THERMAL CONSTRICTION RESISTANCE WITH VARIABLE CONDUCTIVITY NEAR THE CONTACT SURFACE

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ABSTRACT

Expressions are developed for the thermal resistance of an elemental heat flow tube with variable thermal conductivity near the contact interface. This allows the application of the theory of conforming rough surfaces to predict the thermal contact resistance at the interface of dissimilar metals where mass diffusion processes have produced a continuous alloy distribution. The results are presented in the form of dimensionless thermal and geometric constriction parameters which are functions of contact geometry, boundary conditions and thermal conductivity distributions. An example application of a copper-nickel diffusion interface is given.

NOMENCLATURE

a	–	contact area radius
A_n	–	series coefficients for temperature solution in the layer
b	–	radius of heat flux tube
B_n	–	series coefficients for temperature solution in the layer
C_1, C_2	–	constants in the expression for the conductivity distribution within the layer
D_0	–	constant for substrate temperature solution
D_n	–	series coefficients for temperature solution in the substrate
E_0, E_1	–	constant for the layer temperature solution
$f(\cdot)$	–	flux distribution over the contact
F_n	–	series coefficients for radial coordinate solution
$I_0(\cdot)$	–	Modified Bessel function of first kind, zero order
$I_1(\cdot)$	–	Modified Bessel function of first kind, first order
$J_0(\cdot)$	–	Bessel function of first kind, zero order
$J_1(\cdot)$	–	Bessel function of first kind, first order
k_1	–	variable thermal conductivity of the layer
k_2	–	thermal conductivity of the substrate
k_{sfc}	–	thermal conductivity at the surface
$K_0(\cdot)$	–	Modified Bessel function of second kind, zero order
$K_1(\cdot)$	–	Modified Bessel function of second kind, first order
q_0	–	uniform heat flux over the contact
Q	–	total heat flow rate into the contact
r, z	–	cylindrical coordinate system
$\hat{r}, \hat{z}$	–	unit vectors in the r and z direction
R_c	–	constriction resistance of the heat flux tube
t	–	layer thickness
T_1	–	temperature distribution in the layer
T_2	–	temperature distribution in the substrate
$\bar{T}_c$	–	average contact temperature
$\bar{T}_{z=0}$	–	average temperature of the $z = 0$ plane

Greek Symbols

β	–	relative layer thickness ($\equiv t/a$)
Γ	–	Gamma function
δ_n	–	roots of $J_1(\delta_n) = 0$, $(\delta_n \equiv \lambda_n b)$
ϵ	–	dimensionless contact size ($\equiv a/b$)
θ_n	–	function defined in Eqs. (46) and (55)
κ	–	conductivity ratio (k_{sfc}/k_2)
λ_n	–	roots for $J_1(\lambda_n b) = 0$
μ	–	parameter to describe flux distribution over the contact
ϕ_n	–	function defined by Eqs. (34) and (54)
ψ	–	dimensionless constriction factor
ψ_q	–	dimensionless constriction factor for uniform flux
ψ_T	–	dimensionless constriction factor for equivalent isothermal flux
ψ_{qc}	–	dimensionless constriction factor for uniform flux (no layer)
ψ_{Tc}	–	dimensionless constriction factor for equivalent isothermal flux (no layer)

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Subscripts

- n – series coefficients
- 1 – region 1
- 2 – region 2
- sfc – surface
- c – circular contact or constriction

INTRODUCTION

Much research has been devoted to examining the temperature drop that occurs at the interface of two contacting solids. Statistical modelling of the contact surface is used to describe the surface topography and predict the contact geometry when two surfaces are held together under a normal load. From the contact geometry an average contact spot area and contact spot density are used to determine the dimensions of a fundamental contact cell or elemental heat flux tube. The thermal constriction resistance of the elemental heat flux tube is then used to predict the overall contact resistance.

Under the assumptions of vacuum and negligible radiation heat flows through discrete contact spots at the interface of rough surfaces as shown in Fig. 1. However if an interstitial fluid is present and if radiation effects are of concern, then heat transfer in these modes can be modelled as acting in parallel with conduction through the contact spots. For many practical problems involving contact resistance conduction through the discrete contact spots is the predominant mode of heat transfer.

Enhancement of heat transfer across the interface of two contacting solids can be achieved by adding a thin conductive layer. Many techniques are used to add the conductive layer such as the application of thermal greases, insertion of soft foils, or metallic coatings on the surfaces. Antonetti and Yovanovich (1985) developed expressions to predict the contact resistance of two conforming rough surfaces with thin metallic coatings. Negus et al. (1985) presented a solution for circular contact on coated half-space. In both cases it was assumed that the layer under the contact is of homogeneous thermal conductivity.

For many applications where heat is transferred across the interface between two different contacting solids the diffusion of one solid into the other will occur over a sufficient period of time. Figure 2 shows the percentage of a hypothetical material A diffused into another material B in a diffusion zone surrounding the contact interface. As the percentage of A decreases from 100% on the left side of the interface to 0% on the right side a continuous alloy distribution is created. Since most different alloys of two materials are known to have dissimilar thermal conductivities, a continuous distribution of the thermal conductivity from some "bulk" value within material A to some "bulk" value within material B must exist. Branecki (1983) investigated this phenomenon for the specific case of a copper-nickel interface and observed a thermal conductivity distribution as shown in Fig. 3. Such a thermal conductivity distribution could be modelled approximately as several layers of homogeneous materials but this becomes impractical for large variations and represents an extremely tedious problem to solve mathematically.

