Comput. Math. Applic. Vol. 16, No. 9, pp. 747–757, 1988
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0097-4943/88 \$3.00 + 0.00

EFFICIENT EVALUATION OF INCOMPLETE ELLIPTIC INTEGRALS AND FUNCTIONS

T. F. LEMCZYK and M. M. YOVANOVICH
Microelectronics Heat Transfer Laboratory, Department of Mechanical Engineering,
University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

(*Received* 10 *March* 1988)

Communicated by E. Y. Rodin

Abstract—The impetus for this work came as a result of finding that evaluation of the complete elliptic integrals using theta-function expansions was computationally faster, for the same accuracy, than the well known conventional method using Landen's transformations, known as the arithmetic–geometric mean (A.G.M.). By using relations between Jacobian elliptic functions and theta-functions, it is shown here that the incomplete elliptic integrals may also be evaluated very efficiently using a Newton–Raphson scheme. The expressions outlined were found to be substantially more efficient and accurate than several infinite series or polynomial expansions provided by Abramowitz and Stegun in 1970. Analysis and algorithms are presented along with accurate tabulated numerical results.

NOMENCLATURE

c_i—constants
E—elliptic integral of the second kind
E'—complementary complete elliptic integral of the second kind
F—incomplete elliptic integral of the first kind
k, k'—modulus, complementary modulus
K—complete elliptic integral of the first kind
K'—complementary complete elliptic integral of the first kind
m, n—integer constants
q, q_1—nome in theta-function series, complementary nome $\equiv q(\pi/2-\alpha)$
u, w, x, y, z—arguments

Greek symbols

α—modulus parameter
β, γ—angular parameters
ϵ—modulus quotient
Λ_0—Heuman lambda function
ν, ω, Ω—general functions
π—constant $= 3.14159265\ldots$
Π—elliptic integral of the third kind
σ, ϕ, ψ, θ—angular parameters
θ_i, Θ—theta-functions

INTRODUCTION

Integrals of the form

$$\int R(x, \sqrt{y})\,\mathrm{d}x,$$

where R denotes a rational function of x and y and some constant modulus k, and y is generally a quartic function of x, are of a *non-standard* type. They are referred to as *elliptic integrals* in the literature, and were first studied in Ref. [1]. Inverses of certain types of these integrals are known as *elliptic functions*, and they were first studied by Gauss, Abel, Jacobi and Weierstrass at the turn of the nineteenth century. As outlined in Ref. [2], every elliptic integral can be evaluated by aid of functions termed *theta-functions*, and it is this approach which is adopted here. The theta-functions themselves satisfy certain types of differential equations which are outlined by Refs [2, 3].

Numerous representations of theta-functions have been adopted over the years and perhaps the best summary of these is outlined by Ref. [2, Chap. XXI]. Evaluation of complete elliptic integrals of the first and second kind using theta-function theory is very efficient (see Ref. [4]), involves no iteration, and is slightly superior in computational speed compared to the process of the arithmetic–geometric mean (A.G.M.) described by Ref. [5]. This theory has actually been known for some time, as was outlined in Ref. [6]. More recently, Fenton and Gardiner-Garden [7] returned to this theory and re-established that theta-function expansions give very convergent methods for evaluating complete elliptic integrals and their related functions. Numerous other non-standard integrals may often be expressed in terms of elliptic integrals, as noted in Refs [3, 8]. The applications are many and, in particular, thermophysics problems are a rich source of these, since they usually involve Lipshitz–Hankel integrals [9], as studied by Ref. [10], which may be written in terms of elliptic functions.

We note that the evaluation of the complete elliptic integral of the first kind, $K(k)$, is paramount, since all other complete elliptic integrals may be expressed in terms of it. Correspondingly, in this work, first emphasis is placed on the evaluation of the first *incomplete* elliptic integral $F(\theta, k)$. In the same manner as for the complete elliptic integrals, the remaining incomplete elliptic integrals may then be found.

In this work we outline a procedure for the efficient evaluation of the incomplete elliptic integrals using theta-functions. Numerical results are presented in tabulated form for several cases, including some incomplete elliptic integrals of the third kind, for which tables exist only to limited accuracy in the literature (i.e. Ref. [5]). Complex values of parameters are not treated here, but for these, and additional special cases not covered in Appendix A, refer to Refs [2, 3, 5] for excellent reviews.

EVALUATION OF COMPLETE ELLIPTIC INTEGRALS

It is important to outline first the efficient procedure one may use to evaluate the complete elliptic integrals. This was studied in Ref. [7], and also used by one of the authors (M.M.Y.) for many years in applied engineering courses.

The four types of theta-functions we will be using are defined by the nome q and Fourier series (Ref. [5, Section 16.7]) as follows:

$$\theta_1(z, q) = 2q^{1/4}\sin z - 2q^{9/4}\sin 3z + 2q^{25/4}\sin 5z - \cdots \tag{1}$$

$$\theta_2(z, q) = 2q^{1/4}\cos z + 2q^{9/4}\cos 3z + 2q^{25/4}\cos 5z + \cdots \tag{2}$$

$$\theta_3(z, q) = 1 + 2q\cos 2z + 2q^4\cos 4z + 2q^9\cos 6z + \cdots \tag{3}$$

$$\theta_4(z, q) = 1 - 2q\cos 2z + 2q^4\cos 4z - 2q^9\cos 6z + \cdots. \tag{4}$$

These are used for the evaluation of elliptic integrals, and may be found in different notation in various references. Here we have adopted the notation of Refs [2, 5] [Note: Jahnke and Emde [6], as well as Byrd and Friedman [3], use $\theta_0(z, q)$, the "zero-theta", in place of $\theta_4(z, q)$.]

