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CHARACTERISTIC LENGTH OF COMPLEX BODIES FOR TRANSIENT CONDUCTION

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ABSTRACT

A general expression is presented for determining the characteristic body length of convex bodies for transient conduction. The proposed definition of the conduction characteristic length contains a geometry index which depends on two aspect ratios. It is consistent with the characteristic lengths required by the analytical solutions and the lumped-capacitance model for the classical bodies: infinite slab, infinitely long circular cylinder and sphere. The proposed length was verified by comparing the normalized heat loss from infinitely long square prismatic rods and circular cylinders for a wide range of the Fourier number and all values of the Biot number. The maximum percent difference was found to be less than 4.4 %; and it occurred at $Fo = 0.2$, the demarcation between short and long dimensionless times. The percent difference is less than 1 % for very short and very long times for all values of Bi. The square and circle results bound the solutions for all infinitely long prismatic rods of regular polygonal cross-sections. The square solution can be used to estimate the normalized heat loss from rods of equilateral triangle cross-section with an estimated maximum percent difference of less than 3 %. The proposed characteristic length is equivalent to the radius of the inscribed circle for the regular polygonal prismatic rods.

NOMENCLATURE

A	- surface area, m^2
AR_{21}, AR_{32}	- aspect ratios of a parallelepiped
A_n, B_n	- Fourier and Fourier-Bessel coefficients
$2a, 2b, 2c$	- sides of a parallelepiped, m
Bi	- Biot number, dimensionless thermal resistance ratio, $h\mathcal{L}/k$
c	- specific heat, $kJ/(kg \cdot K)$
Fo	- Fourier number, dimensionless time, $\alpha t/\mathcal{L}^2$
G	- geometry index, $G = 1 + 1/AR_{21} + 1/(AR_{21} \cdot AR_{32})$
h	- convective heat transfer coefficient, $W/(m^2 \cdot K)$
$J_0(\cdot), J_1(\cdot)$	- Bessel functions of the first kind of order 0 and 1 respectively
k	- thermal conductivity, $W/(m \cdot K)$
$\mathcal{L}$	- arbitrary characteristic body length, m
N	- number of sides of regular polygon, $3 \le N < \infty$
Q	- energy remaining in body, $\rho c V \bar{\theta}$, kJ
Q_i	- initial energy of body, $\rho c V \theta_i$, kJ
r_c	- infinite circular cylinder radius, m
r_s	- sphere radius, m
T	- constant temperature, K
T_f	- uniform fluid temperature, K
T_i	- initial temperature, K
t	- time, s
V	- volume, m^3
α	- thermal diffusivity, m^2/s
δ_n	- eigenvalues, roots of transcendental equations
ρ	- mass density, kg/m^3
θ	- temperature excess, $T - T_f$, K
θ_i	- initial temperature excess, $T_i - T_f$, K

INTRODUCTION

Transient conduction from bodies subjected to convective boundary conditions is characterized by the Biot and Fourier numbers. The Fourier number is an expression of dimensionless time,

$$Fo = \frac{\alpha t}{\mathcal{L}^2} \tag{1}$$

and the Biot number is the ratio of the internal resistance to the external film resistance,

$$Bi = \frac{h\mathcal{L}}{k} \tag{2}$$

Both dimensionless numbers require a length $\mathcal{L}$ characteristic of the conduction within and convection from the body. For analysis and evaluation purposes, the appropriate choice of characteristic body length is essential.

A survey of the literature leads one to the conclusion that there is no general method to determine the required characteristic length. For simple bodies such as infinite slabs (or plates), cylinders and spheres, the analytical solutions dictate appropriate characteristic lengths. These characteristic lengths have been used by a number of authors including Gurney and Lurie (1923), Newman (1936) and Heisler (1947) in their classic studies, and others [Jacob (1949), Schneider (1955, 1963), Pashkis and Hlinka (1957), Sha and Ganic

(1981)]. This method, however, cannot be used to provide correct values for more complex bodies such as infinitely long prismatic rods of polygonal cross-section and polyhedra.

A number of authors [Eckert (1959), Levenspiel (1984), Colakyan et al. (1984), Incropera and DeWitt (1990)] recommend the use of a general expression derived from the lumped-capacitance solution. However, this expression conflicts with the characteristic lengths of the simple bodies. Other definitions of characteristic lengths have been proposed [Olsen and Schultz (1942), Normington and Blackwell (1964), Schneider (1985)]; but these serve to add confusion because they are not general expressions and sometimes conflict with other definitions.

Therefore, the purpose of this paper is to present a general expression to determine the characteristic lengths of complex bodies which reduce to characteristic lengths commonly used for simple bodies such as infinite slabs, circular cylinders and spheres.

CHARACTERISTIC LENGTHS OF SIMPLE BODIES

Numerous studies have dealt with transient conduction from simple bodies such as infinite slabs, infinite circular cylinders and spheres. For these simple bodies, the choice of characteristic dimension is straight-forward; and from the geometry, the characteristic length is

$$(i) \quad \mathcal{L} = a \quad \text{(infinite slab)} \tag{3}$$
$$(ii) \quad \mathcal{L} = r_c \quad \text{(circular cylinder)} \tag{4}$$
$$(iii) \quad \mathcal{L} = r_s \quad \text{(sphere)} \tag{5}$$

where each length is the distance from the central adiabat to the solid-fluid or convection boundary.

These characteristic lengths are consistent with those required by the corresponding analytical solutions. For these simple bodies, the application of the convective boundary condition to the general solution produces transcendental equations whose roots are the solution eigenvalues δ_n. From Carslaw and Jaeger (1959), the transcendental equations are given as

$$(i) \quad \delta_n \tan \delta_n = \frac{ha}{k} \quad \text{(infinite slab)} \tag{6}$$
$$(ii) \quad \delta_n J_1(\delta_n) = \frac{hr_c}{k} J_0(\delta_n) \quad \text{(circular cylinder)} \tag{7}$$
$$(iii) \quad \left(1 - \frac{hr_s}{k}\right) \tan \delta_n = \delta_n \quad \text{(sphere)} \tag{8}$$

The dimensionless thermal resistance ratio in these equations is the Biot number. Therefore, the analytical solutions provide characteristic lengths which are the same as those suggested by the geometry.

Although the geometry and the analytical solutions provide suitable characteristic dimensions for these bodies, these results cannot be readily applied to more complex bodies. Consequently, a general expression for determining the characteristic length for complex bodies is required.

