

Shape factors for hollow prismatic cylinders bounded by isothermal inner circles and outer regular polygons

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INTRODUCTION

PRISMATIC cylinders with inner circular concentric holes find use in many process industries. For heat transfer purposes it is of interest to know the heat flow rate through the walls of these cylinders. Several papers have been published on the determination of shape factors for this problem [1-4]. However, most of these are valid only for cylinders with holes of small radii. Smith *et al.* [1] correlated electric analogue results for a hollow square cylinder. Their correlation is valid for cylinders of small radius/apothem ratio and is useful up to a ratio of about 0.80. Other workers have developed approximate analytical solutions, notably, Balcerzak and Raynor [2], who based their solution on approximate mapping and point matching on the outer boundary. Laura and Susemihl [3] developed a similar solution using conformal mapping. Both methods give nearly identical results which are accurate only up to a radius/apothem ratio of about 0.8 for the case of a hollow square cylinder. Dugan [4] developed a much more accurate solution using the boundary residual technique. This solution, however, requires the solution of simultaneous equations whose number must be increased as the inner hole radius increases to improve accuracy. Consequently, the method requires a computer to solve the equations.

In this paper an approximate analytical expression for

evaluating the conduction shape factors of hollow prismatic cylinders is derived. The method is based on the 'parallel flux tube' heat flow model developed in circular cylinder coordinates. The equation is valid for cylinders having both small and large inner holes. The accuracy of the results improves with increasing hole size. Therefore, the expression gives accurate results for cylinders with large holes; this is the region where existing simple solutions are inaccurate or fail.

PROBLEM STATEMENT

Consider long prismatic cylinders of uniform thermal conductivity and a concentric circular hole as shown in Fig. 1. It is required to determine the heat transfer rate through the walls of the cylinders when the boundaries are maintained at uniform temperature T_i and T_o . Steady state heat transfer rate per unit length, Q can be expressed as

$$Q = kS(T_i - T_o) = \frac{(T_i - T_o)}{R} \quad (1)$$

where

$$S = \frac{1}{kR}$$

is the conduction shape factor which depends on geometry

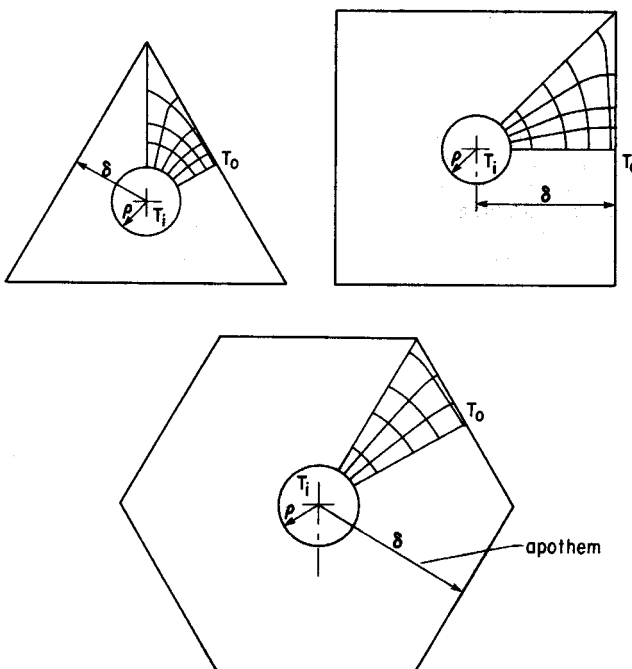


FIG. 1. Temperature distribution and flow lines in long hollow regular prismatic cylinders.

NOMENCLATURE

A $(\ln \delta/\rho)^{1/2}$
 B $1/\sqrt{2}$
 F a function as defined in the text
 k thermal conductivity
 N number of sides of polygon
 Q heat flow rate per unit length of cylinder
 R thermal resistance
 r radial coordinate
 R_n thermal resistance of a flux tube
 R_L lower bound of the thermal resistance

R_u upper bound of the thermal resistance
 S conduction shape factor per unit length $(1/kR)$
 S_L lower bound of the conduction shape factor
 T temperature.

Greek symbols

ρ radius of the inner circular hole
 δ the apothem—defined as the perpendicular distance from the centre to a side of the polygon
 θ angular coordinate.

only, R is the thermal resistance and k is the thermal conductivity. Knowledge of the shape factor would therefore allow one to compute the heat transfer rate easily.