A major improvement is to assume that the conductivity varies linearly from the interface to the substrate. The adoption of a linearly varying region of thermal conductivity should allow good engineering estimates of the actual thermal contact resistance. By using conventional theory, an elemental tube flux with a layer of linearly varying thermal conductivity as shown in Fig. 4 is examined. Flux specified boundary

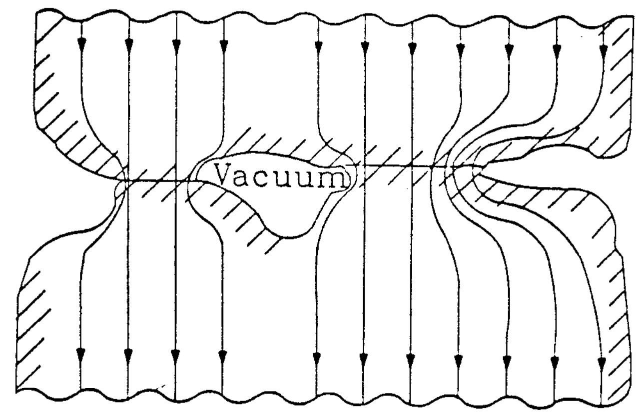

Figure 1: Constriction of Heat Flow Between Contacting Rough Surfaces

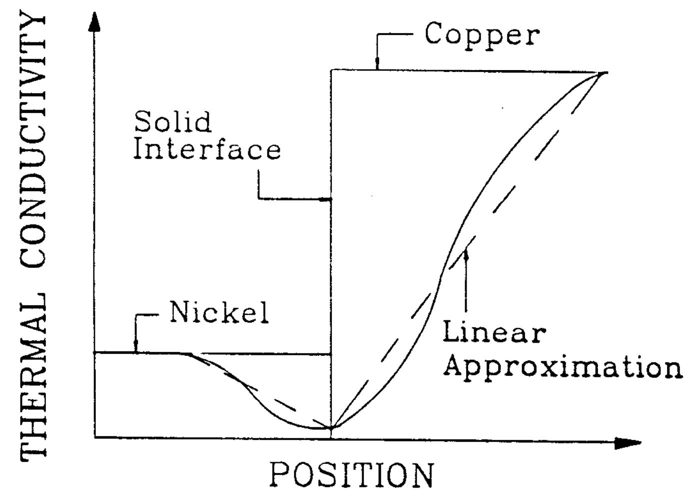

Figure 2: Variation of Thermal Conductivity for a Diffused Nickel-Copper Interface

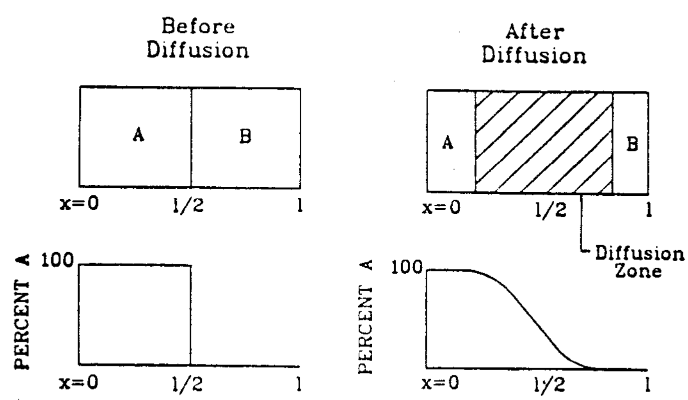

Figure 3: Percentage Distribution of Material B in Material A for a Typical Diffusion Zone

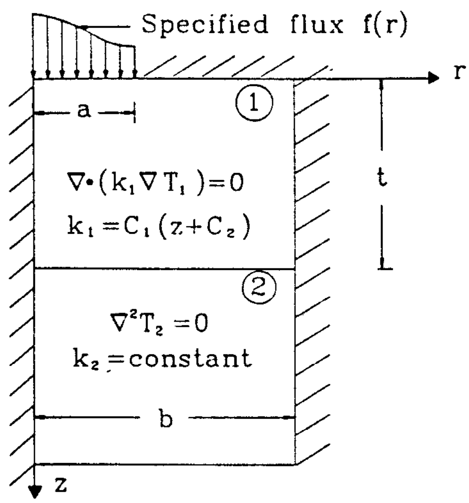

Figure 4: Circular Elemental Heat Flux Tube

conditions are used over the circular contact to avoid solving a mixed boundary value problem. Separability of this problem with spatially varying thermal conductivity has been shown to be possible in the recent work of Negus and Yovanovich (1986). Results are presented for a uniform flux and equivalent isothermal flux to provide upper and lower bounds on the expected resistance.

THEORETICAL DERIVATION

The elemental heat flux tube shown in Fig. 4 contains two regions. Region 1 is a layer of thickness t with a thermal conductivity which varies linearly from a surface value $k_{sfc} = C_1C_2$ to some "bulk" value $k_2 = C_1(t + C_2)$ of the substrate. The second region is a homogeneous substrate with constant thermal conductivity $k_2 = C_1(t + C_2)$.