CHARACTERISTIC LENGTHS FROM LUMPED–CAPACITANCE SOLUTIONS

The lumped-capacitance analysis based on $Bi \to 0$ assumes uniform volumetric temperature and is applied to approximate heat transfer from bodies with negligible internal thermal resistance. The lumped-capacitance temperature solution for a cooling problem is

$$\frac{\theta}{\theta_i} = \frac{T - T_f}{T_i - T_f} = \exp\left(-\frac{hA}{\rho c V} t\right) \tag{9}$$

The thermal energy loss from the system can be obtained from the temperature solution by means of the following integral:

$$Q = \int_0^t hA\theta_i \exp\left(-\frac{hA}{\rho c V} t\right) dt \tag{10}$$

which gives

$$\frac{Q}{Q_i} = 1 - \exp\left(-\frac{hA}{\rho c V} t\right) \tag{11}$$

The argument of the exponential term of the temperature and heat transfer equations can be re-arranged

$$\frac{hA}{\rho c V} t = \left(\frac{h}{k}\right)\left(\frac{V}{A}\right)\left(\frac{k}{\rho c}\right)\left(\frac{A}{V}\right)^2 t \tag{12}$$

Letting

$$\frac{V}{A} = \mathcal{L} \tag{13}$$

we can write Eq. (9) as

$$\frac{\theta}{\theta_i} = \exp(-Bi\ Fo) \tag{14}$$

Equation (13) provides a simple general expression for determining the characteristic length. Colakyan et al. (1984) used this definition and showed that it is possible to *condense* the cooling history of variously shaped objects onto a single plot.

When the characteristic length of the infinite plate, circular cylinder and sphere are calculated using Eq. (13), the results are

$$(i) \quad \mathcal{L} = a \quad \text{(infinite slab)} \tag{15}$$
$$(ii) \quad \mathcal{L} = r_c/2 \quad \text{(infinite circular cylinder)} \tag{16}$$
$$(iii) \quad \mathcal{L} = r_s/3 \quad \text{(sphere)} \tag{17}$$

The characteristic lengths specified by analytical solutions for circular cylinders and spheres differ from those calculated using Eq. (13). Before the cooling charts of Gurney and Lurie (1923), Newman (1936) and Heisler (1947) can be used, the Biot and Fourier numbers whose characteristic lengths are defined by Eq. (13) must be converted.

Equation (13) provides a general expression for calculating the characteristic length. The expression is, however, unsuitable when applied to the circular cylinder and sphere as the length produced differs from that required by the analytical solutions for $Bi >> 0$. To avoid this confusion, a general expression which also provides the characteristic lengths of simple bodies is desired.

ALTERNATE DEFINITIONS OF CHARACTERISTIC LENGTH

Several authors [Olsen and Schultz (1942), Normington and Blackwell (1964), Schneider (1985)] have used other definitions for the characteristic body lengths. However, these definitions are not suitably general.

Olson and Schultz (1942) used the plate thickness $2a$ as the characteristic length for heat transfer from infinite slabs. This choice is made contrary to the results of the analytical solution. Normington and Blackwell (1964) examined heat transfer from prolate and oblate spheroids and used the focal length of the generating ellipse as the characteristic length. This definition cannot be applied to other bodies.

Schneider (1985) used the smallest dimension of interest in the problem as the characteristic length. While this definition gives the same characteristic length as that required by analytical solutions, it cannot be used for more complex irregular shapes.

GENERAL EXPRESSION FOR CHARACTERISTIC BODY LENGTHS

To provide suitable characteristic dimensions for all bodies, the following general expression is proposed:

$$\mathcal{L} = G\frac{V}{A} \tag{18}$$

where the geometry index G is defined with respect to the parallelepiped of dimensions $2a$ by $2b$ by $2c$ where $2a \leq 2b \leq 2c$:

$$G = 1 + 1/AR_{21} + 1/(AR_{21} \cdot AR_{32}) \tag{19}$$

The aspect ratios of the parallelepiped are defined as $AR_{21} = b/a \geq 1$ and $AR_{32} = c/b \geq 1$. The above definition of G reduces for the infinite slab ($AR_{21} \to \infty$) to

$$G = 1$$

for bodies finite in one-dimension; for the infinitely long square rod ($AR_{21} = 1$ and $AR_{32} \to \infty$) it reduces to

$$G = 2$$

for bodies finite in two-dimensions (eg. infinite regular polygonal prismatic rods and circular cylinders), and for $AR_{21} = 1$ and $AR_{32} = 1$ the general expression reduces to

$$G = 3$$

for bodies finite in three-dimensions (eg. cubes and spheres).

The above general expression provides the characteristic lengths for simple bodies such as infinite slabs, circular cylinders and spheres identical to those suggested by geometry and those required by the analytical solutions.

This expression also provides a characteristic dimension for bodies where a suitable length is not immediately evident. Langston (1982) has shown that heat transfer from an infinite rectangular rod with cross-section ($2a \times 2b$) can be calculated by superposing solutions for heat transfer from slabs of thickness $2a$ and $2b$. Since each slab has a characteristic length, the correct choice of characteristic length for the rectangular rod is unclear. For this shape and for the other bodies, the use of Eq. (19) resolves this conflict.

The use of a single, general expression also aids in the study of heat loss by conduction from complex bodies. When comparing normalized heat loss Q/Q_i from different bodies, calculated values are best compared when the dimensionless Biot and Fourier numbers are made equal. Using the general expression of Eq. (19), a consistent characteristic length can be calculated, thereby allowing valid comparisons.

A simple dividing line is needed to distinguish between bodies finite in one-, two- or three-dimensions and to allow the geometry index G to be determined. Olson and Schultz (1942) claim that even if great accuracy is required, a length can be labelled remote (or infinite) if it is 10 times the length of the shortest dimension. For example, a parallelepiped ($2a \times 2b \times 2c$) where $a < b < c$ has a geometry index $G = 3$ when $10 > c/a > b/a$; $G = 2$ when $c/a > 10 > b/a$ (infinite rectangular rod), and $G = 1$ when $c/a > b/a > 10$ (infinite slab).

■

COOLING OF INFINITE SLABS

The cooling (or heating) of a slab is a basic solution or building block for obtaining solutions for rectangular (or square) prismatic rods and parallelepipeds (or cubes). A brief summary of the results will be considered here.