The complete elliptic integrals of the first and second kind, denoted in the literature by K and E, respectively, are given in Legendre notation as,

$$K = \int_0^{\pi/2} \frac{d\psi}{(1 - k^2\sin^2\psi)^{1/2}}, \tag{5}$$

$$E = \int_0^{\pi/2} (1 - k^2\sin^2\psi)^{1/2}\, d\psi. \tag{6}$$

The constant k is referred to as the *modulus*, and $k' = (1 - k^2)^{1/2}$ is the *complementary* modulus. In terms of theta-functions, $z = 0$ or $\pi/2$, and K and E are defined by:

$$K = \frac{\pi}{2}[\theta_3(0, q)]^2 = \frac{\pi}{2}[\theta_4(\pi/2, q)]^2, \tag{7}$$

$$E = K\left[1 - \frac{\theta_4''(0, q)}{\theta_4(0, q)}\right], \tag{8}$$

where

$$q = \exp(-\pi K'/K). \tag{9}$$

The modulus k is defined as the quotient of theta-functions,

$$k = \left(\frac{\theta_2(0, q)}{\theta_3(0, q)}\right)^2. \tag{10}$$

To obtain efficient convergent series for numerical work, it is obvious that we need to determine the *nome* q given k. Hence, using expansions developed by Weierstrass in 1895, from Refs [6, 2], we can deduce the following procedure for the complete elliptic integrals to 16 decimal place accuracy:

(*i*) *For the range* $k \leqslant 1/\sqrt{2}$.

$$\epsilon = \frac{1}{2}\frac{1-\sqrt{k'}}{1+\sqrt{k'}} \tag{11}$$

$$q = \epsilon + 2\epsilon^5 + 15\epsilon^9 + 150\epsilon^{13} + \cdots \tag{12}$$

$$K = \frac{\pi}{2}[1 + 2q + 2q^4 + 2q^9]^2 \tag{13}$$

$$E = \frac{\pi^2}{4K}\left[\frac{1 + 9q^2 + 25q^6 + 49q^{12}}{1 + q^2 + q^6}\right]^{\dagger}. \tag{14}$$

(*ii*) *For the range* $1/\sqrt{2} \leqslant k \leqslant 1$.

$$\epsilon = \frac{1}{2}\frac{1-\sqrt{k}}{1+\sqrt{k}} \tag{15}$$

$$q_1 = \epsilon + 2\epsilon^5 + 15\epsilon^9 + 150\epsilon^{13} + \cdots \tag{16}$$

$$K' = \frac{\pi}{2}(1 + 2q_1 + 2q_1^4 + 2q_1^9)^2 \tag{17}$$

$$E' = \frac{\pi^2}{4K'}\frac{1 + 9q_1^2 + 25q_1^6 + 49q_1^{12}}{1 + q_1^2 + q_1^6} \tag{18}$$

$$K = -\frac{K'}{\pi}\ln q_1 \tag{19}$$

$$E = \frac{1}{K'}\left[\frac{\pi}{2} + K(K' - E')\right]. \tag{20}$$

An important relation used in equation (20) is *Legendre's relation*,

$$EK' + E'K - KK' = \frac{\pi}{2}. \tag{21}$$

We note that for the range (i), the nome q as defined by equation (9) is identical to the form (12). For the range (ii), the form (9) must be used to evaluate q after determining K', K. This will be required to evaluate the incomplete elliptic integrals of the second and third kind to be shown later.

EVALUATION OF THE FIRST INCOMPLETE ELLIPTIC INTEGRAL $F(\theta, k)$

In Legendre's notation we have

$$u = F(\theta, k) = \int_0^\theta \frac{d\psi}{(1 - k^2\sin^2\psi)^{1/2}}, \tag{22}$$

or, in Jacobi's notation also in the literature, we may write

$$u = \int_0^\alpha (1 - t^2)^{-1/2}(1 - k^2t^2)^{-1/2}\,dt, \tag{23}$$

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†This is found after some manipulation of the form (8).

where

$$\mathrm{sn}(u,k) = \alpha = \sin\theta, \tag{24}$$

and sn is referred to as the *Jacobian elliptic sine* function. In terms of theta-functions, we have the relation:

$$\mathrm{sn}(u,k) = \frac{\theta_3}{\theta_2}\frac{\theta_1(w,q)}{\theta_4(w,q)} = \alpha, \tag{25}$$

where $w = u/\theta_3^2(0,q)$. The quotient θ_2/θ_3 is shown in Ref. [3] and by equation (10) to be equal to the square root of the modulus k, and thus we obtain

$$\sqrt{k} = \frac{\theta_1(w,q)}{\alpha\theta_4(w,q)}. \tag{26}$$

Expansions for $\theta_1(w,q)$ and $\theta_4(w,q)$ are given by equations (1) and (4). Now, we proceed to reduce the trigonometric quantities to a simple series in $\sin^n w$, and with this we may reduce equation (26) to

$$0 = c_0 + \sum_{n=1}^{\infty} c_n \sin^n w, \tag{27}$$

or in *nested notation*, setting $x \equiv \sin w$,

$$0 = x(c_1 + x(c_2 + x(c_3 + x(c_4 + x(c_5 + x(c_6 + x(c_7 + x(c_8 + \cdots)))))))) + c_0. \tag{28}$$