Under the assumption of steady state conduction, the governing partial differential equations and boundary conditions are

$$\text{div}\cdot(k_1 \text{grad}\, T_1) = 0 \tag{1}$$

$$\nabla^2 T_2 = 0 \tag{2}$$

$$\frac{\partial T_1}{\partial r}(0,z) = \frac{\partial T_2}{\partial r}(0,z) = 0 \qquad z \geq 0 \tag{3}$$

$$\frac{\partial T_1}{\partial r}(b,z) = \frac{\partial T_2}{\partial r}(b,z) = 0 \qquad z \geq 0 \tag{4}$$

$$-k_2\frac{\partial T_2}{\partial z}(r, z \to \infty) = \frac{Q}{\pi b^2} \qquad 0 \leq r \leq b \tag{5}$$

$$k_1\frac{\partial T_1}{\partial z}(r,t) = k_2\frac{\partial T_2}{\partial z}(r,t) \qquad 0 \leq r \leq b \tag{6}$$

$$T_1(r,t) = T_2(r,t) \qquad 0 \leq r \leq b \tag{7}$$

$$-k_{\text{sfc}}\frac{\partial T_1}{\partial z}(r,0) = f(r) \qquad 0 \leq r \leq a \tag{8}$$

$$\frac{\partial T_1}{\partial z}(r,0) = 0 \qquad a < r \leq b \tag{9}$$

where the vector operators are

$$\text{div} \equiv \frac{1}{r}\frac{\partial}{\partial r}\hat{r} + \frac{\partial}{\partial z}\hat{z} \tag{10}$$

$$\text{grad} \equiv \frac{\partial}{\partial r}\hat{r} + \frac{\partial}{\partial z}\hat{z} \tag{11}$$

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} \tag{12}$$

and $f(r)$ represents the heat flux distribution over the circular contact on the top surface of the flux tube.

In region 2 an expression for the temperature distribution is determined by using the separation of variables technique and considering the boundary conditions of Eqs. (3–5) to give

$$T_2(r,z) = D_0 - \frac{Qz}{k_2\pi b^2} + \sum_{n=1}^{\infty} D_n J_0(\lambda_n r)e^{-\lambda_n z} \tag{13}$$

where $\lambda_n b$ are the roots of

$$J_1(\lambda_n b) = 0 \tag{14}$$

and Q is the total heat flow rate in the flux tube.

The governing partial differential equation in region 1 is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) + \frac{\partial^2 T_1}{\partial z^2} + \frac{1}{k_1}\frac{\partial k_1}{\partial z}\frac{\partial T_1}{\partial z} = 0 \tag{15}$$

where $k_1 = C_1(z + C_2)$ is the thermal conductivity of the layer for any axial position z. By assuming that the solution is separable the temperature distribution has the form

$$T_1(r,z) = R(r)\cdot Z(z) \tag{16}$$

and then Eq. (15) becomes

$$\frac{1}{R}\cdot\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{Z}\cdot\left(\frac{d^2Z}{dz^2} + \frac{1}{k_1}\frac{dk_1}{dz}\frac{dZ}{dz}\right) = 0, \pm\lambda^2 \tag{17}$$

Constriction Factor For A Resistive Layer

Before solving Eq. (17) the behaviour of the ordinary differential equation in z must be examined more closely. The term dk_1/dz in Eq. (17) represents the slope of the thermal conductivity distribution in the layer. When the surface conductivity is less than the substrate conductivity the slope is positive and there is a resistive layer on top of the substrate. A negative slope implies a conductive layer on top of the substrate and thus

$$\frac{1}{k_1}\frac{dk_1}{dz} = \frac{1}{z + C_2} \tag{18}$$

has a sign change. The axial coordinate z is always a positive number and therefore the value of C_2 is negative and larger than z for a conductive layer. Due to this sign change, solutions are developed for the separate cases of either a conductive or a resistive layer on the substrate.

For the case where region 1 represents a resistive layer the value of C_2 is positive and therefore the solution is obtained in a straightforward manner as follows. If the constant in Eq. (17) is chosen to be $+\lambda^2$, a trivial solution results. However, both $-\lambda^2$ and zero give non-trivial solutions and the true solution is obtained by superposition of both.

Consider the case where the separation constant is zero. The boundary conditions of Eqs. (3) and (4) require $R(r) =$ constant, but in the z–direction

$$\frac{d}{dz}\left(k_1\frac{dZ}{dz}\right) = 0 \tag{19}$$

$$C_1(z + C_2)\frac{dZ}{dz} = E_0 \tag{20}$$

$$Z(z) = \frac{E_0}{C_1}\ln(z + C_2) + E_1 \tag{21}$$

For the case of the separation constant being $-\lambda^2$, the result is two ordinary differential equations with respect to r and z

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda^2 R = 0 \tag{22}$$

$$\frac{d^2 Z}{dz^2} + \frac{1}{z + C_2}\frac{dZ}{dz} - \lambda^2 Z = 0 \tag{23}$$

After considering the boundary conditions of Eqs. (3) and (4) the solution to Eq. (22) is

$$R(r) = \sum_{n=1}^{\infty} F_n J_0(\lambda_n r) \tag{24}$$

A change of variable is introduced by letting $x = \lambda(z + C_2)$ which results in a Modified Bessel' s Equation with the solution

$$Z(z) = A' K_0(\lambda(z + C_2)) + B' I_0(\lambda(z + C_2)) \tag{25}$$

From the superposition of Eq. (21) and the combination of Eqs. (24) and (25), the temperature distribution in region 1 is given by

$$T_1(r, z) = \frac{E_0}{C_1}\ln(z + C_2) + E_1 + \sum_{n=1}^{\infty}[A_n K_0(\lambda_n(z + C_2)) + B_n I_0(\lambda_n(z + C_2))]\, J_0(\lambda_n r) \tag{26}$$