The solution to the one-dimensional diffusion equation, $\alpha\partial^2\theta/\partial x^2 = \partial\theta/\partial t$, where $\theta(x,t) = T(x,t) - T_f$ is the instantaneous local temperature excess is [16]

$$\frac{\theta}{\theta_i} = \sum_{n=1}^{\infty} A_n \exp(-\delta_n^2 Fo)\cos(\delta_n \frac{x}{a}) \tag{20}$$

The Fourier coefficients A_n, obtained from the initial condition, $\theta(x,0) = \theta_i = T_i - T_f$, are given by

$$A_n = \frac{2\sin\delta_n}{\delta_n + \sin\delta_n \cos\delta_n} \tag{21}$$

and δ_n are the roots of the transcendental equation

$$\delta_n \sin\delta_n = Bi\cos\delta_n \tag{22}$$

where $Bi = ha/k$, is the slab Biot number and a is the slab half-width.

The instantaneous area-mean temperature excess defined as

$$\bar{\theta}(t) = \frac{1}{a}\int_0^a \theta(x,t)dx \tag{23}$$

is obtained from

$$\frac{\bar{\theta}(t)}{\theta_i} = \sum_{n=1}^{\infty} B_n \exp(-\delta_n Fo) \tag{24}$$

and the Fourier coefficients, B_n are given by

$$B_n = \frac{2\sin^2\delta_n}{\delta_n(\delta_n + \sin\delta_n\cos\delta_n)} \tag{25}$$

During the cooling (or heating) process the slab loses (or gains) a definite quantity of thermal energy, Q, and three methods can be used to find Q. The first method is based on the temporal integration of the conduction rate out of the slab at the solid-fluid boundaries; the second method is based on the temporal integration of the convection loss through the solid-fluid boundary; and the third method is based on an enthalpy balance of the slab between the initial and final cooling times.

The three methods give the same result:

$$\frac{Q}{Q_i} = 1 - \sum_{n=1}^{\infty} B_n \exp(-\delta_n^2 Fo) \tag{26}$$

where $Q_i = 2aA\rho c\theta_i$ is the initial enthalpy of the slab.

Computation of Eigenvalues

The above equations for the instantaneous and area-mean temperature excess, and the thermal energy loss from the slab require the evaluation of the Fourier coefficients, A_n and B_n. The Fourier coefficients are dependent on the eigenvalues δ_n which are the roots of Eq. (22).

The iterative Newton-Raphson technique is one method which can be used to calculate the roots accurately. To compute each root δ_n for a particular value of Bi, one can iterate until convergence the following expression:

$$\delta_n^{i+1} = \delta_n^i - \frac{\delta_n^i \sin\delta_n^i - Bi\cos\delta_n^i}{\delta_n^i\cos\delta_n^i + (1+Bi)\sin\delta_n^i} \tag{27}$$

The number of iterations can be reduced significantly by means of the following approximations of the first root of the transcendental equation:

$$\delta_1 = \left\{\frac{3Bi}{3+Bi}\left[1 - \frac{1}{45}\left(\frac{3Bi}{3+Bi}\right)^2\right]\right\}^{1/2} \tag{28}$$

for small values of $0 \le Bi \le 2$, and for all $Bi \ge 2$ use

$$\delta_1 = \frac{\pi Bi}{2(1+Bi)}\left[1 + \frac{\pi^2}{12(1+Bi)^3 + \pi^2(2Bi-1)}\right] \quad (29)$$

The maximum error for both approximations of δ_1 is less than 0.1% and it occurs at $Bi = 2$. To initiate the iterative procedure for the second and higher order roots let the value of the first guess be $\delta_2^1 = \delta_1(converged) + \pi$, $\delta_3^1 = \delta_2(converged) + \pi$, $\cdots$, etc.

A computational study shows that only 5 terms of the summation are required to obtain four decimal place accuracy for all Bi provided the dimensionless time, $Fo \ge 0.02$; otherwise more terms are required to achieve the same accuracy.

The thermal energy loss from the slab during very short time intervals $Fo \le 0.02$ can be accurately computed from the half-space solution [16] which when divided by $Q_i = \rho c a A \theta_i$ gives:

$$\frac{Q}{Q_i} = \frac{2}{\sqrt{\pi}}\sqrt{Fo} - \frac{1}{Bi}\left[1 - \text{erfc}(Bi\sqrt{Fo})\exp(Bi^2 Fo)\right] \quad (30)$$

For $Bi \le 2$, the above result can be used for maximum dimensionless times determined by the following relationship:

$$Fo_{max} = \left[\frac{1.0 + \sqrt{1.0 - 0.5Bi}}{Bi}\right]^2 \quad (31)$$

and for $Bi > 2$, $Fo_{max} = 0.0625$.

The half-space result should not be used for very small Bi and very small Fo when negligible heat loss has occurred; the lumped-capacitance model is more suitable for these cases. The half-space result is recommended for all $Bi \ge 2$ and all $Fo \le 0.06$.

TWO– AND THREE– DIMENSIONAL SOLUTIONS

One can combine one-dimensional solutions developed for the slab in a straight-forward manner to obtain solutions for multidimensional systems such as rectangular (or square) prismatic rods and parallelepipeds (or cubes). Simple analysis shows that if $(\theta/\theta_i)_{2a}$ and $(\theta/\theta_i)_{2b}$ are temperature solutions for infinite slabs of thickness $2a$ and $2b$ respectively, the temperature solution for a rectangular prismatic rod of dimensions $2a$ by $2b$ is given by the product of the two independent slab solutions:

$$\left(\frac{\theta}{\theta_i}\right)_{2D} = \left(\frac{\theta}{\theta_i}\right)_{2a} \cdot \left(\frac{\theta}{\theta_i}\right)_{2b} \quad (32)$$

Similarly, for a parallelepiped of dimensions $2a$ by $2b$ by $2c$, the temperature solution is given by the product of three slab solutions:

$$\left(\frac{\theta}{\theta_i}\right)_{3D} = \left(\frac{\theta}{\theta_i}\right)_{2a} \cdot \left(\frac{\theta}{\theta_i}\right)_{2b} \cdot \left(\frac{\theta}{\theta_i}\right)_{2c} \quad (33)$$

Langston (1982) has demonstrated that one can obtain the thermal energy loss for two- and three-dimensional systems from the above equations. For one-dimensional systems of width $2a$ and $2b$ respectively, one has

$$\left(\frac{Q}{Q_i}\right)_{2a} = 1 - \frac{1}{a}\int_0^a \left(\frac{\theta}{\theta_i}\right)_{2a} dx \quad (34)$$

and

$$\left(\frac{Q}{Q_i}\right)_{2b} = 1 - \frac{1}{b}\int_0^b \left(\frac{\theta}{\theta_i}\right)_{2b} dy \quad (35)$$

