Simplified Solutions to Circular Annular Fins with Contact Resistance and End Cooling

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One- and two-dimensional solutions are obtained for circular annular fins of constant cross-section having uniform base, end, and side conductances. The solutions are dependent upon one geometric parameter and three fin parameters which relate the internal conductive resistance to the three boundary resistances. The two and one-dimensional solutions are compared by means of their fin efficiency ratios. For fins used in typical microelectronic applications, the analytical expressions are also reduced to simple polynomial expressions which converge to unity for large values of the arguments. Numerical computations were performed on an IBM-PC and some typical results are reported in graphical form. These plots give the heat loss ratio as a function of the dimensionless geometric and fin parameters.

Nomenclature

A_n	= Fourier temperature coefficients, Eq. (8)
a	= fin inner radius
B_n	= Fourier heat flow rate coefficients, Eq. (15)
Bi, Bi_c, Bi_e	= Biot numbers
b	= fin outer radius
h, h_c, h_e	= side, contact and end conductances
$I_0(\cdot)$, $I_1(\cdot)$	= modified Bessel functions of first kind, order zero and one
$K_0(\cdot)$, $K_1(\cdot)$	= modified Bessel functions of second kind, order zero and one
k	= thermal conductivity
m	= one-dimensional fin parameter, $m = \sqrt{Bi}$
P_0, P_1	= polynomials used in Eq. (44)
Q	= total heat flow rate
R_2	= parameter defined by Eq. (46)
r	= cylindrical coordinate
T	= fin temperature
t	= fin half-thickness
W_0, W_1	= polynomials defined by Eq. (44)
z	= cylindrical coordinate
α	= relative inner radius (a/t)
β	= relative outer radius (b/t)
γ, γ_n	= function defined by Eq. (9)
δ_n	= eigenvalues, roots of Eq. (11)
ζ	= relative position (z/t)
η	= fin efficiency
θ	= temperature excess, $(T - T_\infty)$
Λ	= fin efficiencies ratio $\equiv \eta_2/\eta_1$
ν	= arbitrary order of modified Bessel functions
ρ	= relative position (r/t)
ϕ, ϕ_n	= function defined by Eq. (10)
ψ	= dimensionless temperature excess, (θ/θ_b)
Ω	= function defined by Eq. (31)
π	= pi

Subscripts

b	= base condition
c	= contact condition
e	= end condition
I	= ideal condition
n	= order of term $(n = 1,2,3,\ldots)$
∞	= ambient condition
1	= one-dimensional fin
2	= two-dimensional fin

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Introduction

CIRCULAR annular fins are used extensively in heat exchange devices to increase the heat transfer rate from a heat source for a given temperature difference or to decrease the temperature difference between the heat source and the heat sink for a given heat flow rate. Interest in the use of fins is found in many fields of thermal engineering, such as air conditioning, heat-exchangers and microelectronics where, using extended surfaces, thermal analysts have succeeded in designing more compact and efficient heat transfer systems.

When fins are added to a surface in contact with a fluid as shown in Fig. 1, a number of factors must be considered: the additional resistance due to the mechanical contact between the fin base and the previously exposed surface, the conductive resistance to heat flow within the fin itself, and the resistance to heat flow through the convective film of the surrounding fluid.

Several solutions to the problem of steady conduction within an annular fin of constant thickness have been presented.[1-4] These solutions are based upon the assumption of one-dimensional conduction in the radial direction, perfect contact at the fin base, and either insulated and/or some approximation to account for end cooling. The two-dimensional solution in which the temperature depends upon the thickness direction as well as the radius direction was obtained for the case of an isothermal fin base and some end cooling.[5] The fin efficiency was determined and reported in graphical form versus a design parameter group, which included the inner and outer fin radii, the total fin thickness, its thermal conductivity, and the film coefficient. The film coefficient was assumed to be uniform and of the same magnitude over the sides and the end. The other geometric parameter employed in the plots was the fin height to thickness ratio. The two-dimensional results were compared with the one-dimensional results for a range of fin height to thickness ratios. It was observed that the one-dimensional results were accurate when the height to thickness ratio is of the order of 10 or greater, and the two-dimensional results for rectangular fins are useful as approximations to annular fins when fin curvature is not large. With height to thickness ratios less than 10 and for annular fins with large curvature, the full two-dimensional solution must be employed.