This is the functional equation for x, to which we can apply a Newton–Raphson scheme to evaluate x given the constants c_n. The constants c_n are functions of α and k, which need to be specified beforehand. The first nine constants, truncated to give double precision accuracy, can be shown to be:

$$c_0 = \frac{-\alpha\sqrt{k}}{2}(1 - 2q + 2q^4 - 2q^9 + 2q^{16}) \tag{29}$$

$$c_1 = q^{1/4} - 3q^{9/4} + 5q^{25/4} - 7q^{49/4} \tag{30}$$

$$c_2 = \frac{-\alpha\sqrt{k}}{2}(4q - 16q^4 + 36q^9 - 64q^{16}) \tag{31}$$

$$c_3 = 4q^{9/4} - 20q^{25/4} + 56q^{49/4} \tag{32}$$

$$c_4 = \frac{-\alpha\sqrt{k}}{2}(16q^4 - 96q^9 + 320q^{16}) \tag{33}$$

$$c_5 = 16q^{25/4} - 112q^{49/4} \tag{34}$$

$$c_6 = \frac{-\alpha\sqrt{k}}{2}(64q^9 - 512q^{16}) \tag{35}$$

$$c_7 = 64q^{49/4} \tag{36}$$

$$c_8 = \frac{-\alpha\sqrt{k}}{2}256q^{16}. \tag{37}$$

The nome q is a function of the modulus k, and can be evaluated as was shown for the complete elliptic integrals in the previous section.

In order to achieve accuracy to double precision (16 decimal places) as compared with the process of the A.G.M., over certain values of k we need to perform a Gauss transformation as given by Ref. [3, Section 164.02]. This transformation is outlined as follows:

$$F(\phi, k_1) = F(\theta, k)/(1 + k_1), \tag{38}$$

or

$$F(\theta, k) = (1 + k_1)F(\phi, k_1), \tag{39}$$

where

$$k_1 = \frac{1-k'}{1+k'}; \quad k' = (1-k^2)^{1/2} \tag{40}$$

and

$$\sin\theta(1 + k_1 \sin^2\phi) = (1 + k_1)\sin\phi \tag{41}$$

$$2\sin\phi = \frac{1+k_1}{k_1 \sin\theta} - \left[\left(\frac{1+k_1}{k_1\sin\theta}\right)^2 - \frac{4}{k_1}\right]^{1/2}. \tag{42}$$

It was found that for values of ϕ less than 45° (note: $\phi = \sin^{-1} k$), no transformations were necessary for double precision accuracy. For $45° < \phi \leqslant 80°$, one Gauss transformation was necessary, and for $80° < \phi$, 2 successive Gauss transformations were required. For this latter case, the procedure is similar to equation (38) as follows:

$$F(\theta, k) = (1 + k_1)(1 + k_2)F(\phi_2, k_2), \tag{43}$$

where k_1 is as in equations (40) and

$$k_2 = \frac{1-k_1'}{1+k_1'}, \tag{44}$$

$$2\sin\phi_2 = \frac{1+k_2}{k_2\sin\phi_1} - \left[\left(\frac{1+k_2}{k_2\sin\phi_1}\right)^2 - \frac{4}{k_2}\right]^{1/2}. \tag{45}$$

A summary of these transformations is shown in Table 1, along with initial starting values, x_s, for the iteration process. The remarkable consequence of all this work is the fact that convergence of the Newton–Raphson scheme is very efficient. This is shown in Table 2. It requires, on average, about 3 or 4 iterations for the scheme to converge over the entire range of θ and ϕ.

It is important to note that other transformations were attempted, but failed to yield reasonable results. It is not clearly understood at this point why the Gauss transformation works so well, and why other transformations in the literature do not. Also, on a real time comparison with the process of the A.G.M., it was found that the method outlined here was about 10% slower. This could be substantially improved if a relationship between the constants c_n could be found. All computations were performed in double precision on an IBM PC in BASIC and FORTRAN 77.