However, for a rectangular prismatic rod one has

$$\left(\frac{Q}{Q_i}\right)_{2D} = 1 - \frac{1}{ab}\int_0^a\int_0^b \left(\frac{\theta}{\theta_i}\right)_{2D} dydx \quad (36)$$

which can be written as

$$\left(\frac{Q}{Q_i}\right)_{2D} = 1 - \left[\frac{1}{a}\int_0^a \left(\frac{\theta}{\theta_i}\right)_{2a} dx\right]\left[\frac{1}{b}\int_0^b \left(\frac{\theta}{\theta_i}\right)_{2b} dy\right] \quad (37)$$

The above results lead to the following expression:

$$\left(\frac{Q}{Q_i}\right)_{2D} = 1 - \left[1 - \left(\frac{Q}{Q_i}\right)_{2a}\right]\left[1 - \left(\frac{Q}{Q_i}\right)_{2b}\right] \quad (38)$$

Therefore, expanding the previous result gives

$$\left(\frac{Q}{Q_i}\right)_{2D} = \left(\frac{Q}{Q_i}\right)_{2a} + \left(\frac{Q}{Q_i}\right)_{2b} - \left(\frac{Q}{Q_i}\right)_{2a} \cdot \left(\frac{Q}{Q_i}\right)_{2b} \quad (39)$$

In a similar manner one can extend the above two-dimensional results to three-dimensional systems to obtain for the parallelepiped:

$$\left(\frac{Q}{Q_i}\right)_{3D} = 1 - \left[1 - \left(\frac{Q}{Q_i}\right)_{2a}\right]\left[1 - \left(\frac{Q}{Q_i}\right)_{2b}\right]\left[1 - \left(\frac{Q}{Q_i}\right)_{2c}\right] \quad (40)$$

The expressions for the square prismatic rod and the cube can be obtained from the above general two- and three-dimensional results:

$$\left(\frac{Q}{Q_i}\right)_{square} = 2\left(\frac{Q}{Q_i}\right)_{2a} - \left(\frac{Q}{Q_i}\right)_{2a}^2 \quad (41)$$

and

$$\left(\frac{Q}{Q_i}\right)_{cube} = 3\left(\frac{Q}{Q_i}\right)_{2a} - 3\left(\frac{Q}{Q_i}\right)_{2a}^2 + \left(\frac{Q}{Q_i}\right)_{2a}^3 \quad (42)$$

INFINITE CIRCULAR CYLINDER SOLUTION

Heat transfer from infinite circular cylinders is reviewed because it is an important analytic solution, and also the circular cylinder represents an N–sided regular polygon with $N \to \infty$. The temperature solution of the one-dimensional diffusion equation $\alpha[\partial^2\theta/\partial r^2 + (1/r)\partial\theta/\partial r] = \partial\theta/\partial t$ within a solid cylinder of radius a is [16]:

$$\frac{\theta}{\theta_i} = \sum_{n=1}^{\infty} A_n \exp(-\delta_n^2 Fo) J_0(\delta_n \frac{r}{a}) \quad (43)$$

where $Fo = \alpha t/a^2$, and the Fourier-Bessel coefficients A_n are given by

$$A_n = \frac{2Bi}{(\delta_n^2 + Bi^2)J_0(\delta_n)} \quad (44)$$

with $Bi = ha/k$. The eigenvalues δ_n are the roots of the transcendental equation:

$$\delta_n J_1(\delta_n) = Bi J_0(\delta_n) \quad (45)$$

where $J_0(\cdot)$ and $J_1(\cdot)$ are Bessel functions of the first kind of order zero and one respectively. These special functions can be easily and accurately computed using the polynomial approximations presented in Abramowitz and Stegun (1980).

The area-mean instantaneous temperature excess is defined as

$$\bar{\theta} = \frac{1}{\pi a^2}\int_0^a \theta(r,t) 2\pi r dr \quad (46)$$

which leads to

$$\frac{\bar{\theta}}{\theta_i} = \sum_{n=1}^{\infty} B_n \exp(-\delta_n^2 Fo) \quad (47)$$

The thermal energy loss from a cylinder of length $2L$ and of radius $2a$ during the time interval from $t = 0$ to $t = t$ is

$$Q = \int_0^t 4\pi L\left(-\frac{\partial\theta}{\partial r}\right) r dr \quad (48)$$

which gives

$$\frac{Q}{Q_i} = 1 - \sum_{n=1}^{\infty} B_n \exp(-\delta_n^2 Fo) \tag{49}$$

where $Q_i = 2\pi a^2 L\rho c\theta_i$ is the initial internal thermal energy of the cylinder. The Fourier-Bessel coefficients B_n are found from the following reduced equation:

$$B_n = \frac{4Bi^2}{\delta_n^2(\delta_n^2 + Bi^2)} \tag{50}$$

The eigenvalues can be computed using the iterative Newton-Raphson method applied to the following equation:

$$\delta_n^{i+1} = \delta_n^i - \frac{\delta_n^i J_1(\delta_n^i) - Bi J_0(\delta_n^i)}{\delta_n^i J_0(\delta_n^i) + Bi J_1(\delta_n^i)} \tag{51}$$

The first eigenvalue lies in the interval $0 \leq \delta_1 \leq 2.40483$ for $0 \leq Bi < \infty$. One can reduce the number of iterations for $n \geq 2$ by letting the first guess of $\delta_2^1 = \delta_1(converged) + \pi$, $\delta_3^1 = \delta_2(converged) + \pi, \cdots$, etc. For $Fo \geq 0.02$, only 5 terms of the summation are required to give four decimal place accuracy.

COMPARISON OF NORMALIZED HEAT LOSS FROM INFINITELY LONG SQUARE RODS AND CYLINDERS

The results of the above reviews and the proposed general characteristic body length will be used to compare the heat loss from infinitely long square rods and circular cylinders for a wide range of Fo and all values of Bi. This is an important comparison because the square and circular cross-sections can be classified as regular polygons of $N = 4$ and $N \to \infty$ respectively; therefore, these geometries bound an infinite set of regular polygons.

Heat loss from the square and circular cross-sections can be determined accurately using the known analytical solutions. However, both solutions require a characteristic body length, namely the half-width for the square and the radius for the circle. The proposed general characteristic body length, $\mathcal{L} = GV/A$, which becomes

$$\mathcal{L} = 2\,\frac{cross - sectional\ area}{convection\ \ perimeter}$$

will be used in the two independent parameters: $Bi = h\mathcal{L}/k$ and $Fo = \alpha t/\mathcal{L}^2$ which determine the solution.