One objective of this paper is to obtain general, one and two-dimensional solutions for circular annular fins of con-

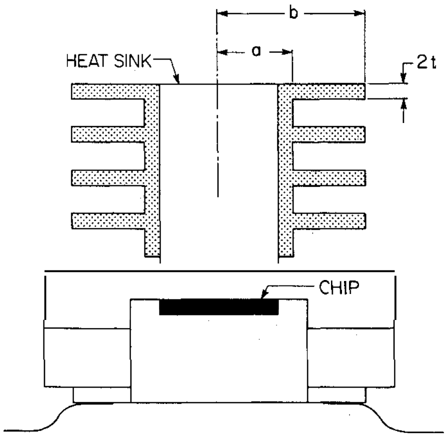

Fig. 1 Heat sink for a microelectronic chip.

stant thickness with boundary conditions of the third kind (Robin condition) applied to the fin base, sides and end. The film coefficients along the fin sides and end are assumed to be uniform and different. The contact conductance at the fin base will also be uniform. These assumptions will introduce three dimensionless parameters called Biot numbers that characterize the relative magnitudes of the fin conductive resistance to the base, sides, and end film resistance.

A second objective is to compare the two and one-dimensional heat flow rates for a range of the pertinent Biot numbers and fin aspect ratio, which is defined as the ratio of the fin outer to inner radius. This study will show that the simpler one-dimensional solution can be used over a rather wide range of the fin parameters with an acceptable error.

A third objective of this study is to demonstrate how the relatively complex special functions which appear in the two solutions can be computed with great accuracy and ease on an IBM-PC without requiring special mathematical software.

Problem Statements and Solutions

Two-Dimensional Solution

Steady-state conduction with an isotropic, circular annular fin and in the absence of internal heat sources must satisfy the two-dimensional Laplace equation, which in cylindrical coordinates has the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+\frac{\partial^2\theta}{\partial z^2}=0 \tag{1}$$

Because of symmetry of the boundary conditions of the third kind which are imposed over the fin base, sides and end, we can write the following conditions for the temperature excess $\theta(r,z)=T(r,z)-T_\infty$:

$$z=0,\ a\le r\le b,\qquad \frac{\partial\theta}{\partial z}=0 \tag{2a}$$

$$z=t,\ a\le r\le b,\qquad \frac{\partial\theta}{\partial z}=-\frac{h}{k}\theta \tag{2b}$$

$$r=a,\ 0\le z\le t,\qquad \frac{\partial\theta}{\partial z}=-\frac{h_c}{k}(\theta_b-\theta) \tag{2c}$$

$$r=b,\ 0\le z\le t,\qquad \frac{\partial\theta}{\partial z}=-\frac{h_e}{k}\theta \tag{2d}$$

The first equation represents the symmetry condition along the mid-plane, the second equation represents the side cooling through a uniform film coefficient h, the third equation represents the base contact condition with a uniform base temperature excess θ_b and uniform contact conductance h_c, and the last equation represents the end cooling condition through a uniform film coefficient h_e. The sink temperature T_∞ is assumed to be constant over the entire fin surface. The boundary conditions as well as the geometry of the problem are shown in Fig. 2.

For convenience, and to simplify the two-dimensional analysis, the fin half-thickness t is chosen to be the characteristic dimension in the geometry and the boundary conditions. Introducing the dimensionless parameters

$$\rho=r/t,\qquad \zeta=z/t \tag{3a}$$

$$\alpha=a/t,\qquad \beta=b/t \tag{3b}$$

and the dimensionless Biot numbers

$$Bi=ht/k \tag{4a}$$

$$Bi_c=h_ct/k \tag{4b}$$

$$Bi_e=h_et/k \tag{4c}$$

Laplace's equation and the boundary conditions become

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{\partial^2\psi}{\partial\zeta^2}=0\qquad \alpha\le\rho\le\beta,\ 0\le\zeta\le1 \tag{5}$$

$$\zeta=0,\quad \alpha\le\rho\le\beta,\qquad \frac{\partial\psi}{\partial\zeta}=0 \tag{6a}$$