RELATED INTEGRALS AND FUNCTIONS

The incomplete elliptic integral of the second kind is defined by

$$E(\theta, k) = \int_0^\theta (1 - k^2\sin^2\psi)^{1/2}\,\mathrm{d}\psi. \tag{46}$$

Table 1. Range of transformations for evaluating $F(\theta, \phi)$

$0° < \phi \leq 45°$	$45° < \phi \leq 80°$	$80° < \phi < 90°$
No transformations required. Accuracy exact with A.G.M to double precision. $x_s = 0.004$	1 Gauss transformation required. Accuracy to double precision with A.G.M. $x_s = 0.012$	2 Gauss transformations required. Accuracy to double precision with A.G.M. $x_s = 0.022$

Table 2. Convergence of method (Newton–Raphson iterations)†

	$0° < \phi < 5°$	$5° < \phi < 45°$	$45° < \phi < 80°$	$80° < \phi < 90°$
$80° < \theta < 90°$	3	4	4	4
$45° < \theta < 80°$	3	4	4	3
$5° < \theta < 45°$	3	3	3	3
$0° < \theta < 5°$	2	3	3	3

† Represented by average number of iterations

From Refs [2, p. 518; 5, Section 17.2.13]

$$E(\theta,k)=\frac{\pi}{2K}\frac{\theta_4'\left(\dfrac{\pi u}{2K},q\right)}{\theta_4\left(\dfrac{\pi u}{2K},q\right)}+\frac{uE}{K},\tag{47}$$

where $u = F(\theta, k)$, K and E are the complete elliptic integrals of the first and second kind with modulus k, and θ_4 is defined by equation (4). To achieve double precision accuracy, only five terms are required in equation (4), and four terms for the derivative θ_4'. Noting that the nome q is a function of the modulus k as in equations (12) and (16), we need only evaluate $u = F(\theta, k)$, E and K, outlined earlier, and we may then determine $E(\theta, k)$ from equation (47). Twelve decimal place values for $F(\theta, k)$ and $E(\theta, k)$ are provided in Table 3.

As a direct result of being able to compute efficiently the incomplete elliptic integrals of the first and second kind, we can now efficiently compute elliptic integrals of the third kind, $\Pi(\theta, \gamma^2, k)$, and also functions such as the Heuman lambda-function $\Lambda_0(\beta, k)$, and the Jacobian zeta-function $Z(\beta, k)$. These are outlined below in terms of known functions and limiting forms are also given in Appendix A.

Table 3. Selected values for $F(\theta, k)$, $E(\theta, k)$

$\theta°$ \ $\phi°$	1	10	30	45
	$F(\theta,\phi)$			
1	0.017453292790	0.017453319237	0.017453514038	0.017453735571
10	0.174533193454	0.174559492848	0.174753855140	0.174976301923
30	0.523605673662	0.524284017289	0.529428627052	0.535622732805
45	0.785419897053	0.787564937491	0.804366101232	0.826017876249
60	1.047244324488	1.051879112762	1.089550670052	1.142429058046
80	1.396356715044	1.405645220554	1.484554552055	1.608476732060
88	1.536004057997	1.547397952699	1.645446429580	1.804719328423
90	1.570915958127	1.582842804338	1.685750354813	1.854074677301
	60	80	88	90
1	0.017453957120	0.017454151959	0.017454177604	0.017454178684
10	0.175200286348	0.175398542412	0.175424727014	0.175425829652
30	0.542229109804	0.548425344543	0.549270415213	0.549306144334
45	0.851223749071	0.877408330406	0.881211426058	0.881373587020
60	1.212596615255	1.301353213761	1.316305100453	1.316957896925
80	1.812529534398	2.265273260789	2.427180030034	2.436246053716
88	2.086744929901	2.953656299014	3.861075154349	4.048125418683
90	2.156515647500	3.153385251888	4.742717265279	∞
	$E(\theta,\phi)$			
	1	10	30	45
1	0.017453292250	0.017453265802	0.017453071007	0.017452849489
10	0.174532656946	0.174506364812	0.174312496773	0.174091565468
30	0.523591877695	0.522915112409	0.517881934860	0.512049322350
45	0.785376430775	0.783241622961	0.767195985711	0.748186504178
60	1.047150781365	1.042550471931	1.007555555144	0.964951457643
80	1.396170097845	1.386978856068	1.316058404877	1.226610499417
88	1.535775438373	1.524510704027	1.437230174207	1.325956187678
90	1.570676709128	1.558887196602	1.467462209339	1.350643881048
	60	80	88	90
1	0.017452627966	0.017452433157	0.017452407517	0.017452406437
10	0.173870127161	0.173674975302	0.173649260229	0.173648177667
30	0.506092072466	0.500742319368	0.500030025084	0.500000000000
45	0.728224155457	0.709723805114	0.707212890400	0.707106781187
60	0.918393294316	0.872755203913	0.866299900681	0.866025403784
80	1.122485895679	1.005432946316	0.985689154039	0.984807753012
88	1.193592110305	1.034013578241	1.001185987678	0.999390827019
90	1.211056027568	1.040114395706	1.002584065528	1.000000000000