SOLUTION

The conduction shape factor for long prismatic hollow cylinders can be expressed in the general form [1-3]

$$S = \frac{2\pi}{\ln(\delta/\rho) \pm F} \quad (2)$$

where F is a function of the number of sides of the polygon. This equation, however, gives poor results, for cylinders with large diameter holes, when F is determined by the current approximate methods [1-3]. A better approximate analytic solution is developed below which results in a different, but much more accurate form of the equation for determining this conduction shape factor.

APPROXIMATE ANALYTICAL SOLUTION

The resistance of an element of an isotropic homogeneous material to uniform one-dimensional heat flow can be expressed as

$$\text{Resistance} = \frac{\text{Flow distance}}{(\text{Flow area} \times \text{thermal conductivity})} \quad (3)$$

When this definition is applied to cylinders in which heat is considered to flow in the radial direction only, an element of the form shown in Fig. 2(a) is used. The elemental heat flow distance is dr and the elemental flow area is $r d\theta$ per unit length. The resistance of the element to heat flow can then be expressed as

$$dR = \frac{dr}{kr d\theta} \quad (4)$$

A flux tube consists of the elements added in series radially (Fig. 2(b)). The resistance of the flux tube is

$$dR_n = \int_{r_1}^{r_2} \frac{dr}{kr d\theta} \quad (5)$$

where r_1 and r_2 are the inner and outer radii of the flux tube. This equation can be rewritten as

$$dR_n = \frac{1}{k d\theta} \int_{r_1}^{r_2} \frac{dr}{r} \quad (6)$$

The flux tubes are arranged in parallel by integrating in the angular direction. This gives the total resistance of a sector (Fig. 2(c)) as

$$\frac{1}{kR_u} = \int_{\theta_1}^{\theta_2} \frac{d\theta}{\int_{r_1}^{r_2} dr/r} \quad (7)$$

where θ_1 and θ_2 are the angular limits of the sector [5]. The

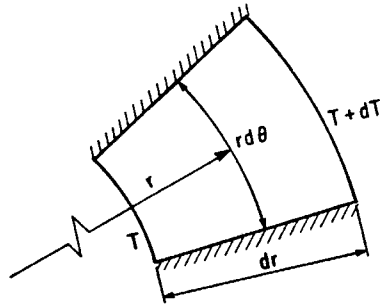


FIG. 2(a). Element in cylindrical coordinates.

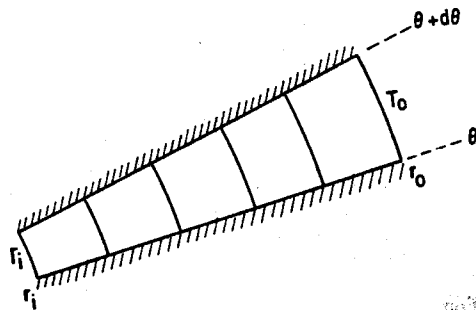


FIG. 2(b). Flux tube.

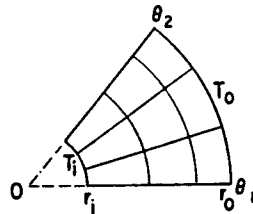


FIG. 2(c). Parallel flux tubes.

resistance R_u in equation (7), as shown in ref. [7], is the upper bound on the resistance in a circular cylinder system [6]. In terms of the conduction shape factor it is written as

$$S_L = \int_{\theta_1}^{\theta_2} \frac{d\theta}{\int_{r_1}^{r_2} dr/r} \quad (8)$$

where S_L is the lower bound on the shape factor of the sector. The lower bound on the resistance can be found by integrating the elements in the angular direction first and then radially in series [6, 7]. The result is given by

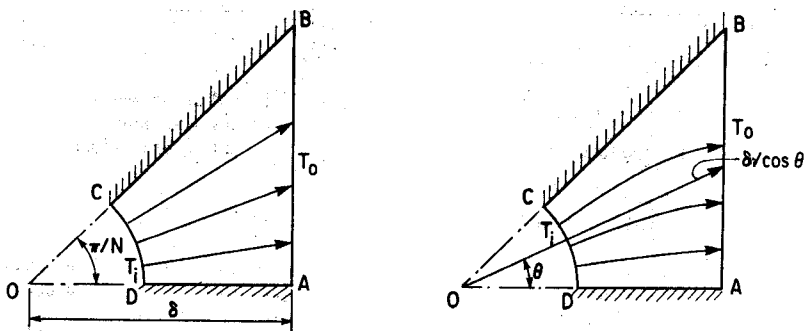


FIG. 3. Basic cell.