$$\zeta=1,\quad \alpha\le\rho\le\beta,\qquad \frac{\partial\psi}{\partial\zeta}=-(Bi\,\psi) \tag{6b}$$

$$\rho=\alpha,\quad 0\le\zeta\le1,\qquad \frac{\partial\psi}{\partial\rho}=-[Bi_c(1-\psi)] \tag{6c}$$

$$\rho=\beta,\quad 0\le\zeta\le1,\qquad \frac{\partial\psi}{\partial\rho}=-(Bi_e\psi) \tag{6d}$$

where we have introduced the dimensionless excess temperature $\psi=\theta/\theta_b$.

Two-Dimensional Solution and Heat Flow Rate

The two-dimensional temperature distribution which satisfies the dimensionless Laplace's equation, Eq. (5), and the dimensionless boundary conditions, **Eqs. (6a–6d),** can be obtained by a straightforward application of the separation of variables method. The interested reader is referred to the conduction texts by Ozisik[6] and Mikhailov and Ozisik[7] for the details of this method.

The solution is

$$\psi(\rho,\zeta)=\sum_{n=1}^{\infty}\frac{A_n}{\phi_n}[I_0(\delta_n\rho)+\gamma_nK_0(\delta_n\rho)]\cos(\delta_n\zeta) \tag{7}$$

The parameters A_n, γ_n and ϕ_n which appear in Eq. **(7)** are defined below:

$$A_n=\frac{2\sin(\delta_n)}{\delta_n+\sin(\delta_n)\cos(\delta_n)} \tag{8}$$

$$\gamma_n=\frac{\delta_nI_1(\delta_n\beta)+Bi_eI_0(\delta_n\beta)}{\delta_nK_1(\delta_n\beta)-Bi_eK_0(\delta_n\beta)} \tag{9}$$

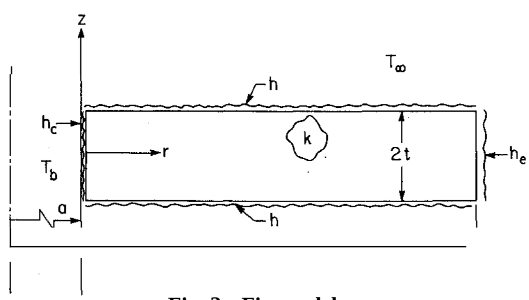

Fig. 2 Fin model.

$$\phi_n = I_0(\delta_n \alpha) + \gamma_n K_0(\delta_n \alpha) + \left[\frac{\gamma_n \delta_n}{Bi_c} K_1(\delta_n \alpha)\right] - \left[\frac{\delta_n}{Bi_c} I_1(\delta_n \alpha)\right] \tag{10}$$

The functions which appear in the solution are the modified Bessel functions of the first and second kinds of order zero and unity.

The additional parameter which appears in the solution is δ_n, which is the set of roots of the transcendental equation,[6,8]

$$\delta_n \sin(\delta_n) = Bi \cos(\delta_n) \tag{11}$$

The roots of Eq. *(11)* depend upon the magnitude of the parameter Bi, which represents the boundary condition along the fin sides. This parameter also represents the relative magnitude of the internal fin conductive resistance to the external film resistance. In theory its magnitude can range between zero and infinity. The first six roots of Eq. *(11)* are given in Carslaw and Jaeger[8] for a wide range of Bi where it is seen that for $0 < Bi < \infty$, in general, $(n-1)\pi < \delta_n < (2n-1)\pi/2$, $n = 1, 2, \ldots$. However, for applications of the analysis to fins of interest to the microelectronics industry, the magnitude of Bi will be small because the fins are thin and made of high conductivity metal. The film coefficient will also be relatively small when air cooling is present. It will be shown later how δ_1 can be computed approximately by direct methods for an important range of Bi.