Table 4. Incomplete $\Pi(\theta, \gamma^2, \phi)$; $\gamma^2 = 0.1,\ 0.5$

$\theta°$ \ $\phi°$	1	15	30	45
	$\gamma^2 = 0.1$			
1	0.017453470001	0.017453529087	0.017453691254	0.017453912791
15	0.262392651443	0.262590556405	0.263137864406	0.263895537974
30	0.528205419022	0.529750956640	0.534119286520	0.540411178855
45	0.800152692676	0.805143972639	0.819719226671	0.842096304336
60	1.079536324835	1.090576712217	1.124054916306	1.179796546823
75	1.365687829054	1.385201841526	1.446495439636	1.557387474808
88	1.617104864312	1.645207887991	1.735523375692	1.908422904221
90	1.655894132724	1.685358775764	1.780303494655	1.963259707143
	60	75	88	90
1	0.017454134344	0.017454296541	0.017454354831	0.017454355911
15	0.264665130496	0.265236285116	0.265443180145	0.265447019880
30	0.547123080287	0.552336708043	0.554278260274	0.554314570939
45	0.868168661072	0.890401264284	0.899218368451	0.899386364852
60	1.253930907389	1.329257564848	1.363848295364	1.364541468240
75	1.731212785137	1.972040903456	2.137926356569	2.142013900670
88	2.216030126452	2.816582198285	4.175310739802	4.382938078332
90	2.293549650346	2.966009011167	5.154873005005	∞
	$\gamma^2 = 0.5$			
	1	15	30	45
1	0.017454178913	0.017454238003	0.017454400181	0.017454621735
15	0.264811133186	0.265012314941	0.265568698104	0.266338966933
30	0.548151967556	0.549801656749	0.554466014962	0.561188594815
45	0.870445649697	0.876209098974	0.893065728905	0.919022739166
60	1.253165008864	1.267259462391	1.310168161246	1.382180357781
75	1.709309567121	1.736946979963	1.824333511097	1.984641750819
88	2.151844214223	2.194604893074	2.333096359495	2.602617110801
90	2.221639684918	2.266850425642	2.413671504201	2.701287762095
	60	75	88	90
1	0.017454843304	0.017455005513	0.017455063808	0.017455064888
15	0.267121390856	0.267702093116	0.267912451904	0.267916355942
30	0.568365562104	0.573944469434	0.576022955047	0.576061831288
45	0.949385473370	0.975378987023	0.985713913052	0.985910974827
60	1.479063558781	1.578813355307	1.625064106611	1.625993807386
75	2.241555968376	2.606458401726	2.868209335253	2.874678895261
88	3.095288629009	4.097333673743	6.480005245197	6.851017961617
90	3.234773471249	4.366205147481	8.242640572377	∞

Heuman's lambda-function $\Lambda_0(\beta, k)$ and Jacobian zeta-function $Z(\beta, k)$

Complete elliptic integrals of the third kind can be expressed in terms of $\Lambda_0(\beta, k)$ and $Z(\beta, k)$, and therefore these will be summarized first. From Ref. [11], we note

$$\Lambda_0(\beta, k) = \frac{2}{\pi}[(E - K)F(\beta, k') + KE(\beta, k')], \tag{48}$$

$$Z(\beta, k) = E(\beta, k) - EF(\beta, k)/K, \tag{49}$$

where $k' = (1 - k^2)^{1/2}$, $E \equiv E(\pi/2, k)$, $K \equiv F(\pi/2, k)$.

Limiting cases are listed in Appendix A.

Elliptic integrals of the third kind

The elliptic integral of the third kind is given by the Legendre and Jacobi forms respectively,

$$\Pi(\theta, \gamma^2, k) = \int_0^\theta \frac{\mathrm{d}\psi}{(1 - \gamma^2 \sin^2 \psi)(1 - k^2 \sin^2 \psi)^{1/2}}, \tag{50}$$

$$= \int_0^y \frac{\mathrm{d}t}{(1 - \gamma^2 t^2)[(1 - t^2)(1 - k^2 t^2)]^{1/2}}, \tag{51}$$

where $y = \sin\theta$, $t = \sin\psi$ and $\gamma^2 \neq 1$, $\gamma^2 \neq k^2$.

This integral is *complete* when $\theta = \pi/2$ (or $y = 1$), and then the notation $\Pi(\gamma^2, k)$ is often used in the literature. Following Ref. [11], various cases of the elliptic integral of the third kind can be reduced to combinations of the first and second kind elliptic integrals. The *hyperbolic* cases are defined if (i) $\gamma^2 > 1$ or (ii) $0 < \gamma^2 < k^2$, and the *circular* cases occur when (iii) $\gamma^2 < 0$ and (iv) $k^2 < \gamma^2 < 1$. Both cases (i) and (iii) can be reduced to cases (ii) and (iv) respectively using

