2), collect terms and simplify after evaluation of integrals and derivatives to obtain, respectively,

$$(3a+5)C_u^2 G_m(n) + C_u C_\eta^2 \frac{2}{p(1+p)} = 1 \qquad \text{(A.10)}$$

$$(5a+3)p^2 C_u G_e(n) = C_\eta^2 \qquad \text{(A.11)}$$

where

$$G_m(n) = \frac{\Lambda(n+1)}{\Lambda(n)}\int_0^\infty f_u^2(\eta)\,d\eta \qquad \text{(A.12)}$$

$$G_e(n) = \frac{\Lambda(n-1)}{\Lambda(n)}\int_0^\infty f_T(\eta) f_u(\eta)\,d\eta\,. \qquad \text{(A.13)}$$

Substitute Eq. (A.11) for C_η^2 into Eq. (A.10), and a minor manipulation results in

$$C_u = [(3a+5)G_m(n) + (5a+3)\frac{2p}{1+p}G_e(n)]^{-1/2} \qquad \text{(A.14)}$$

$$C_\eta = [(5a+3)p^2 C_u Ge(n)]^{1/2}\,. \qquad \text{(A.15)}$$

A recurrence relation for $G_e(n)$ and an equation for $G_m(n)$ are obtained by solving integrals in Eqs. (A.13) and (A.12). They are, for Pr = 1 ,

$$G_e(n) = \frac{2n}{2n+3}\left[\frac{1}{8n(n+1)} - G_e(n-1)\right] \quad ; n \ge 1 \qquad \text{(A.16)}$$

$$G_m(n) = \frac{1}{2n+5}\left[(n+1)(4n+9)G_e(n) - \frac{1}{4}\right]; n \ge 0 \qquad \text{(A.17)}$$

with $G_e(0) = \dfrac{\sqrt{2}-1}{12}$.

Or, for Pr $\neq 1$,

$$G_e(n) = \frac{2}{1-p^2}[D(n,1) - D(n,p)] \qquad \text{(A.18)}$$

$$G_m(n) = \frac{4}{(1-p^2)^2}\left[\frac{2(n+1)(1+p)}{2n+5}D(n,1) + 2pD(n+1,p) - \frac{p}{2(n+2)(n+3)}\right] \qquad \text{(A.19)}$$

where

$$D(n,p) = \frac{1}{4(n+1)(2n+3)}\left[p^2 + \frac{2n+1}{n+1}p + \frac{n+1}{n+2} - 4n(n-1)(1+p^2)D(n-2,p) - \frac{4n}{p}\{n+1+(3n+2)p^2\}D(n-1,p)\right] \quad ; n \ge 2 \qquad \text{(A.20)}$$

with $D(0,p) = \dfrac{2p^3+3p^2+2-2(1+p^2)^{3/2}}{24p^2}$

$D(1,p) = \dfrac{2(1+p^2)^{5/2} - 2p^5 - 5p^2 - 2}{120p^3}$.

Equations (A.14) and (A.15) combined with Eqs. (A.16) through (A.20) complete a set of equations needed in evaluation of $C_u(a,\mathrm{Pr})$ and $C_\eta(a,\mathrm{Pr})$. The exponent a may be a value from Eq. (13) as $a = n/(n+4)$ where n is non-negative integer. A direct calculation for the case of $n = -1$ is relatively simple and is not included here.

When T_{w1} is uniform, Eqs. (A.18) to (A.20) written for $\mathrm{Pr} \neq 1$, simply reduce to

$$G_e(0) = \frac{(1+p^2)^{3/2} - p^3 - 2(\sqrt{2}-1)p^2 - 1}{6p^2(1-p^2)} \tag{A.21}$$

$$G_m(0) = \frac{(1+p^2)^{5/2} - p^5 - (2\sqrt{2}-1)(p^3+p^2) - 1}{7.5\,p^2(1-p^2)^2}\,. \tag{A.22}$$

<u>Derivation of ϕ_1 and ϕ_2.</u> The modifying functions ϕ_1 and ϕ_2 , required in the $t - x$ transformations for $x > x_o$, are evaluated by using the von Kármán-Pohlhausen integral method for a step change in constant wall temperature. Substitute solutions given by Eqs. (18) and (19) for T and u into Eqs. (A.1) and (A.2), evaluate, collect terms and rearrange to obtain a set of differential equations as

$$\frac{4}{5}\frac{1}{(\phi_1^2\xi)^{1/4}}\frac{d}{d\xi}\left[(\phi_1^2\xi)^{5/4} H_m\right] = 1 + (\theta_{w2}-1)\gamma \tag{A.23}$$

$$\frac{4}{3}(\phi_1^2\xi)^{1/4}\frac{d}{d\xi}\left[(\phi_1^2\xi)^{3/4} H_e\right] = 1 + \frac{\theta_{w2}-1}{\gamma} \tag{A.24}$$