Total Heat Flow Rate

The heat flow rate through the circular annular fin is of primary interest and can be determined at the fin base when using Fourier's law of conduction, where we have

$$Q = -4\pi k a \int_0^t -k 2\pi a \frac{\partial \theta(a, z)}{\partial r} dz \tag{12}$$

in dimensional form, or

$$Q = -4\pi k a \theta_b \int_0^1 \frac{\partial \theta(a, \zeta)}{\partial \rho} d\zeta \tag{13}$$

in dimensionless form. Substitution of Eq. *(7)* into Eq. *(13)* gives

$$Q = -4\pi k t \alpha \theta_b \sum_{n=1}^{\infty} \frac{B_n}{\phi_n} [I_1(\delta_n \alpha) - \gamma_n K_1(\delta_n \alpha)] \tag{14}$$

where the new parameter B_n is

$$B_n = A_n \sin(\delta_n) \tag{15}$$

Limiting Values of the Two-Dimensional Solution

The general two-dimensional solution reduces to the solutions obtained by other investigators. If perfect contact is assumed at the fin base, then $Bi_c \to \infty$ and the parameter ϕ_n, Eq. (10), reduces to:

$$\phi_n = I_0(\delta_n \alpha) + \gamma_n K_0(\delta_n \alpha) \tag{16}$$

If the fin end is assumed to be insulated, then $Bi_e \to 0$ and the parameter γ_n, Eq. (9), reduces to

$$\gamma_n = I_1(\delta_n \beta)/K_1(\delta_n \beta) \tag{17}$$

On the other hand, if the film resistance at the fin end is assumed to be negligible, then $Bi_e \to \infty$ and γ_n reduces to

$$\gamma_n = -I_0(\delta_n \beta)/K_0(\delta_n \beta) \tag{18}$$

These limiting expressions are important when parametric studies are performed to examine the sensitivity of the two-dimensional solution to the fin parameters Bi_c, Bi_e, α, and β.

When the internal conductive resistance to the external film resistance becomes very small, i.e., $Bi \to 0$, the general two-dimensional solution can be approximated by the leading term of Eqs. (7) and (14). When Bi is sufficiently small the solution can be approximated by

$$\psi(\rho, \zeta) = \frac{A_1}{\phi_1} [I_0(\delta_1 \rho) + \gamma_1 K_0(\delta_1 \rho)] \cos(\delta_1 \zeta) \tag{19}$$

with

$$A_1 = \frac{2 \sin(\delta_1)}{\delta_1 + \sin(\delta_1) \cos(\delta_1)} \tag{20}$$

$$\gamma_1 = \frac{\delta_1 I_1(\delta_1 \beta) + Bi_e I_0(\delta_1 \beta)}{\delta_1 K_1(\delta_1 \beta) - Bi_e K_0(\delta_1 \beta)} \tag{21}$$

$$\phi_1 = I_0(\delta_1 \alpha) + \gamma_1 K_0(\delta_1 \alpha) + \left[\frac{\gamma_1 \delta_1}{Bi_c} K_1(\delta_1 \alpha)\right] - \left[\frac{\delta_1}{Bi_c} I_1(\delta_1 \alpha)\right] \tag{22}$$

and the parameter δ_1 which is the first root of Eq. *(11)* can be approximated by[9]

$$\delta_1 = \{A[1 - \Delta^2/45]\}^{1/2} \tag{23}$$

where

$$A = 3Bi/(3 + Bi) \tag{24}$$

with an error less than .09% provided $Bi \leq 2$. In the limit of $Bi \to 0$, $\delta_1 \to \sqrt{Bi}$.

The two-dimensional solution for the total heat flow rate, Eq. (14), can be approximated by the leading term when $Bi \ll 1$; therefore,

$$Q = -4\pi k t \alpha \theta_b \frac{B_1}{\phi_1} [I_1(\delta_1 \alpha) - \gamma_1 K_1(\delta_1 \alpha)] \tag{25}$$

and $B_1 = A_1 \sin(\delta_1)$.