Before applying the non-homogeneous boundary condition on the $z = 0$ plane, B_n must be eliminated by invoking the perfect contact conditions of Eqs. (6) and (7). The condition of Eq. (7) gives

$$\frac{E_0}{C_1}\ln(t + C_2) + E_1 - D_0 + \frac{Qt}{k_2\pi b^2} + \sum_{n=1}^{\infty}\left[A_n K_0(\lambda_n(t + C_2)) + B_n I_0(\lambda_n(t + C_2)) - D_n e^{-\lambda_n t}\right] J_0(\lambda_n r) = 0 \tag{27}$$

Since $J_0(\lambda_n r)$ is generally non-zero and varying over $0 \le r \le b$ then

$$\frac{E_0}{C_1}\ln(t + C_2) + E_1 - D_0 + \frac{Qt}{k_2\pi b^2} = 0 \tag{28}$$

$$A_n K_0(\lambda_n(t + C_2)) + B_n I_0(\lambda_n(t + C_2)) - D_n e^{-\lambda_n t} = 0 \tag{29}$$

Similarly the condition of Eq. (6) gives

$$\frac{E_0}{C_1(t + C_2)} + \frac{Q}{k_2\pi b^2} + \sum_{n=1}^{\infty}\lambda_n\left[-A_n K_1(\lambda_n(t + C_2)) + B_n I_1(\lambda_n(t + C_2)) + D_n e^{-\lambda_n t}\right] J_0(\lambda_n r) = 0 \tag{30}$$

and again since $J_0(\lambda_n r)$ is generally non-zero and varying over $0 \le r \le b$ then

$$\frac{E_0}{C_1(t + C_2)} + \frac{Q}{k_2\pi b^2} = 0 \tag{31}$$

$$-A_n K_1(\lambda_n(t + C_2)) + B_n I_1(\lambda_n(t + C_2)) + D_n e^{-\lambda_n t} = 0 \tag{32}$$

From the addition of Eqs. (29) and (32):

$$B_n = \phi_n A_n \tag{33}$$

where

$$\phi_n = \frac{K_1(\lambda_n(t + C_2)) - K_0(\lambda_n(t + C_2))}{I_1(\lambda_n(t + C_2)) + I_0(\lambda_n(t + C_2))} \tag{34}$$

and the temperature distribution of region 1 can be expressed as

$$T_1(r, z) = \frac{E_0}{C_1}\ln(z + C_2) + E_1 + \sum_{n=1}^{\infty} A_n\left[K_0(\lambda_n(z + C_2)) + \phi_n I_0(\lambda_n(z + C_2))\right] J_0(\lambda_n r) \tag{35}$$

The temperature distribution of Eq. (35) is substituted in the flux boundary condition of Eqs. (8) and (9), multiplied by $rJ_0(\lambda_m r)dr$ and integrated from $r = 0$ to $r = b$ to give the left hand of Eq. (8) as

$$-E_0\int_0^b rJ_0(\lambda_m r)dr + C_1 C_2\sum_{n=1}^{\infty} A_n\lambda_n[K_1(\lambda_n C_2) - \phi_n I_1(\lambda_n C_2)]\int_0^b rJ_0(\lambda_n r)J_0(\lambda_m r)dr \tag{36}$$

Since $J_1(0) = J_1(\lambda_m b) = 0$ the first term of Eq. (36) simply integrates to zero. By using the orthogonality of Bessel functions the second term is zero for $\lambda_m \neq \lambda_n$ and after substituting the right hand side of Eq. (8) the expression is

$$C_1 C_2 A_n \lambda_n\left[K_1(\lambda_n C_2) - \phi_n I_1(\lambda_n C_2)\right]\int_0^b rJ_0^2(\lambda_n r)dr = \int_0^a rJ_0(\lambda_n r)f(r)dr \tag{37}$$

which gives

$$A_n = \frac{2\int_0^a rJ_0(\lambda_n r)f(r)dr}{C_1 C_2\lambda_n b^2 J_0^2(\lambda_n b)[K_1(\lambda_n C_2) - \phi_n I_1(\lambda_n C_2)]} \tag{38}$$

The thermal resistance for the heat flux tube is given by the definition

$$R_c = \frac{\bar{T}_c - \bar{T}_{z=0}}{Q} \tag{39}$$

where Q is the heat flow crossing the top surface, $\bar{T}_c$, the average temperature over the contact spot and, $\bar{T}_{z=0}$, the average temperature over the top of the flux tube. A dimensionless constriction factor is defined as

$$\psi = 4k_{\mathrm{sfc}}aR_c \tag{40}$$

The average contact temperature is

$$\bar{T}_c = \frac{1}{\pi a^2}\int_0^a 2\pi r T_1(r, 0)dr \tag{41}$$

or after substituting for $T_1(r, 0)$ and integrating

$$\bar{T}_c = \frac{E_0}{C_1}\ln(C_2) + E_1 + \sum_{n=1}^{\infty} A_n\left[K_0(\lambda_n C_2) + \phi_n I_0(\lambda_n C_2)\right]\frac{2J_1(\lambda_n a)}{\lambda_n a} \tag{42}$$