$$kR_L = \int_{r_1}^{r_2} \frac{dr}{\int_{\theta_1}^{\theta_2} r d\theta} \quad (9)$$

when r is dependent upon θ . This is called the 'parallel isotherm model'. In some simple shapes, such as an infinite plate or an infinite circular cylinder the two models, equations (7) and (9), yield the same resistance. For complicated shapes, however, the results of these two models differ [7]. There are cases where neither model can be evaluated analytically. When they can be evaluated, the two resistances bound the actual resistance. The arithmetic mean of the two resistance values has been shown to be a good estimate of the actual resistance [6, 7].

It is possible to extend the use of the above models to cases where only one can be evaluated. This is done if the actual heat flow pattern nearly corresponds to that model. The value of the resistance from the model will be nearly equal to the actual resistance. The derivation in this paper uses this concept and only the upper bound on the resistance is evaluated. It is not possible to evaluate the lower bound of the resistance of hollow prismatic cylinders because the outer boundary is not compatible with the radial flow model.

To apply equation (8) to regular hollow prismatic cylinders a basic cell (see Fig. 3) is defined. The element ABCD is common to all regular hollow prismatic cylinders. Applying equation (8) to this cell gives

$$S_L = \int_0^{\pi/N} d\theta \int_0^{\delta/\cos\theta} \frac{dr}{r} \quad (10)$$

where zero and π/N are the angular limits, ρ and $\delta/\cos\theta$ the radii and N is the number of sides of the polygon. The complete cylinder cross-section contains $2N$ basic cells. Therefore, the lower bound on the shape factor for the hollow prismatic cylinder after carrying out the integration in the denominator is

$$S_L = 2N \int_0^{\pi/N} \frac{d\theta}{\ln(\delta/\rho) - \ln \cos \theta} \quad (11)$$

The integration of the above equation is not easy, it can however, be carried out if $-\ln(\cos\theta)$ is replaced by $1/2 \sin^2(\theta/2)$. This is a combination of the first and second terms in its trigonometric series [8]. The equation is then

$$S = 2N \int_0^{\pi/N} \frac{d\theta}{\ln(\delta/\rho) + 1/2 \sin^2(\theta/2)} \quad (12)$$

which can be integrated; therefore, the shape factor is given as

$$S = 2N \left[\frac{1}{A\sqrt{A^2+B^2}} \tan^{-1} \left\{ \frac{\sqrt{A^2+B^2}}{A} \tan \left(\frac{\pi}{N} \right) \right\} \right] \quad (13)$$

where $A^2 = \ln(\delta/\rho)$ and $B^2 = 1/2$. The subscript L has been

dropped because equations (12) and (13) give values of the shape factor which are slightly greater than the lower bound values because of the substitution.

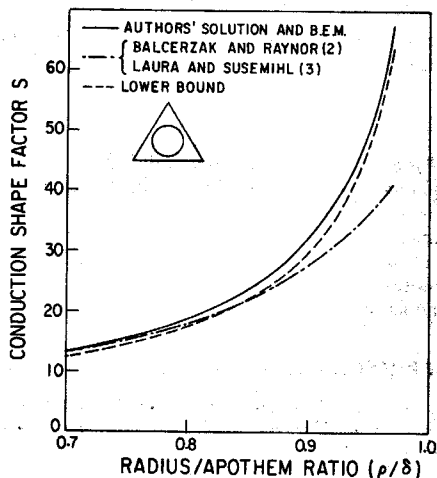


FIG. 4. Variation of conduction shape factor with radius/apothem ratio for hollow triangular cylinders.

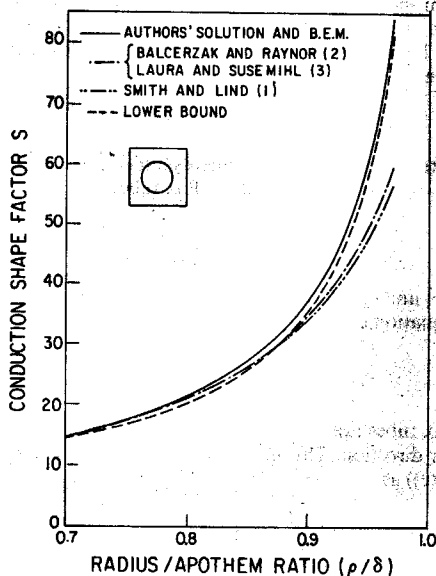


FIG. 5. Variation of conduction shape factor with radius/apothem ratio for hollow square cylinders.