One-Dimensional Solution

The one-dimensional formulation of the circular annular fin assumes that the temperature excess depends upon the radial coordinate only because $Bi \ll 1$. For this problem the governing equation is the second order ordinary differential equation:[10]

$$\frac{1}{\rho} \frac{d}{d\rho} \left[\rho \frac{\partial \psi}{\partial \rho}\right] - m^2 \psi = 0 \qquad \alpha \leq \rho \leq \beta \tag{26}$$

where the fin parameter $m^2 = Bi = \delta_1^2$ of the two-dimensional solution. The base and end boundary conditions are

$$\rho = \alpha, \; \partial\psi/\partial\rho = -Bi_c[1 - \psi] \tag{27}$$

$$\rho = \beta, \; \partial\psi/\partial\rho = -Bi_e \psi \tag{28}$$

The solution of Eq. *(26)* is[10]

$$\psi(\rho) = C_1 I_0(m\rho) + C_2 K_0(m\rho) \tag{29}$$

and the gradient of ψ is

$$\partial\psi/\partial\rho = m C_1 I_1(m\rho) - m C_2 K_1(m\rho) \tag{30}$$

The constants C_1 and C_2 are determined by the boundary conditions, Eqs. *(27)* and *(28).* The condition at the fin end gives

$$\Omega = \frac{C_2}{C_1} = \frac{mI_1(m\beta) + Bi_e I_0(m\beta)}{mK_1(m\beta) - Bi_e K_0(m\beta)} \quad (31)$$

The condition at the fin base gives

$$\frac{1}{C_1} = I_0(m\alpha) + m\Omega K_0(m\alpha) + \frac{m\Omega}{Si_c}K_1(m\alpha) - \frac{m}{Bi_c}I_1(m\alpha) \quad (32)$$

The above general relationships reduce to the following special cases: For an insulated fin end, $Si_e = 0$, we find

$$\frac{C_2}{C_1} = \frac{I_1(m\beta)}{K_1(m\beta)} \quad (33)$$

For an infinitely long fin $b/a \rightarrow \infty$ or $Si_e \rightarrow \infty$; or for negligible end film resistance on a finite fin $Si_e = \infty$; therefore,

$$\frac{C_2}{C_1} = -\frac{I_0(m\beta)}{K_0(m\beta)} \quad (34)$$

When there is perfect contact at the fin base, $Si_c \rightarrow \infty$ and

$$\frac{1}{C_1} = I_0(m\alpha) + m\Omega K_0(m\alpha) \quad (35)$$

Total Heat Flow Rate

The total heat flow rate can be determined by application of Fourier's law of conduction at the fin base, where

$$Q = -[4\pi kat\,\partial\theta(a)/\partial r] \quad (36)$$

or

$$Q = -[4\pi ka\theta_b\,\partial\psi(\alpha)/\partial\rho] \quad (37)$$

and with Eqs. *(30)* and (31), Eq. *(37)* becomes

$$Q = -\{4\pi kt\theta_b m\alpha C_1[I_1(m\alpha) - \Omega K_1(m\alpha)]\} \quad (38)$$

where C_1 in general is determined by Eq. *(32).*

Comparison of One and Two-Dimensional Solutions

The one-dimensional solution is obviously simpler than the two-dimensional solution, which in general consists of a summation of two or more terms each consisting of the modified Bessel functions I_0, I_1, K_0, and K_1. These functions require special attention, particularly when found in ratios.

For one-dimensional fins it is customary to use a fin efficiency, defined as the ratio of the actual heat flow rate divided by the ideal heat flow rate, which in the circular annular fin case is

$$Q_i = 2\pi(b^2 - a^2)h\theta_b + 4\pi bth_e\theta_b \quad (39)$$

This expression can be put into a form which is more convenient for subsequent comparison and analysis. By introducing Bi and Si_e into Eq. (39), it becomes

$$Q_i = 2\pi kt\theta_b[(\beta^2 - \alpha^2)Bi + 2\beta Bi_e] \quad (40)$$

The one-dimensional fin efficiency is defined as

$$\eta_1 = Q_1/Q_i \quad (41)$$

and the two-dimensional fin efficiency is similarly defined to be

$$\eta_2 = Q_2/Q_i \quad (42)$$

Obviously the ratio of the fin efficiencies is equal to the heat flow rate ratio:

$$\Lambda = \frac{\eta_2}{\eta_1} = \frac{Q_2}{Q_1} = \frac{\sum_{n=1}^{\infty} \frac{B_n}{\phi_n}[I_1(\delta_n\alpha) - \gamma_n K_1(\delta_n\alpha)]}{\frac{\sqrt{Bi}}{\phi}[I_1(\sqrt{Bi}\alpha) - \gamma K_1(\sqrt{Bi}\alpha)]} \quad (43)$$

where γ and ϕ are defined by Eqs. *(9)* and (10), respectively, and γ_n is replaced by $\sqrt{Bi}$. This ratio **A** will be examined in greater detail in a subsequent section.