The average temperature over the top surface of the flux tube is

$$\bar{T}_{z=0} = \frac{1}{\pi b^2}\int_0^b 2\pi r T_1(r, 0)dr \tag{43}$$

or after substituting for $T_1(r, 0)$ and integrating

$$\bar{T}_{z=0} = \frac{E_0}{C_1}\ln(C_2) + E_1 \tag{44}$$

Thus a general expression for the constriction factor after substituting Eqs. (39), (42), and (44) into Eq. (40) is given by

$$\psi = \frac{16}{Q}\sum_{n=1}^{\infty}\frac{\theta_n J_1(\lambda_n a)\int_0^a rJ_0(\lambda_n r)f(r)dr}{(\lambda_n b)^2 J_0^2(\lambda_n b)} \tag{45}$$

where

$$\theta_n = \frac{K_0(\lambda_n C_2) + \phi_n I_0(\lambda_n C_2)}{K_1(\lambda_n C_2) - \phi_n I_1(\lambda_n C_2)} \tag{46}$$

$$Q = \int_0^b 2\pi r f(r)dr \tag{47}$$

Constriction Factor For A Conductive Layer

As mentioned previously a different solution is obtained for the resistive or conductive layer on top of the substrate. For the conductive layer case, the ordinary differential equation with respect to r does not change but the differential equation with respect to z changes because the slope of the conductivity distribution k_1 becomes negative. The quantity $z + C_2$ will become negative for the conductive layer case because C_2 is a negative quantity with a magnitude greater than z in region 1. Since the logarithmic and Bessel functions of Eq. (35) are only valid for positive arguments then a different solution must exist for the case of a conductive layer on the substrate .

Consider the differential equation for the case of the separation constant being equal to zero. By starting with Eq. (20), a new solution is derived as follows:

$$\frac{dZ}{dz} = \frac{-E_0}{C_1(-z - C_2)} \tag{48}$$

$$\int dZ = \int \frac{E_0 d(-z)}{C_1(-z - C_2)} \tag{49}$$

$$Z = \frac{E_0}{C_1}\ln(-(z + C_2)) + E_1 \tag{50}$$

The next case to consider is when the separation is equal to $-\lambda^2$. An approach similar to the one for a zero separation constant is used. A first step is to use a change of variable by letting $v = -z$ and using the fact that

$$\frac{dZ}{dz} = \frac{-dZ}{dv}, \quad \frac{d^2Z}{dz^2} = \frac{d^2Z}{dv^2} \tag{51}$$

then Eq. (23) is changed to

$$\frac{d^2Z}{dv^2} + \frac{1}{v - C_2}\frac{dZ}{dv} - \lambda^2 Z = 0 \tag{52}$$

The solution used previously is applied except that for this case $x = \lambda(z+C_2)$ is replaced by $x = \lambda(v-C_2)$ which results in the same Modified Bessel' s Equation and solution as before . After substituting the expressions for x and v and superposing the solutions as above, the temperature distribution in region 1 becomes

$$T_1(r,z) = \frac{E_0}{C_1}\ln(-(z + C_2)) + E_1$$

$$\sum_{n=1}^{\infty} A_n\left[K_0(-\lambda_n(z + C_2)) + \phi_n I_0(-\lambda_n(z + C_2))\right] J_0(\lambda_n r) \tag{53}$$

but for the conductive layer case

$$\phi_n = \frac{K_1(-\lambda_n(t + C_2)) + K_0(-\lambda_n(t + C_2))}{I_1(-\lambda_n(t + C_2)) - I_0(-\lambda_n(t + C_2))} \tag{54}$$

An expression for the constriction factor is developed using the procedure discussed earlier. The expression for the conductive layer case is the same as the resistive layer case given by Eq. (45) except

$$\theta_n = \frac{K_0(-\lambda_n C_2) + \phi_n I_0(-\lambda_n C_2)}{\phi_n I_1(-\lambda_n C_2) - K_1(-\lambda_n C_2)} \tag{55}$$

Constriction Factor for $f(r) = (1 - (r/a)^2)^\mu$

Another general expression for the constriction factor ψ defined by Eq. (45) is obtained for the case of

$$f(r) = q_0(1 - (\frac{r}{a})^2)^\mu \quad \mu \geq -1 \tag{56}$$

Yovanovich (1975) presented an expression for the constriction factor of an unlayered flux tube with the same flux distribution given by Eq. (56) which has the same integrals as Eq. (45). By using the integrals evaluated by Yovanovich (1975) the constriction factor for the flux distribution in Eq. (56) is given by

$$\psi = \frac{16}{\pi\epsilon}(\mu + 1)2^\mu\Gamma(\mu + 1)\sum_{n=1}^{\infty}\frac{\theta_n J_1(\delta_n\epsilon)J_{\mu+1}(\delta_n\epsilon)}{\delta_n^3\delta_n^\mu J_0^2(\delta_n)} \tag{57}$$

where $\Gamma(\mu + 1)$ is the Gamma function, ϵ, the dimensionless contact size a/b, $\delta_n = \lambda_n b$, the roots of $J_1(\delta_n) = 0$, and $J_{\mu+1}(\delta_n\epsilon)$ the Bessel function of the first kind of order $\mu + 1$.