For both solutions described above, the eigenvalues δ_n are calculated using the Newton-Raphson method discussed earlier. Successive iterations on the roots are continued until

$$\left| \frac{\delta_n^{new} - \delta_n^{old}}{\delta_n^{new}} \right| \leq 10^{-5}$$

In order to obtain very accurate solutions for this comparison, checks for the convergence of the infinite summations are made by comparing the result after 10 additional terms have been included in the summation, and applying a maximum tolerance on the difference:

$$\left| \frac{\left(\frac{Q}{Q_i}\right)_{n+10} - \left(\frac{Q}{Q_i}\right)_n}{\left(\frac{Q}{Q_i}\right)_{n+10}} \right| \leq 10^{-10}$$

An upper limit of 2000 terms was applied to the infinite summations.

The heat loss from the square and circular cross-sections are compared by computing the percent difference:

$$\%\ difference = \left| \frac{\left(\frac{Q}{Q_i}\right)_{circle} - \left(\frac{Q}{Q_i}\right)_{square}}{\left(\frac{Q}{Q_i}\right)_{square}} \right| \times 100\% \tag{52}$$

The percent difference was computed for Fo ranging from 10^{-4} to 7.0×10^3 for Bi ranging from 0.001 to 160 which essentially corresponds to an infinite value of Bi. The normalized heat loss Q/Q_i ranged from 2.0×10^{-4} to 1. Tabulated results are presented in Tables 1 through 13 for selected values of Bi.

From these tables it can be seen that the percent difference is less than 1 % for all values of Fo when $Bi \leq 0.1$. This is to be expected because the lumped-capacitance model applies to both geometries when $Bi \leq 0.1$. The percent difference continues to be less than 1 % for very short times and very long times for all values of Bi. The percent difference exceeds 1 % but never rises above 4.4 % for all other values of Fo and Bi. The maximum percent difference appears to occur in the range $0.2 \leq Fo \leq 0.5$ for $1 \leq Bi \leq 160$. In general, the maximum percent difference for $Bi \geq 2$ occurs at $Fo = 0.2$ which is often said to be the demarcation between short and long time solutions. The heat loss from the circular cross-section exceeds that of the square cross-section for all values of Bi and for all $Fo > 0$.

DISCUSSION AND CONCLUSIONS

A general expression is required to provide characteristic body lengths for transient conduction within complex bodies. For simple bodies such as infinite slabs, infinite circular cylinders and spheres, the general expression must reduce to the characteristic length provided by the analytical series solutions. Present expressions are either insufficiently general or provide conflicting characteristic lengths for simple bodies.

A novel expression has been proposed to calculate the characteristic body length of non-classical two- and three-dimensional bodies such as infinitely long prismatic rods of regular polygonal cross-section, and parallelepipeds. This simple expression is based on the lumped-capacitance model as well as the analytical solutions for slabs, infinitely long circular cylinders, spheres and cubes. The expression was derived from the lump-capacitance model applied to a parallelepiped, and, therefore, it contains a geometry index which is a function of two aspect ratios which are defined to be equal to or greater than unity.

It can be easily applied to all bodies; when applied to simple bodies, the proposed body length agrees with those body lengths required by the analytical solutions.

The percent difference between the normalized heat loss for the infinitely long square and circular cross-sections, whose solutions were based on the proposed characteristic body length, was found to be less than 4.4 %. This maximum difference occurs at a dimensionless time of approximately 0.2 for Biot numbers greater than 2. Otherwise, the percent difference was found to be less than 1 % for very small and very long dimensionless times.

Solutions for infinitely long prismatic rods of regular polygonal shape $4 < N < \infty$, where N denotes the number of sides, are presently unavailable. Since the square and circular cross-sections bound the infinite set of regular polygonal geometries, i.e., $4 \leq N < \infty$, the normalized heat loss for these geometries is expected to lie between the results for the square and the circle reported in this study. The arithmetic mean of the square and circle results can provide a good approximation for these two-dimensional geometries. Indeed, the square results can provide a good approximation for the prismatic rod whose cross-section is an equilateral triangle ($N = 3$) with a maximum percent difference estimated to be less than three percent provided the proposed characteristic body length is used.

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Percent Difference Between $(Q/Q_i)_{square}$ and $(Q/Q_i)_{circle}$

Dimensionless Time Fo	$\frac{Q}{Q_i}$		Percent Difference
	Square Rod	Circular Cylinder	
1.0×10^{-1}	1.9994×10^{-4}	1.9995×10^{-4}	3.7×10^{-3}
1.5	2.9989	2.9990	4.3
2.0	3.9983	3.9984	4.9
3.0	5.9966	5.9970	5.8
5.0	9.9921	9.9928	7.0
7.0	1.3986×10^{-3}	1.3987×10^{-3}	7.7
1.0×10^{0}	1.9974	1.9975	8.1
1.5	2.9946	2.9948	8.5
2.0	3.9907	3.9911	8.6
3.0	5.9801	5.9806	8.8
5.0	9.9469	9.9478	9.0
7.0	1.3898×10^{-2}	1.3899×10^{-2}	9.0
1.0×10^{1}	1.9795	1.9797	9.0
1.5	2.9545	2.9547	9.0
2.0	3.9198	3.9201	9.0
3.0	5.8217	5.8222	8.9
5.0	9.5132	9.5141	8.8
7.0	1.3060×10^{-1}	1.3061×10^{-1}	8.7
1.0×10^{2}	1.8121	1.8123	8.4
1.5	2.5911	2.5913	8.0
2.0	3.2959	3.2962	7.7
3.0	4.5108	4.5111	7.0
5.0	6.3200	6.3203	5.7
7.0	7.5329	7.5332	4.7
1.0×10^{3}	8.6457	8.6460	3.5
1.5	9.5016	9.5018	2.2
2.0	9.8166	9.8167	1.5
3.0	9.9752	9.9753	1.0
5.0	9.9995	9.9996	8.9×10^{-4}
7.0	1.0000×10^{0}	1.0000×10^{0}	0.0