From the limiting values reported earlier, it can be seen that as $Bi \rightarrow 0$, the summation in the numerator of Eq. *(43)* goes to the leading term, which finally approaches the denominator, so that the ratio **A** approaches unity. When the ratio is close to unity, the one-dimensional solution can be used to approximate the two-dimensional solution. This ratio will be examined as a function of the parameters Bi, Si_c, Si_e, and β/α.

Simplification of Analytical Results

The analytical results for the one and two-dimensional fins can be simplified considerably when typical fins used in microelectronics are considered. Most circular annular fins are designed to have $\beta > \alpha \le 10$; therefore, the arguments of the modified Bessel functions which appear in the dimensionless temperature excess and heat flow rate expressions, are $x > (n-1)\pi\alpha$ and $y > (n-1)\pi\beta$ for $n \ge 2$. With $\alpha = 10$, the arguments for the second term of the solutions are $y > x > 10\pi$, which are considered to be large values. Asymptotic expansions of I_ν and K_ν can then be used to compute all terms after the leading term of the temperature and heat flow rate solutions.'' Thus for $n \ge 2$, the parameter γ_n, Eq. (9), can be expressed as

$$\gamma_n = \frac{e^{2y}}{\pi}\left[\frac{\delta_n P_1(y) + Bi_e P_0(y)}{\delta_n W_1(y) + Bi_e W_0(y)}\right] \quad (44)$$

where the polynomials $P_0(y)$, $P_1(y)$, $W_0(y)$, and $W_1(y)$ can be obtained from the asymptotic expansion for the modified Bessel functions.'' For very large values of y, the polynomials in Eq. *(44)* are approximately one, and the expression reduces to

$$\gamma_n = \frac{e^{2y}}{\pi}\left[\frac{\delta_n + Bi_e}{\delta_n - Bi_e}\right] \quad (45)$$

For either $Si_e = 0$ or $Si_e = 10^3$, $|\gamma_n| > e^{20\pi}/\pi \approx 6.2 \times 10^{26}$. This important fact will be used to simplify for computational purposes. The numerator in Eq. *(43)* can be simplified considerably. It can be shown that

$$R_2 = \frac{I_1(\delta_n\alpha) - \gamma_n K_1(\delta_n\alpha)}{\phi_n} \quad (46)$$

can be rewritten as

$$\frac{1}{R_2} = \frac{\delta_n}{Bi_c} - \frac{[I_0(\delta_n\alpha) + \gamma_n K_0(\delta_n\alpha)]}{[I_1(\delta_n\alpha) - \gamma_n K_1(\delta_n\alpha)]} \quad (47)$$

Factoring the function γ_n we obtain

$$\frac{1}{R_2} = \frac{\delta_n}{Bi_c} - \frac{[I_0(x)/\gamma_n(y) + K_0(x)]}{[I_1(x)/\gamma_n(y) + K_1(x)]} \quad (48)$$

When $y > x = 30\pi$, $I_0(x)/\gamma_n(y) << 1$ and $I_1(x)/\gamma_n(y) << 1$, therefore, Eq. *(48)* reduces to

$$\frac{1}{R_2} = \frac{\delta_n}{Si_c} + \frac{K_0(x)}{K_1(x)} \quad (49)$$

which for $x > 10\pi$ and $n \ge 2$, reduces further to

$$\frac{1}{R_2} = \frac{\delta_n}{Si_c} - \frac{W_0(x)}{W_1(x)} \quad (50)$$

The ratio $W_0(x)/W_1(x)$ goes to unity as n increases. For example, when $n = 2$ and $x = 10\pi$, $W_0(x)/W_1(x)$ is approximately 0.981.