where we have used Eqs. (A.14) and (A.15), $\xi = x/x_o$ and $\gamma = \sqrt{(1-1/\xi)\phi_2/\phi_1}$. The dimensionless functions H_m and H_e are given by

$$H_m = 1 + (\theta_{w2}-1)^2\gamma^5 + (\theta_{w2}-1)h_m \tag{A.25}$$

$$H_e = 1 + (\theta_{w2}-1)^2\gamma^3 + (\theta_{w2}-1)h_e \tag{A.26}$$

where

$$h_m = \frac{2\gamma^3\int_0^\infty f_u(\gamma\eta)f_u(\eta)\,d\eta}{\int_0^\infty f_u^2(\eta)\,d\eta} \tag{A.27}$$

$$h_e = \frac{\gamma^3\int_0^\infty f_T(\gamma\eta)f_u(\eta)\,d\eta + \int_0^\infty f_T(\eta/\gamma)f_u(\eta)\,d\eta}{\int_0^\infty f_T(\eta)f_u(\eta)\,d\eta}\,. \tag{A.28}$$

After evaluation of integrals for $n = 0$, Eqs. (A.27) and (A.28) become, for $\mathrm{Pr} = 1$,

$$h_m = \frac{4(1+\gamma^2)^{5/2} - 10\gamma^2(1+\gamma^2)^{1/2} - 4\gamma^5 - 4}{3\sqrt{2}-4} \tag{A.29}$$

$$h_e = \frac{(1+\gamma^2)^{3/2} - \gamma^3 - 1}{\sqrt{2}-1} \tag{A.30}$$

or, for $\mathrm{Pr} \neq 1$,

$$\begin{aligned} h_m = {} & \Big[(p^2+\gamma^2)^{5/2} + (1+p^2\gamma^2)^{5/2} \\ & -p^2(1+p)(1+\gamma^2)^{5/2} - (1-p^3)(1-p^2)(1+\gamma^5)\Big] \\ & \div \Big[(1+p^2)^{5/2} - p^5 - (2\sqrt{2}-1)(p^3+p^2) - 1\Big] \end{aligned} \tag{A.31}$$

$$\begin{aligned} h_e = {} & \Big[(p^2+\gamma^2)^{3/2} + (1+p^2\gamma^2)^{3/2} \\ & -2p^2(1+\gamma^2)^{3/2} - (1+p-p^2)(1-p)(1+\gamma^3)\Big] \\ & \div \Big[(1+p^2)^{3/2} - p^3 - 2(\sqrt{2}-1)p^2 - 1\Big]\,. \end{aligned} \tag{A.32}$$

The initial conditions needed to solve the above differential equations can be obtained by considering the conditions at $\xi = 1$ or $x = x_o$. It is clear that at $x = x_o$, the $t - x$ transformations for $x \leq x_o$ and those for $x > x_o$ should be continuous, resulting in $\phi_1 = 1$. Also, $\lim_{\xi \to 1^+} \gamma = 0$.

For simplicity, let $\psi = \phi_1^2\xi$, and expand derivatives in Eqs. (A.23) and (A.24) to obtain

$$H_m\frac{d\psi}{d\xi} + \frac{4}{5}\psi\frac{dH_m}{d\gamma}\frac{d\gamma}{d\xi} = 1 + (\theta_{w2}-1)\gamma \tag{A.33}$$

$$H_e\frac{d\psi}{d\xi} + \frac{4}{3}\psi\frac{dH_e}{d\gamma}\frac{d\gamma}{d\xi} = 1 + \frac{\theta_{w2}-1}{\gamma} \tag{A.34}$$

with $\psi = 1$ and $\gamma = 0$ at $\xi = 1$.

Solving these equations numerically however poses a number of singularity problems at $\xi = 1$. Thus a simple modification to the above equations is made to remedy the problem. A resulting set of equations after multiplying Eq. (A.34) by γ, and rearranging both equations in a manner that eliminates the problem is

$$\frac{d\psi}{d\xi} = \frac{1}{H_m}\left[1 + (\theta_{w2}-1)\gamma - \frac{4}{3}\psi\left(\frac{dH_m}{5\gamma^2\,d\gamma}\right)\frac{d\gamma^3}{d\xi}\right] \tag{A.35}$$

$$\frac{d\gamma^3}{d\xi} = \frac{3}{4}\,\frac{\left(1 - H_e\frac{d\psi}{d\xi}\right)\gamma + \theta_{w2} - 1}{\psi\left(\frac{dH_e}{3\gamma\,d\gamma}\right)} \tag{A.36}$$

with $\psi = 1$ and $\gamma^3 = 0$ at $\xi = 1$.

These equations, where H_m and H_e are given from Eqs. (A.25) through (A.32) represent a set of coupled but computationally simple first order differential equations and they can be easily solved numerically for ψ and γ^3 with fixed values of θ_{w2} and Pr. Upon finding ψ and γ^3, ϕ_1 and ϕ_2 can be subsequently determined by using the definitions of ψ and γ stated earlier.