Two particular cases are of interest because these provide the upper and lower bounds of the constriction factor. If μ is chosen to be zero then there will be a uniform flux applied over the contact and the expression for the constriction factor is

$$\psi_q = \frac{16}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{\theta_n J_1^2(\delta_n\epsilon)}{\delta_n^3 J_0^2(\delta_n)} \tag{58}$$

By letting $\mu = -1/2$, the special case of an equivalent isothermal flux applied over the contact exists which is the exact flux distribution for an isothermal contact on a homogeneous half-space. This flux distribution is used to approximate an isothermal boundary condition over the contact to avoid the solution of a mixed boundary problem. The expression for the constriction factor when $\mu = -1/2$ is

$$\psi_T = \frac{8}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{\theta_n J_1(\delta_n\epsilon)\sin(\delta_n\epsilon)}{\delta_n^3 J_0^2(\delta_n)} \tag{59}$$

PRESENTATION AND DISCUSSION OF RESULTS

The dimensionless constriction factors ψ_q or ψ_T are a function of six independent parameters

$$\psi = \psi(a, b, t, k_{\text{sfc}}, k_2, \mu) \tag{60}$$

Three dimensionless parameters are defined as

$$\epsilon = a/b \tag{61}$$

$$\beta = t/a \tag{62}$$

$$\kappa = k_{\text{sfc}}/k_2 \tag{63}$$

to reduce the number of independent parameters from six to four. In the presentation of the results, a value of $\epsilon = 0.1$ is chosen while the conductivity ratio, the relative layer thickness, and the contact boundary conditions are varied.

The expressions for the constriction factors in Eqs. (58) and (59) have only been partially non-dimensionalized because the arguments of the expressions for θ_n and subsequently ϕ_n are still in dimensional form. By using the dimensionless parameters defined in Eqs (61–63) the arguments of the Modified Bessel's functions are given in dimensionless form as

$$\lambda_n(t + C_2) = \frac{\delta_n \beta \epsilon}{1-\kappa} \quad (64)$$

$$\lambda_n C_2 = \frac{\delta_n \beta \epsilon \kappa}{1-\kappa} \quad (65)$$

for the case of a resistive layer, and for the case of a conductive layer,

$$-\lambda_n(t + C_2) = \frac{\delta_n \beta \epsilon}{\kappa - 1} \quad (66)$$

$$-\lambda_n C_2 = \frac{\delta_n \beta \epsilon \kappa}{\kappa - 1} \quad (67)$$

In order to evaluate the effect of the layer, the relative constriction factors ψ_q/ψ_{qc} and ψ_T/ψ_{Tc} are plotted against the relative layer thickness β. The constriction factors ψ_{qc} and ψ_{Tc} represent the constriction factor with no layer when there is a uniform flux and equivalent isothermal flux applied over the contact. The numerical results for the plots were computed quickly and efficiently using a program written with Microsoft Fortran on an IBM-PC. Results for the uniform flux case are presented in Fig. 5 and the results for the equivalent isothermal flux are presented in Fig. 6.

From Figs. 5 and 6, it is obvious that the constriction factor decreases for a resistive layer while the opposite effect occurs for a conductive layer. For thin layers, as $\kappa \to \infty$ the constriction factor becomes infinite because the substrate appears as an insulated boundary condition for a conductive layer. The constriction factor decreases for $\kappa < 1$ because the heat does not spread out as much as in the unlayered flux tube. Therefore the effective heat path length is reduced which causes a reduction in the constriction factor. Even though the constriction factor decreases when a resistive layer is added to the flux tube, the thermal resistance R_c increases as expected. This can be easily seen from the definition of the constriction factor in Eq. (40). For a fixed geometry, as the surface conductivity is decreased the constriction factors decrease, but not as quickly as the surface conductivity. Since a decrease in the surface conductivity is more significant than the decrease in the constriction factor, then the thermal resistance R_c must increase.

The results presented in Figs. 5 and 6 are for two different flux specified boundary conditions over the contact, yet the plots show the same trends. To investigate these results further the percentage difference between the relative constriction factor ψ_q/ψ_{qc} and ψ_T/ψ_{Tc} are presented in Table 1 for various values of κ and β. The maximum percentage difference is 3.6% for $\kappa = 100$ and $\beta = 0.1$. With such a small error the conclusion is that boundary conditions have little effect on the primary effect of adding a layer to the flux tube. The expression for the constriction factor ψ_T in Eq. (59) requires thousands of terms to obtain a converged result for $\epsilon \ll 0.1$. The values of ψ_{Tc} and ψ_{qc} are easily computed using the correlations presented by Negus and Yovanovich (1985). Since the value of ψ_q requires typically 200 terms, an approximate value of ψ_T is quickly evaluated by computing ψ_{Tc} from a simple correlation and multiplying by the ratio ψ_q/ψ_{qc}.