The fin efficiencies ratio **A,** Eq. (43), can now be expressed in a form which permits accurate and efficient numerical computations. Equation **(43)** can be written as

$$\mathrm{A} = \frac{\frac{B_1}{\phi_1}[I_1(\delta_1\alpha) - \gamma_1 K_1(\delta_1\alpha)] + \sum_{n=2}^{\infty} B_n R_2}{\frac{\sqrt{Bi}}{\phi}[I_1(\sqrt{Bi}\alpha) - \gamma K_1(\sqrt{Bi}\alpha)]} \tag{51}$$

Also note that when $\sqrt{Bi}\alpha > 10\pi$, then by similar arguments we have

$$\frac{I_1(\sqrt{Bi}\alpha) - \gamma K_1(\sqrt{Bi}\alpha)}{\phi} = \left[\frac{\sqrt{Bi}}{Bi_c} + 1\right]^{-1} \tag{52}$$

Equation (51) simplifies further as Bi —0 because 6, $-(n-1)\pi$ and the Fourier coefficients approach zero for **n ≥ 2,** and therefore the summation term disappears. Therefore,

$$\mathrm{A} = \frac{\frac{B_1}{\phi_1}[I_1(\delta_1\alpha) - \gamma_1 K_1(\delta_1\alpha)]}{\frac{\sqrt{Bi}}{4}[I_1(\sqrt{Bi}\alpha) - \gamma K_1(\sqrt{Bi}\alpha)]} \tag{53}$$

Finally, when $Bi < 0.2$, 6, $\approx \sqrt{Bi}$ and B, $= \delta_1$, the ratio **A** goes to unity.

Discussion of Computational Results and Conclusions

Numerous numerical computations were performed on an IBM-PC using the simplified expressions developed in the previous sections. The fin efficiencies ratio was computed for a wide range of the independent geometric and fin parameters. The end Biot number was taken to be either zero or one thousand, which is nearly equivalent to an infinite value. The contact Biot number ranged from **0.1** to 20, which is considered to be an upper bound corresponding to a soldered joint. Typical values for microelectronic applications are expected to lie in the 0.1 to 1 range. The side Biot number ranged from **.001** to **5;** however, it is expected that it will not exceed 0.5 for most microelectronic applications. The geometric parameter is the radii ratio or β/α; it ranged from 1.1 to **5.** The parameter a was set to **10** for all cases. The cases reported here are listed in Table 1.

It was observed that when $\beta/\alpha \geq 2$, the end condition has negligible influence on **A** for all values of the other fin parameters. This means that the fin performance is the same whether the end parameter Bi_e is zero or infinite. On the other hand, it was observed that the end condition becomes very important when β/α approaches unity, Bi —0, and $Bi_c \rightarrow 20$. It was observed that **A** is less than unity for all values of the geometric and fin parameters, demonstrating that the one-dimensional fin solution overpredicts the heat transfer rate, especially for large values of Bi and Bi_c. Figure **3** shows how **A** depends upon increasing Bi as a function of Bi_c for a fixed value of $\beta/\alpha = 1.2$ for the two extreme end conditions. Results are similar for other values of $\beta/\alpha \leq 2$. Also, one can see that **A** approaches unity for small values of Bi_c for all values of Bi when $Bi_e = 0$ or 1000. Figures 4 and **5** show **A** for $\beta/\alpha = 1.2$, $Bi_e = 0$ or 1000, $0.1 \leq Bi_c < 20$, for the typical range of Bi, i.e., $0 < Bi \leq 0.25$. It is noted that the largest difference between the one and two-dimensional solutions occurs when $Bi_e = 0$, at $Bi = 0.25$ and $Bi_c = 20$; however, the difference is approximately **5%.** On the other hand, when $Bi_e = 1000$ the largest difference between the two solutions is approximately 1% at $Bi = 0.25$ and $Bi_c = 20$.

To show clearly the effect of the geometric parameter β/α upon **A** for selected fixed values of Bi_c, Bi_e, and the typical range of Bi, the results are given in Figs. **6** and **7.** It is interesting to see in Fig. **6** that **A** increases with increasing β/α when $Bi_c = 0.1$ and $Bi_e = 0$, but decreases with increasing β/α when $Bi = 1000$. It appears that $\beta/\alpha = 1.5$ is approximately the limiting value independent of Bi_e. One should also note that the difference between the one- and two-dimensional solutions is less than **3%** when $Bi \leq 5$.