Table 1 Biot = 0.001

Dimensionless Time Fo	$\frac{Q}{Q_i}$		Percent Difference
	Square Rod	Circular Cylinder	
1.0×10^{-1}	1.9943×10^{-3}	1.9947×10^{-3}	2.1×10^{-2}
1.5	2.9890	2.9899	2.9
2.0	3.9826	3.9840	3.6
3.0	5.9664	5.9692	4.7
5.0	9.9216	9.9275	6.0
7.0	1.3861×10^{-2}	1.3870×10^{-2}	6.6
1.0×10^{0}	1.9740	1.9754	7.1
1.5	2.9462	2.9484	7.4
2.0	3.9087	3.9117	7.6
3.0	5.8052	5.8096	7.7
5.0	9.4866	9.4939	7.7
7.0	1.3024×10^{-1}	1.3034×10^{-1}	7.6
1.0×10^{1}	1.8073	1.8086	7.4
1.5	2.5845	2.5863	7.1
2.0	3.2879	3.2901	6.7
3.0	4.5010	4.5037	6.0
5.0	6.3090	6.3120	4.8
7.0	7.5225	7.5254	3.8
1.0×10^{2}	8.6376	8.6399	2.6
1.5	9.4971	9.4984	1.3
2.0	9.8144	9.8150	6.2×10^{-3}
3.0	9.9747	9.9748	1.3
5.0	9.9995	9.9995	4.0×10^{-5}
7.0	1.0000×10^{0}	1.0000×10^{0}	0.0

Table 2 Biot = 0.01

Dimensionless Time Fo	$\frac{Q}{Q_i}$		Percent Difference
	Square Rod	Circular Cylinder	
1.0×10^{-2}	1.9841×10^{-3}	1.9845×10^{-3}	2.3×10^{-2}
1.5	2.9704	2.9714	3.5
2.0	3.9539	3.9557	4.6
3.0	5.9139	5.9179	6.7
5.0	9.8101	9.8207	1.1×10^{-1}
7.0	1.3679×10^{-2}	1.3699×10^{-2}	1.5
1.0×10^{-1}	1.9439	1.9478	2.0
1.5	2.8936	2.9016	2.8
2.0	3.8319	3.8451	3.4
3.0	5.6781	5.7035	4.5
5.0	9.2603	9.3119	5.5
7.0	1.2705×10^{-1}	1.2782×10^{-1}	6.0
1.0×10^{0}	1.7629	1.7740	6.3
1.5	2.5225	2.5385	6.3
2.0	3.2121	3.2319	6.1
3.0	4.4063	4.4315	5.7
5.0	6.2014	6.2304	4.6
7.0	7.4205	7.4482	3.7
1.0×10^{1}	8.5565	8.5787	2.6
1.5	9.4514	9.4641	1.3
2.0	9.7915	9.7980	6.6×10^{-2}
3.0	9.9699	9.9713	1.4
5.0	9.9994	9.9994	4.9×10^{-4}
7.0	1.0000×10^{0}	1.0000×10^{0}	0.0

Table 3 Biot = 0.1

Dimensionless Time Fo	$\frac{Q}{Q_i}$		Percent Difference
	Square Rod	Circular Cylinder	
1.0×10^{-3}	1.9525×10^{-3}	1.9531×10^{-3}	3.1×10^{-2}
1.5	2.9127	2.9138	4.0
2.0	3.8656	3.8675	5.1
3.0	5.7532	5.7574	7.2
5.0	9.4695	9.4804	1.1×10^{-1}
7.0	1.3122×10^{-2}	1.3143×10^{-2}	1.6
1.0×10^{-2}	1.8503	1.8544	2.2
1.5	2.7257	2.7343	3.1
2.0	3.5787	3.5932	4.0
3.0	5.2295	5.2598	5.8
5.0	8.3558	8.4307	8.9
7.0	1.1297×10^{-1}	1.1430×10^{-1}	1.18×10^{0}
1.0×10^{-1}	1.5434	1.5673	1.55
1.5	2.1747	2.2194	2.06
2.0	2.7480	2.8148	2.44
3.0	3.7574	3.8664	2.90
5.0	5.3610	5.5262	3.08
7.0	6.5501	6.7364	2.84
1.0×10^{0}	7.7873	7.9665	2.30
1.5	8.9445	9.0757	1.47
2.0	9.4965	9.5799	8.7×10^{-1}
3.0	9.8854	9.9132	2.8
5.0	9.9941	9.9963	2.2×10^{-2}
7.0	9.9997	9.9998	1.5×10^{-3}
1.0×10^{0}	1.0000×10^{0}	1.0000×10^{0}	0.0

Table 4 Biot = 1.0

Dimensionless Time Fo	$\frac{Q}{Q_i}$		Percent Difference
	Square Rod	Circular Cylinder	
1.0×10^{-3}	3.8138×10^{-3}	3.8160×10^{-3}	5.8×10^{-2}
1.5	5.6596	5.6640	7.8
2.0	7.4781	7.4854	9.9
3.0	1.1048×10^{-2}	1.1063×10^{-2}	1.4×10^{-1}
5.0	1.7975	1.8014	2.2
7.0	2.4677	2.4749	2.9
1.0×10^{-2}	3.4394	3.4531	4.0
1.5	4.9873	5.0154	5.6
2.0	6.4629	6.5093	7.1
3.0	9.2432	9.3356	9.9
5.0	1.4291×10^{-1}	1.4503×10^{-1}	1.49×10^{0}
7.0	1.8831	1.9190	1.90
1.0×10^{-1}	2.4940	2.5543	2.42
1.5	3.3745	3.4782	3.07
2.0	4.1281	4.2731	3.51
3.0	5.3623	5.5718	3.91
5.0	7.0882	7.3462	3.64
7.0	8.1691	8.4091	2.94
1.0×10^{0}	9.0870	9.2616	1.92
1.5	9.7137	9.7946	8.3×10^{-1}
2.0	9.9102	9.9429	3.3
3.0	9.9912	9.9956	4.5×10^{-2}
5.0	9.9999	1.0000×10^{0}	8.9×10^{-4}
7.0	1.0000×10^{0}	1.0000	0.0

Table 5 Biot = 2.0

Dimensionless Time Fo	$\frac{Q}{Q_i}$		Percent Difference
	Square Rod	Circular Cylinder	
1.0×10^{-4}	7.7640×10^{-4}	7.7667×10^{-4}	3.5×10^{-2}
1.5	1.1568×10^{-3}	1.1572×10^{-3}	3.3
2.0	1.5338	1.5343	3.4
3.0	2.2791	2.2800	3.8
5.0	3.7425	3.7445	5.2
7.0	5.1769	5.1804	6.7
1.0×10^{-3}	7.2849	7.2915	9.0
1.5	1.0704×10^{-2}	1.0718×10^{-2}	1.3×10^{-1}
2.0	1.4028	1.4052	1.7
3.0	2.0447	2.0496	2.4
5.0	3.2584	3.2705	3.7
7.0	4.4011	4.4227	4.9
1.0×10^{-2}	6.0137	6.0533	6.5
1.5	8.4963	8.5734	9.0
2.0	1.0783×10^{-1}	1.0905×10^{-1}	1.13×10^{0}
3.0	1.4923	1.5150	1.52
5.0	2.2009	2.2481	2.15
7.0	2.8019	2.8758	2.64
1.0×10^{-1}	3.5681	3.6828	3.22
1.5	4.6042	4.7822	3.87
2.0	5.4375	5.6672	4.22
3.0	6.7057	6.9962	4.33
5.0	8.2656	8.5504	3.45
7.0	9.0853	9.3002	2.37
1.0×10^{0}	9.6496	9.7653	1.20
1.5	9.9292	9.9620	3.3×10^{-1}
2.0	9.9857	9.9939	8.2×10^{-2}
3.0	9.9994	9.9999	4.7×10^{-3}
5.0	1.0000×10^{0}	1.0000×10^{0}	0.0

Table 6 Biot = 4.0

Percent Difference Between $(Q/Q_i)_{square}$ and (Q/Q_i)...