Table 5. Incomplete $\Pi(\theta, \gamma^2, \phi)$; $\gamma^2 = 0.9$, 1

$\theta°$ \ $\phi°$	1	15	30	45
	$\gamma^2 = 0.9$			
1	0.017454887928	0.017454947022	0.017455109213	0.017455330783
15	0.267311610497	0.267516201343	0.268082026367	0.268865400118
30	0.571068620117	0.572840946669	0.577854045902	0.585084455240
45	0.968565073983	0.975465627078	0.995689639173	1.026954262326
60	1.584686188326	1.605155590664	1.667876624374	1.774526374757
75	2.744639915491	2.799900256217	2.977101484511	3.312107513623
88	4.620017519437	4.739522277316	5.133937179732	5.933811793915
90	4.967868999231	5.099584555503	5.535513209603	6.425573644196
	60	75	88	90
1	0.017455552368	0.017455714589	0.017455772889	0.017455773968
15	0.269661175528	0.270251812985	0.270465776257	0.270469747206
30	0.592810387583	0.598820907975	0.601061292363	0.601103202435
45	1.063715776463	1.095352365590	1.107973699411	1.108214594931
60	1.920812149907	2.074876579981	2.147527913792	2.148996317919
75	3.876614376125	4.744332058197	5.404323238959	5.421258204038
88	7.505693899885	11.124061618136	21.517134981934	23.284483567519
90	8.200869161724	12.464091505630	30.304518759221	∞
	$\gamma^2 = 1$			
	1	15	30	45
1	0.017455065198	0.017455124293	0.017455286487	0.017455508061
15	0.267950129014	0.268155593835	0.268723839483	0.269510573079
30	0.577358455473	0.579164983705	0.584275373072	0.591647537839
45	1.000032684912	1.007311426564	1.028657249209	1.061695675463
60	1.732155119238	1.755647021548	1.827809262659	1.951138930286
75	3.732419888196	3.816547721377	4.088637756786	4.612796113312
88	28.640381402161	29.590854673433	32.802148252251	39.675239854077
90	∞	∞	∞	∞
	60	75	88	90
1	0.017455729650	0.017455891874	0.017455955376	0.017455951254
15	0.270309771690	0.270902956534	0.271117909952	0.271121832123
30	0.599526819407	0.605657988621	0.607943717461	0.607986405500
45	1.100604787410	1.134143595278	1.147537853149	1.147793574696
60	2.121599132946	2.302764655626	2.388787111951	2.390529756031
75	5.525541968744	7.003718597607	8.192303774545	8.223563231008
88	54.689422357519	98.276543996369	341.910456760807	412.291487581163
90	∞	∞	∞	∞

transformations given by Ref. [5; Section 17.7]. Expressions for limiting cases of the elliptic integral of the third kind are summarized in Appendix B. Here we note the hyperbolic case (ii) for the incomplete elliptic integral of the third kind, which may be expressed in terms of theta-function expansions.

Incomplete $\Pi(\theta, \gamma^2, k)$, $0 < \gamma^2 \leqslant k^2$, {hyperbolic}

When $\gamma^2 = k^2$, the integral is defined by equation (A.26). For $0 < \gamma^2 < k^2$, the integral reduces to

$$\Pi(\theta, \gamma^2, k) = F(\theta, k) + \frac{\gamma[F(\theta, k)Z(\beta, k) - \Omega_2]}{[(1-\gamma^2)(k^2-\gamma^2)]^{1/2}}, \tag{52}$$

where

$$\beta = \sin^{-1}(\gamma/k), \tag{53}$$

$$\Omega_2 = \frac{1}{2}\ln\left(\frac{\theta_4[v + \omega(\beta), q]}{\theta_4[v - \omega(\beta), q]}\right), \tag{54}$$

$$v = \pi F(\theta, k)/2K, \tag{55}$$

$$\omega(\beta) = \pi F(\beta, k)/2K \tag{56}$$

and $\theta_4(z, q)$ is defined in equation (4). Tabulations for $\Pi(\theta, \gamma^2, k)$ are shown in Tables 4 and 5 for $\gamma^2 \leqslant 1$.

CONCLUSIONS

An efficient and accurate methodology for computing incomplete elliptic integrals using theta-function expansions has been summarized and results have been provided in tabular form for several cases. Software has been provided with interactive codes based on the outlined material. Special forms in Appendices A and B have also been included in the codes.

Table 3 values can be compared to results in Ref. [5, Chap. 17]. Tables 4 and 5 were also compared to Ref. [11] whose authors used Simpson numerical integration to provide six decimal place accuracy. For the circular cases occurring when $k^2 < \gamma^2 < 1$, listed in Tables 4 and 5, the form given by Ref. [5, Section 17.7.11] was used. Complex arguments would otherwise occur using theta-function expansions, and these are not within the scope of this work.

Computations were compared to the process of the A.G.M. and found to be sufficiently accurate and efficient. These integrals have numerous applications both old and new and their efficient computation, particularly on a personal computer, provides the analyst with substantial savings over resorting to numerical integration schemes. Although accuracy is usually needed to only a few decimal places, particular applications sometimes require a series of these integrals, or ratios (i.e. Ref. [10]. In these cases, for adequate convergence, substantial decimal accuracy (10–16) is required. We also note a lesser known work by González [12], who provided compact expressions for incomplete elliptic integrals in terms of Legendre polynomial series. These were found to be less efficient, although quite accurate, requiring a convergence acceleration scheme (see Ref. [13]) over certain range of parameters. Although Carlson in Ref. [14] has provided robust schemes for elliptic functions, the object of this work was to summarize and clarify the use of theta-functions for evaluating elliptic integrals. Perhaps further work could be undertaken to compare more rigorously the duplication formulae given by Ref. [15], with the theta-function expansions shown here. Finally, the merit in this work is due to the research that was conducted by the many early mathematicians who devoted time towards functions which are not so well known, albeit remembered, today. Ironically, the use of these theta-functions vastly supersedes many present-day numerical integration techniques. Other applications of these functions can only be the subject of further research.

Acknowledgement—The authors wish to acknowledge the financial support of the Natural Sciences and Engineering Research Council of Canada under operating grant No. A7455.