COMPARISON WITH OTHER WORKS

The variation of the shape factor as a function of the radius/apothem ratio for ratios above 0.7 is given in Figs. 4 and 5 for a hollow triangular cylinder and hollow square cylinder, respectively. The results from the various works are in good agreement for radius/apothem ratios below 0.7; therefore this region is not plotted. For ratios above 0.8 all the previous solutions [1-3] give lower values of the shape factor than equation (13). The results were checked against values obtained by the numerical boundary element method (BEM) and the finite element method (FEM). The two numerical methods gave nearly identical results. However, the FEM requires post-processing to determine the temperature gradients at the boundaries while the BEM gives these gradients directly, therefore, only BEM results were used for further comparison in the rest of the work since they were easier to obtain. In both Figs. 4 and 5 the numerical results are plotted on the same curve as those of equation (13), showing the accuracy of the derived equation. The plots of equation (13) were also compared with those plotted by Dugan [4] and were seen to be comparable, even for cylinders with large inner holes.

The lower bound from equation (11), evaluated numerically, is also plotted on both Figs. 4 and 5. The results of equation (13) are always above this lower bound showing that equation (13) yields shape factor values slightly above the lower bound.

A comparison of the percentage differences between the shape factors from the other methods and those from the BEM for the hollow square cylinder are shown in Fig. 6. The results from refs. [1-3] are seen to be good up to a radius/apothem ratio of 0.8 after which their error increases rapidly. For very small ($\rho/\delta < 0.1$) and very large holes ($\rho/\delta > 0.97$), the BEM results become less accurate and cannot be used as a datum. The results from refs. [2, 3] and equation (13) are more accurate for very small holes, while equation (13) has small error for radius/apothem ratio higher than 0.97 and its accuracy should improve as the ratio approaches one. The argument for this is given in the discussion below.

It is hence seen that equation (13) can be used to determine the conduction shape factor for hollow prismatic cylinders of any radius/apothem ratio between zero and one.

For ratios below 0.8 the approximate equations of Balcerzak and Raynor [2], Smith *et al.* [1] or Laura and Susemihl [3], which are summarized in Table 1, can also be used.

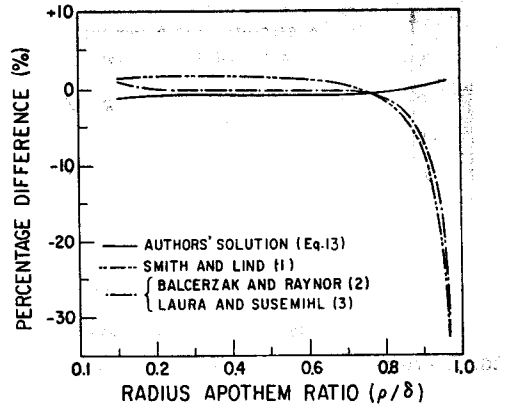


FIG. 6. Percentage difference of approximate solutions compared to the numerical BEM solution for hollow square cylinders.

DISCUSSION

The approximate lower bound of the shape factor from equation (13) gives good results because of three facts.

(1) The heat flow pattern from the hole to the outside surface is close to the parallel flux tube model, even for the critical case of a triangular cylinder.

(2) For cylinders with large holes most of the heat goes out through the region where the angle θ is small. The value of $-\ln \cos \theta$ is nearly equal to $1/2 \sin^2 \theta/2$ for small angle θ , therefore this substitution does not introduce much error. As the hole gets larger this region becomes more important and the accuracy of equation (13) therefore increases.

(3) Although the heat flow at the inner circular boundary is spread over a large angle θ when the hole is very small the term $1/2 \sin^2 \theta/2$ becomes very small compared to $\ln \delta/\rho$ and hence the dependence on θ is negligible and the solution tends to that of a hollow circular cylinder. The flux tubes around the hole are essentially parallel and the solution is not influenced very much by the outside boundary.