Dimensionless Time Fo	Q/Q_i		Percent Difference
	Square Rod	Circular Cylinder	
1.0×10^{-4}	1.1476×10^{-3}	1.1480×10^{-3}	3.6×10^{-2}
1.5	1.7044	1.7050	3.7
2.0	2.2538	2.2546	3.9
3.0	3.3341	3.3357	4.8
5.0	5.4370	5.4407	6.9
7.0	7.4794	7.4862	9.1
1.0×10^{-3}	1.0453×10^{-2}	1.0466×10^{-2}	1.2×10^{-1}
1.5	1.5221	1.5248	1.8
2.0	1.9799	1.9844	2.3
3.0	2.8513	2.8605	3.2
5.0	4.4628	4.4846	4.9
7.0	5.9459	5.9839	6.3
1.0×10^{-2}	7.9946	8.0617	8.3
1.5	1.1067×10^{-1}	1.1192×10^{-1}	1.13×10^{0}
2.0	1.3828	1.4019	1.38
3.0	1.8692	1.9029	1.80
5.0	2.6710	2.7366	2.46
7.0	3.3281	3.4262	2.95
1.0×10^{-1}	4.1420	4.2872	3.50
1.5	5.2085	5.4215	4.09
2.0	6.0414	6.3049	4.36
3.0	7.2666	7.5776	4.28
5.0	8.6830	8.9543	3.13
7.0	9.3644	9.5484	1.96
1.0×10^{0}	9.7869	9.8718	8.6×10^{-1}
1.5	9.9655	9.9843	1.9
2.0	9.9944	9.9981	3.6×10^{-2}
3.0	9.9999	9.9999	9.4×10^{-4}
5.0	1.0000×10^{0}	1.0000×10^{0}	0.0

Table 7 Biot = 6.0

Dimensionless Time Fo	Q/Q_i		Percent Difference
	Square Rod	Circular Cylinder	
1.0×10^{-4}	1.5080×10^{-3}	1.5086×10^{-3}	4.0×10^{-2}
1.5	2.2327	2.2337	4.2
2.0	2.9447	2.9460	4.7
3.0	4.3375	4.3400	5.9
5.0	7.0265	7.0326	8.7
7.0	9.6158	9.6268	1.1×10^{-1}
1.0×10^{-3}	1.3354×10^{-2}	1.3375×10^{-2}	1.6
1.5	1.9281	1.9324	2.2
2.0	2.4911	2.4981	2.8
3.0	3.5491	3.5629	3.9
5.0	5.4684	5.5002	5.8
7.0	7.2018	7.2557	7.4
1.0×10^{-2}	9.5552	9.6478	9.6
1.5	1.3015×10^{-1}	1.3181×10^{-1}	1.28×10^{0}
2.0	1.6067	1.6314	1.54
3.0	2.1342	2.1762	1.97
5.0	2.9827	3.0607	2.61
7.0	3.6635	3.7766	3.09
1.0×10^{-1}	4.4929	4.6552	3.61
1.5	5.5608	5.7913	4.14
2.0	6.3816	6.6599	4.36
3.0	7.5666	7.8818	4.17
5.0	8.8879	9.1444	2.89
7.0	9.4910	9.6543	1.72
1.0×10^{0}	9.8424	9.9112	6.9×10^{-1}
1.5	9.9777	9.9908	1.3
2.0	9.9968	9.9990	2.2×10^{-2}
3.0	9.9999	1.0000×10^{0}	3.0×10^{-4}
5.0	1.0000×10^{0}	1.0000	0.0

Table 8 Biot = 8.0

Dimensionless Time Fo	Q/Q_i		Percent Difference
	Square Rod	Circular Cylinder	
1.0×10^{-4}	1.8581×10^{-3}	1.8589×10^{-3}	4.1×10^{-2}
1.5	2.7427	2.7439	4.5
2.0	3.6080	3.6099	5.1
3.0	5.2927	5.2962	6.7
5.0	8.5198	8.5284	1.0×10^{-1}
7.0	1.1602×10^{-2}	1.1617×10^{-2}	1.3
1.0×10^{-3}	1.6016	1.6045	1.8
1.5	2.2946	2.3004	2.5
2.0	2.9463	2.9557	3.2
3.0	4.1570	4.1755	4.4
5.0	6.3178	6.3590	6.5
7.0	8.2384	8.3068	8.2
1.0×10^{-2}	1.0810×10^{-1}	1.0925×10^{-1}	1.06×10^{0}
1.5	1.4533	1.4732	1.37
2.0	1.7771	1.8062	1.64
3.0	2.3293	2.3774	2.06
5.0	3.2028	3.2890	2.69
7.0	3.8940	4.0167	3.15
1.0×10^{-1}	4.7275	4.9001	3.65
1.5	5.7896	6.0295	4.06
2.0	6.5981	6.8832	4.32
3.0	7.7514	8.0661	4.06
5.0	9.0076	9.2523	2.72
7.0	9.5615	9.7108	1.56
1.0×10^{0}	9.8712	9.9304	6.0×10^{-1}
1.5	9.9833	9.9935	1.0
2.0	9.9978	9.9994	1.5×10^{-2}
3.0	1.000000×10^{0}	1.000000×10^{0}	0.0