REFERENCES

1. A. M. Legendre, *Traité des Fonctions Elliptiques et des Integrales Euleriennes*, Vol. 1. Paris (1825).
2. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (4th edn). University Press, New York (1927).
3. P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*. Springer, Berlin (1954).
4. M. M. Yovanovich, Special Functions with Applications. ME 755 Course Notes, Department of Mechanical Engineering, University of Waterloo (1982).
5. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1970).
6. E. Jahnke and F. Emde, *Tables of Functions* (4th edn). Dover, New York (1945).
7. J. D. Fenton and R. S. Gardiner-Garden, Rapidly-convergent methods for evaluating elliptic integrals and theta and elliptic functions. *J. Aust. math. Soc.* (Series B), **24**, 47–58 (1982).
8. S. Okui, Complete elliptic integrals resulting from infinite integrals of Bessel functions. Parts I and II. *J. Res. natn. Bur. Stand.* (U.S.), **78B**, **79B** (1974), (1975).
9. G. Eason, B. Noble and I. N. Sneddon, On certain integrals of Lipschitz–Hankel type involving products of Bessel functions. *Phil. Trans. R. Soc.*, Ser. A, **247**, 529–551 (1955).
10. T. F. Lemczyk and M. M. Yovanovich, Thermal constriction resistance with convective boundary conditions. Parts 1 and 2. *Int. J. Heat Mass Transfer* (in press).
11. R. G. Selfridge and J. E. Maxfield, *A Table of the Incomplete Elliptic Integral of the Third Kind*. Dover, New York (1958).
12. M. O. González, Elliptic integrals in terms of Legendre polynomials. *Proc. Glasg. math. Ass.* **2**, 97–99, (1954).
13. D. Levin and A. Sidi, Two new classes of nonlinear transformations for accelerating the convergence of infinite integrals and series. *Appl. Math. Comput.* **9**, 175–215 (1981).
14. B. C. Carlson and E. M. Notis, ALGORITHM 577, algorithms for incomplete elliptic integrals. *ACM Comput. Mach.* 7(3), 398–403 (1981).
15. B. C. Carlson, Computing elliptic integrals by duplication. *Num. Math.* **33**, 1–16 (1979).

APPENDIX A

Special Values

Complete elliptic integral K(k) and E(k)

$$E(1) = E'(0) = 1 \tag{A.1}$$

$$K(1) = K'(0) = \infty \tag{A.2}$$

$$K(0) = K'(1) = \pi/2 \tag{A.3}$$

$$E(0) = E'(1) = \pi/2. \tag{A.4}$$

Other special values and limiting cases can be found in Byrd and Friedman Ref. [3, Section 111].

Incomplete elliptic integrals $F(\theta, k)$ and $E(\theta, k)$

$$E(0, k) = F(0, k) = 0 \tag{A.5}$$

$$E(\theta, 0) = F(\theta, 0) = \theta \tag{A.6}$$

$$E(\theta, 1) = \sin\theta \tag{A.7}$$

$$F(\theta, 1) = \ln(\tan\theta + \sec\theta) \tag{A.8}$$

$$F(-\theta, k) = -F(\theta, k) \tag{A.9}$$

$$E(-\theta, k) = -E(\theta, k) \tag{A.10}$$

$$F(m\pi \pm \theta, k) = 2mK(k) \pm F(\theta, k) \tag{A.11}$$

$$E(m\pi \pm \theta, k) = 2mE(k) \pm E(\theta, k). \tag{A.12}$$

Complete elliptic integral $\Pi(\pi/2, \gamma^2, k)$

$$\Pi(\pi/2, \gamma^2, 1) = \Pi(\pi/2, 1, k) = \infty \tag{A.13}$$

$$\Pi(\pi/2, 0, k) = K(k) \tag{A.14}$$

$$\Pi(\pi/2, 0, 0) = \pi/2 \tag{A.15}$$

$$\Pi(\pi/2, \gamma^2 < 1, 0) = \frac{\pi}{2(1-\gamma^2)^{1/2}}. \tag{A.16}$$

Incomplete elliptic integral $\Pi(\theta, \gamma^2, k)$

$$\Pi(0, \gamma^2, k) = 0 \tag{A.17}$$

$$\Pi(\theta, 0, k) = F(\theta, k) \tag{A.18}$$

$$\Pi(\theta, 0, 1) = F(\theta, 1) = \ln(\tan\theta + \sec\theta) \tag{A.19}$$

$$\Pi(\theta, 1, 0) = \tan\theta \tag{A.20}$$

$$\Pi(\theta, \gamma^2 > 1, 0) = \frac{\tanh^{-1}[(\gamma^2-1)^{1/2}\tan\theta]}{(\gamma^2-1)^{1/2}} \tag{A.21}$$

$$\Pi(\theta, \gamma^2 < 1, 0) = \frac{\tanh^{-1}((1-\gamma^2)^{1/2}\tan\theta)}{(1-\gamma^2)^{1/2}} \tag{A.22}$$

$$\Pi(\theta, 1, k) = \frac{k'^2 F(\theta, k) - E(\theta, k) + \tan\theta(1-k^2\sin^2\theta)^{1/2}}{k'^2}; \quad k \neq 1 \tag{A.23}$$

$$\Pi(\theta, \gamma^2 > 0, 1) = \frac{\ln(\tan\theta + \sec\theta) - \gamma\ln\left[\dfrac{1+\gamma\sin\theta}{1-\gamma\sin\theta}\right]^{1/2}}{1-\gamma^2}; \quad \gamma^2 \neq 1 \tag{A.24}$$