The results of equation (13) and those from refs. [2, 3] should improve in accuracy as the number of sides of the polygon increases. This is shown in Fig. 7 where the percentage difference between the previous approximate sol-

Table 1. Summary of approximate equations for determining conduction shape factors of hollow prismatic cylinders

Authors	Shape factor S	Parameters												
Smith <i>et al.</i> [1]	$S = \frac{c}{\log(\delta/\rho) + d}$	$c = 2.79 \pm 0.010$ $d = 0.036 \pm 0.007$												
Balcerzak and Raynor [2]	$S = \frac{2\pi}{\ln\{\delta/\rho \cos(\pi/N)\} - A}$	$N =$ number of sides of prism <table border="1"> <tr><th>N</th><th>A</th></tr> <tr><td>3</td><td>0.56958</td></tr> <tr><td>4</td><td>0.27079</td></tr> <tr><td>5</td><td>0.16068</td></tr> <tr><td>6</td><td>0.10669</td></tr> <tr><td>etc.</td><td></td></tr> </table>	N	A	3	0.56958	4	0.27079	5	0.16068	6	0.10669	etc.	
N	A													
3	0.56958													
4	0.27079													
5	0.16068													
6	0.10669													
etc.														
Laura and Susemihl [3]	$S = \frac{2\pi}{\ln(A_s \delta/\rho)}$	N A_s <table border="1"> <tr><td>3</td><td>1.13209</td></tr> <tr><td>4</td><td>1.07870</td></tr> <tr><td>5</td><td>1.05246</td></tr> <tr><td>6</td><td>1.03754</td></tr> <tr><td>etc.</td><td></td></tr> </table>	3	1.13209	4	1.07870	5	1.05246	6	1.03754	etc.			
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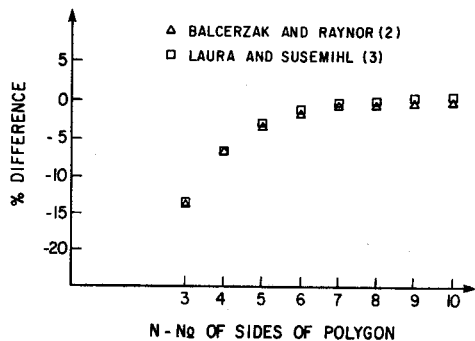


FIG. 7. Percentage difference between equation (13) and previous solutions [2, 3].

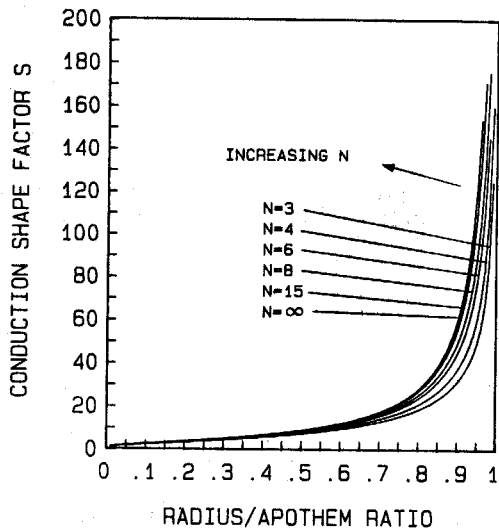


FIG. 8. Conduction shape factors for prisms with different number of sides.

utions and equation (13) is shown to decrease as the number of sides of the cylinder increases for a radius/apothem ratio of 0.9

Figure 8 shows the plots of conduction shape factor as a function of various radius/apothem ratios for prisms with a different number of sides. The results are generated using equation (13). It can be seen that for radius/apothem ratios below 0.5 the conduction shape factors for all the prisms are nearly equal to that of a circular cylinder. For higher radius/apothem ratios, however, the need for a solution such as equation (13) is evident.

SUMMARY

An approximate equation has been derived for estimating the conduction shape factors of long hollow prismatic cylinders with isothermal boundaries. The derivation is based on the 'parallel flux tube' heat flow model. It is valid for cylinders with any size of inner hole. It is particularly useful for calculating shape factors for cylinders with very large inner holes. This is the region where existing approximate solutions fail or are inaccurate and accurate solutions require large computational effort. The results compare well with simple approximate solutions by previous workers [1-3] and are superior for cylinders with very large inner radii. They also agree very well with results obtained numerically by the BEM and the FEM. The accuracy of the equation improves with increasing inner hole size and also with increase in the number of sides of the prismatic cylinders.

This simple equation can therefore, be used to determine shape factors of hollow prismatic cylinders with inner holes of any radii.

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