Table 9 Biot = 10.0

Dimensionless Time Fo	Q/Q_i		Percent Difference
	Square Rod	Circular Cylinder	
1.0×10^{-4}	3.4665×10^{-3}	3.4685×10^{-3}	5.6×10^{-2}
1.5	5.0447	5.0481	6.7
2.0	6.5600	6.5653	8.1
3.0	9.4448	9.4552	1.1×10^{-1}
5.0	1.4786×10^{-2}	1.4811×10^{-2}	1.6
7.0	1.9715	1.9756	2.1
1.0×10^{-3}	2.6547	2.6621	2.8
1.5	3.6860	3.6999	3.7
2.0	4.6204	4.6416	4.6
3.0	6.2872	6.3253	6.0
5.0	9.1026	9.1789	8.3
7.0	1.1485×10^{-1}	1.1602×10^{-1}	1.02×10^{0}
1.0×10^{-2}	1.4550	1.4732	1.25
1.5	1.8807	1.9099	1.55
2.0	2.2387	2.2790	1.80
3.0	2.8307	2.8926	2.14
5.0	3.7349	3.8379	2.76
7.0	4.4316	4.5721	3.17
1.0×10^{-1}	5.2560	5.4459	3.61
1.5	6.2863	6.5394	4.03
2.0	7.0563	7.3475	4.12
3.0	8.1279	8.4314	3.73
5.0	9.2360	9.4496	2.31
7.0	9.6880	9.8068	1.23
1.0×10^{0}	9.9185	9.9598	4.1×10^{-1}
1.5	9.9913	9.9970	5.7×10^{-2}
2.0	9.9991	9.9998	6.8×10^{-3}
3.0	1.000000×10^{0}	1.000000×10^{0}	0.0

Table 10 Biot = 20.0

Percent Difference Between $(Q/Q_i)_{square}$ and $(Q/Q_i)_{circle}$

Dimensionless Time Fo	$\frac{Q}{Q_i}$		Percent Difference
	Square Rod	Circular Cylinder	
1.0×10^{-4}	6.0976×10^{-3}	6.1027×10^{-3}	8.2×10^{-2}
1.5	8.6660	8.6750	1.0×10^{-1}
2.0	1.1060×10^{-2}	1.1074×10^{-2}	1.3
3.0	1.5466	1.5491	1.6
5.0	2.3238	2.3293	2.3
7.0	3.0087	3.0175	2.9
1.0×10^{-3}	3.9210	3.9354	3.7
1.5	5.2385	5.2633	4.7
2.0	6.3870	6.4227	5.6
3.0	8.3605	8.4192	7.0
5.0	1.1548×10^{-1}	1.1654×10^{-1}	9.1
7.0	1.4151	1.4306	1.09×10^{0}
1.0×10^{-2}	1.7420	1.7646	1.30
1.5	2.1858	2.2202	1.58
2.0	2.5529	2.5988	1.80
3.0	3.1517	3.2199	2.14
5.0	4.0537	4.1631	2.70
7.0	4.7418	4.8883	3.09
1.0×10^{-1}	5.5503	5.7450	3.51
1.5	6.5529	6.8078	3.89
2.0	7.2960	7.5846	3.96
3.0	8.3168	8.6088	3.51
5.0	9.3426	9.5371	2.08
7.0	9.7431	9.8459	1.06
1.0×10^{0}	9.9372	9.9704	3.3×10^{-1}
1.5	9.9940	9.9981	4.1×10^{-2}
2.0	9.9994	9.9999	4.3×10^{-3}
3.0	1.0000×10^{0}	1.0000×10^{0}	0.0

Table 11 Biot = 40.0

Dimensionless Time Fo	$\frac{Q}{Q_i}$		Percent Difference
	Square Rod	Circular Cylinder	
1.0×10^{-4}	9.7711×10^{-3}	9.7825×10^{-3}	1.2×10^{-1}
1.5	1.3423×10^{-2}	1.3443×10^{-2}	1.4
2.0	1.6700	1.6728	1.7
3.0	2.2494	2.2542	2.1
5.0	3.2198	3.2291	2.9
7.0	4.0379	4.0518	3.4
1.0×10^{-3}	5.0915	5.1127	4.1
1.5	6.5635	6.5971	5.1
2.0	7.8140	7.8602	5.9
3.0	9.9160	9.9875	7.2
5.0	1.3234×10^{-1}	1.3356×10^{-1}	9.1
7.0	1.5902	1.6073	1.08×10^{0}
1.0×10^{-2}	1.9218	1.9463	1.27
1.5	2.3685	2.4049	1.54
2.0	2.7358	2.7838	1.75
3.0	3.3324	3.4026	2.11
5.0	4.2271	4.3383	2.63
7.0	4.9074	5.0553	3.01
1.0×10^{-1}	5.7045	5.9001	3.43
1.5	6.6902	6.9441	3.80
2.0	7.4179	7.7033	3.85
3.0	8.4108	8.6952	3.38
5.0	9.3936	9.5778	1.96
7.0	9.7685	9.8633	9.6×10^{-1}
1.0×10^{0}	9.9454	9.9748	3.0
1.5	9.9951	9.9985	3.4×10^{-2}
2.0	9.9996	9.9999	3.5×10^{-3}
3.0	1.0000×10^{0}	1.0000×10^{0}	0.0

Table 12 Biot = 80.0

Dimensionless Time Fo	$\frac{Q}{Q_i}$		Percent Difference
	Square Rod	Circular Cylinder	
1.0×10^{-4}	1.3844×10^{-2}	1.3864×10^{-2}	1.4×10^{-1}
1.5	1.8302	1.8334	1.7
2.0	2.2166	2.2209	2.0
3.0	2.8782	2.8851	2.4
5.0	3.9467	3.9587	3.0
7.0	4.8235	4.8407	3.5
1.0×10^{-3}	5.9326	5.9576	4.2
1.5	7.4583	7.4963	5.1
2.0	8.7403	8.7914	5.8
3.0	1.0878×10^{-1}	1.0954×10^{-1}	7.0
5.0	1.4224	1.4352	8.9
7.0	1.6902	1.7080	1.05×10^{0}
1.0×10^{-2}	2.0221	2.0472	1.24
1.5	2.4680	2.5050	1.50
2.0	2.8341	2.8827	1.71
3.0	3.4279	3.4987	2.06
5.0	4.3173	4.4288	2.58
7.0	4.9927	5.1409	2.97
1.0×10^{-1}	5.7834	5.9789	3.38
1.5	6.7597	7.0127	3.74
2.0	7.4792	7.7625	3.79
3.0	8.4576	8.7377	3.31
5.0	9.4184	9.5973	1.90
7.0	9.7806	9.8715	9.2×10^{-1}
1.0×10^{0}	9.9492	9.9769	2.8
1.5	9.9956	9.9987	3.1×10^{-2}
2.0	9.9996	9.9999	3.3×10^{-3}
3.0	1.0000×10^{0}	1.0000×10^{0}	0.0

Table 13 Biot = 160.0