A hypothetical problem is presented to show the effect of a resistive layer on the constriction factor. The case of a copper-nickel diffusion zone is analyzed where the surface of the contact is assumed to be 20% nickel and 80% copper. From the Metals Handbook (1979), the thermal conductivity of the surface is 36 $W/(m \cdot K)$, the conductivity of pure copper is 391 $W/(m \cdot K)$ and the conductivity of pure nickel is 90 $W/(m \cdot K)$. This gives a conductivity ratio $\kappa = 0.092$ for

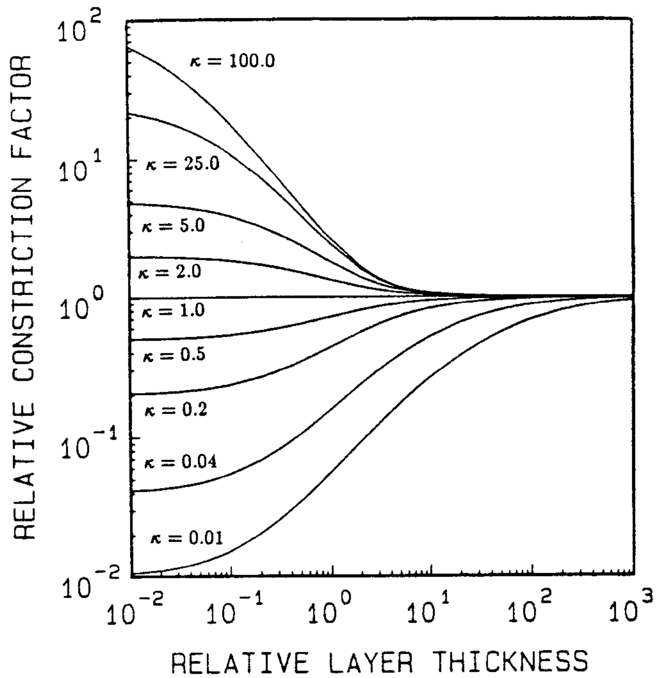

Figure 5: Relative Constriction Factor vs. Relative Layer Thickness for Uniform Flux Over the Contact and $\epsilon = 0.1$

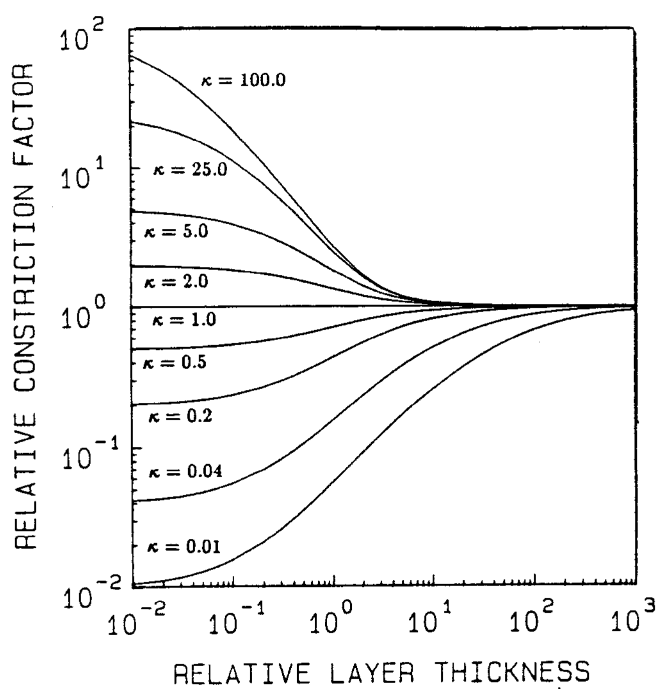

Figure 6: Relative Constriction Factor vs. Relative Layer Thickness for Equivalent Isothermal Flux Over the Contact and $\epsilon = 0.1$

κ	β	ψ_T/ψ_{Tc}	ψ_q/ψ_{qc}	Percentage Difference
0.01	0.01	0.0106	0.0105	0.3
	0.10	0.0156	0.0154	1.4
	1.00	0.0564	0.0567	-0.5
	10.00	0.2586	0.2642	-2.2
	100.00	0.6827	0.6913	-1.3
	1000.00	0.9437	0.9459	-0.2
0.04	0.01	0.0416	0.0415	0.3
	0.10	0.0550	0.0546	0.7
	1.00	0.1554	0.1578	-1.5
	10.00	0.5122	0.5216	-1.8
	100.00	0.8803	0.8847	-0.5
	1000.00	0.9851	0.9857	-0.1
25.00	0.01	21.7284	21.6610	0.3
	0.10	11.3402	11.0676	2.4
	1.00	2.4820	2.4059	3.1
	10.00	1.0716	1.0684	0.3
	100.00	1.0062	1.0059	0.0
	1000.00	1.0006	1.0006	0.0
100.00	0.01	65.7028	64.7848	1.4
	0.10	19.0476	18.3760	3.5
	1.00	2.7219	2.6302	3.4
	10.00	1.0744	1.0710	0.3
	100.00	1.0064	1.0061	0.0
	1000.00	1.0006	1.0006	0.0

Table 1: Percentage Difference Between ψ_T/ψ_{Tc} and ψ_q/ψ_{qc} for $\epsilon = 0.1$

the copper side of the interface and $\kappa = 0.40$ on the nickel side. A typical interface was investigated by Branecki (1983) where the diffusion zone extended approximately 100 μm into the nickel side and approximately 300 μm into the copper side. For a typical contact radius of 5 μm the relative layer thickness for the copper side is $\beta = 60$ and for the nickel side $\beta = 20$. The relative contact size is rarely larger than 0.3 therefore a value of 0.1 is assumed which gives constriction factors for the unlayered case of $\psi_{qc} = 0.9401$ and $\psi_{Tc} = 0.8549$. The constriction factors for the nickel side are $\psi_q = 0.9006$ and $\psi_T = 0.8217$ and the relative constriction factors are $\psi_q/\psi_{qc} = 0.9580$ and $\psi_T/\psi_{Tc} = 0.9561$. For the copper side of the interface, $\psi_q = 0.8608$ and $\psi_T = 0.7840$ which gives values of the relative constriction factors of $\psi_q/\psi_{qc} = 0.9156$ and $\psi_T/\psi_{Tc} = 0.9123$. In this case of a relatively thick layer, there is only a 5–10% reduction in the constriction factor even though the conductivity ratios are quite small. If a large enough diffusion zone exists it has little effect on the constriction factor even for large differences in the surface and substrate conductivities.