Table 1 Summary of parametric studies

Case	b/a	Bi	Bi_c	Bi_e
1	1.2	0–5	.I–20	0
2	1.2	0–5	.1-20	1000
3	1.2	0–.25	.1–20	0
4	1.2	0–.25	.1–20	1000
5	1.1–1.5	0–5	1	0,1000
6	1.1–1.5	0–5	10	0,1000

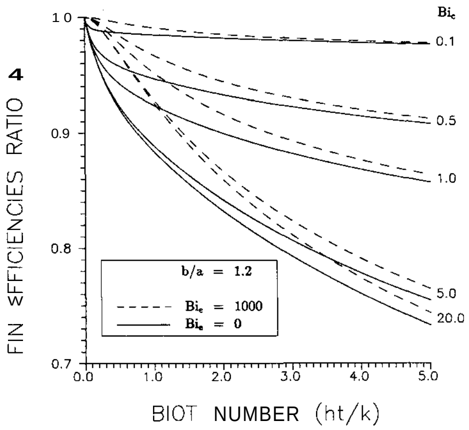

Fig. 3 Heat loss rate ratio vs Bi: case 1.

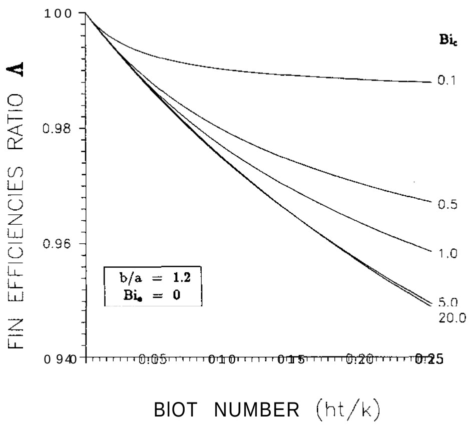

Fig. 4 Heat loss rate ratio vs Bi: case 2.

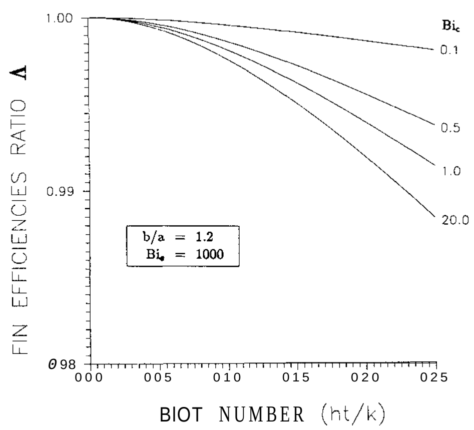

Fig. 5 Heat loss rate ratio vs Bi: case 3.

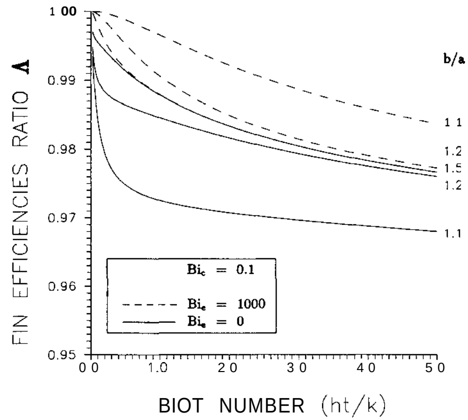

Fig. 6 Heat loss rate ratio vs Bi: case 4.

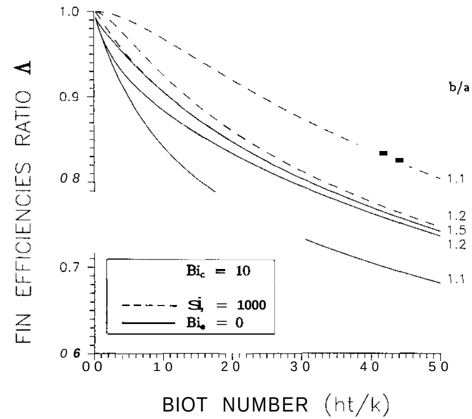

Fig. 7 Heat loss rate ratio vs Bi: case 5.

In Fig. 7 one observes the same trends of Λ with increasing β/α when $Bi_c = 10$ and Bi_e is either zero or 1000. The difference between the one and two-dimensional solutions is much greater for this case.

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