$$\Pi(\theta, \gamma^2 < 0, 1) = \frac{\ln(\tan\theta + \sec\theta) + |\gamma|\tan^{-1}(|\gamma|\sin\theta)}{1-\gamma^2} \tag{A.25}$$

$$\Pi(\theta, k^2, k) = \frac{E(\theta, k) - (k^2\sin\theta\cos\theta)/(1-k^2\sin^2\theta)^{1/2}}{k'^2}; \quad k \neq 1 \tag{A.26}$$

$$\Pi(\theta, 1, 1) = \frac{\sin\theta}{2\cos^2\theta} + \frac{1}{2}\ln\left[\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right]. \tag{A.27}$$

Heuman lambda-function $\Lambda_0(\beta, k)$ and Jacobian zeta-function $Z(\beta, k)$

$$\Lambda_0(\pi/2, k) = 1; \quad \Lambda_0(m\pi/2, k) = m; \quad m = 0, 1, 2, \ldots \tag{A.28}$$

$$Z(\pi/2, k) = Z(0, k) = \Lambda_0(0, k) = Z(\beta, 0) = 0 \tag{A.29}$$

$$\Lambda_0(\beta, 0) = \sin\beta \tag{A.30}$$

$$\Lambda_0(\beta, 1) = 2\beta/\pi \tag{A.31}$$

$$\Lambda_0(-\beta, k) = -\Lambda_0(\beta, k) \tag{A.32}$$

$$\Lambda_0(m\pi \pm \beta, k) = 2m \pm \Lambda_0(\beta, k). \tag{A.33}$$

APPENDIX B

Complete Elliptic $\Pi(\pi/2, \gamma^2, k)$

Complete $\Pi(\gamma^2, k)$, $\gamma^2 < 0$, {*circular*}

If $\gamma^2 = -k$, then the integral reduces to

$$\Pi(-k, k) = \frac{1}{4(1+k)}[\pi + 2(1+k)K]. \tag{B.1}$$

For other cases, we note that there are two equivalent expressions

$$\Pi(\gamma^2, k) = \frac{k^2 K}{k^2 - \gamma^2} - \frac{\pi}{2}\frac{\gamma^2 \Lambda_0(\phi, k)}{[\gamma^2(1-\gamma^2)(\gamma^2 - k^2)]^{1/2}}, \tag{B.2}$$

or

$$\Pi(\gamma^2, k) = \frac{K}{1-\gamma^2} + \frac{\pi}{2}\frac{\gamma^2[\Lambda_0(\beta, k) - 1]}{[\gamma^2(1-\gamma^2)(\gamma^2 - k^2)]^{1/2}}, \tag{B.3}$$

where

$$\phi = \sin^{-1}\left(\frac{\gamma^2}{\gamma^2 - k^2}\right)^{1/2}, \quad \beta = \sin^{-1}\frac{1}{(1-\gamma^2)^{1/2}}. \tag{B.4}$$

Complete $\Pi(\gamma^2, k)$, $k^2 < \gamma^2 < 1$, {*circular*}

When $\gamma^2 = k^2$ or k, the special forms are:

$$\Pi(k^2, k) = \frac{E}{k'^2} \tag{B.5}$$

and

$$\Pi(k, k) = \frac{1}{4(1-k)}[\pi + 2(1-k)k]. \tag{B.6}$$

For other cases, there are two equivalent expressions:

$$\Pi(\gamma^2, k) = K + \frac{\pi}{2}\frac{\gamma(1 - \Lambda_0(\theta, k))}{[(\gamma^2 - k^2)(1-\gamma^2)]^{1/2}}, \tag{B.7}$$

or

$$\Pi(\gamma^2, k) = \frac{\pi}{2}\frac{\gamma \Lambda_0(\xi, k)}{[(\gamma^2 - k^2)(1-\gamma^2)]^{1/2}}, \tag{B.8}$$

where θ and ξ are defined as:

$$\theta = \sin^{-1}\left(\frac{1-\gamma^2}{1-k^2}\right)^{1/2}, \quad \xi = \sin^{-1}\left(\frac{\gamma^2 - k^2}{\gamma^2(1-k^2)}\right)^{1/2}. \tag{B.9}$$

Complete $\Pi(\gamma^2, k)$, $0 < \gamma^2 < k^2$, {*hyperbolic*}

One special case is defined here when $\gamma^2 = k^2$, hence this is given above. For other cases we note

$$\Pi(\gamma^2, k) = K + \frac{\gamma K Z(\beta, k)}{[(1-\gamma^2)(k^2 - \gamma^2)]^{1/2}}, \tag{B.10}$$

where

$$\beta = \sin^{-1}(\gamma/k). \tag{B.11}$$

Complete $\Pi(\gamma^2, k)$, $\gamma^2 > 1$, {*hyperbolic*}

This case is simply defined by

$$\Pi(\gamma^2, k) = -\frac{\gamma K Z(\beta, k)}{[(\gamma^2 - 1)(\gamma^2 - k^2)]^{1/2}}, \tag{B.12}$$

where

$$\beta = \sin^{-1}(1/\gamma). \tag{B.13}$$