However, if the diffusion zone is very thin, there is a significant effect on the constriction factor. Consider a case similar to the previous one except that the relative layer thickness is not as large as before because the time allowed for the diffusion process has been shortened significantly. For example, consider a problem where the diffusion time is shortened to give a diffusion zone on the nickel side of only 5 μm or $\beta = 1$ and 14 μm or $\beta = 2.8$ on the copper side. The conductivity ratios do not change because it is assumed that there is an alloy of 20% nickel and 80% copper at the interface and a value of $\epsilon = 0.1$ is still assumed. After carrying out the same computations as before except for a different β, the constriction factors for the nickel side of the interface are $\psi_q = 0.6077$ and $\psi_T = 0.5495$ and for the copper side $\psi_q = 0.4205$ and $\psi_T = 0.3769$. The relative constriction factors for the nickel side of the interface are now $\psi_q/\psi_{qc} = 0.6464$ and $\psi_T/\psi_{Tc} = 0.6394$ and for the copper side $\psi_q/\psi_{qc} = 0.4473$ and $\psi_T/\psi_{Tc} = 0.4486$. From the relative constriction factors given above it is obvious that there is a significant drop in the constriction factor for thin resistive diffusion zones but an increase in dimensional thermal resistance.

SUMMARY AND CONCLUSIONS

A general expression for the thermal constriction factor of a circular flux tube with a layer of linearly varying thermal conductivity has been determined for a circular contact with flux specified boundary conditions by using the separation of variables technique. Another general expression was developed for a heat flux distribution $q = (1 - (r/a)^2)^\mu$ over the contact. Expressions for the uniform flux ($\mu = 0$) and equivalent isothermal flux ($\mu = -1/2$) boundary conditions were presented.

The values of the constriction factor could be obtained by using a finite difference or finite elements method but it would be very costly to perform a parametric study. By using analytical techniques, values for constriction factors can be obtained quickly and efficiently even on a microcomputer. Thus the effects of contact geometry, boundary conditions and the thermal conductivity ratio can be studied effectively by the thermal designer.

A layer created by the diffusion process can raise or lower the constriction factor depending on whether it is a conductive or resistive layer on top of the flux tube. By having a resistive layer on top of the flux tube, the constriction factor decreases and the opposite effect occurs for a conductive layer. For a given conductivity ratio, a thick layer will raise or lower the constriction factor, but the effect is small compared to a thin layer, which causes a substantial increase or decrease in the constriction factor. A very thick layer causes the constriction factor to approach the value for an unlayered flux tube. The relative effect of adding a layer with linearly varying thermal conductivity is independent of the flux distribution applied over the contact.

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REFERENCES

Abramowitz, M., and Stegun, I., 1965, *Handbook of Mathematical Functions*, Dover, New York.

American Society for Metals, 1979, *Metals Handbook Ninth Edition Volume 2 Properties and Selections: Nonferrous Alloys and Pure Metals*, American Society for Metals, Metals Park, Ohio.

Antonetti, V.W. and Yovanovich, M.M., 1985, "Enhancement of Thermal Contact Conductance by Metallic Coatings: Theory and Experiment", *Journal of Heat Transfer*, Vol. 107, pp.513–519.

Branecki, M., 1983, Personal communications, Department of Mechanical Engineering, Waterloo.

Cooper, M., Mikic, B.B., and Yovanovich, M.M., 1969, " Thermal Contact Conductance", *International Journal of Heat and Mass Transfer* , Vol.12, pp.279–300.

Gradshteyn, I.S., and Ryzhik, I.M., 1965, *Tables of Integrals, Series and Products*, Academic Press, New York.

Negus, K.J., Yovanovich, M.M., 1984, " Application of the Method of Optimized Images to Steady Three-Dimensional Conduction Problems", ASME Paper , No. 84-WA/HT110 ASME Winter Annual Meeting, New Orleans, LA, Dec. 9–13.

Negus, K.J, Yovanovich, M.M., and Thompson, J.C., 1985, " Thermal Constriction Resistance of Circular Contacts on Coated Surfaces: Effect of Contact Boundary Conditions", AIAA Paper No. 85-1014, AIAA 20th Thermophysics Conference, Williamsburg, VA, June 19–21.

Negus, K.J, and Yovanovich, M.M., 1986, " Simple Separability for Steady Heat Conduction with Spatially-Varying Thermal Conductivity", accepted by the *International Journal of Heat and Mass Transfer.*

Ralls, K.M., Courtney, T.H., and Wulff, J., 1976, *Introduction to Materials Science and Engineering*, John Wiley and Sons, New York.

Yovanovich, M.M., 1975, "General Expressions for Circular Constriction Resistances for Arbitrary Flux Distributions", AIAA 13th Aerospace Sciences Meeting, Pasadena, CA, Jan. 20–22.

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