On the Low Rayleigh Number Asymptote for Natural Convection Through an Isothermal, Parallel-Plate Channel

L. Martin

G. D. Raithby
Mem. ASME

M. M. Yovanovich
Fellow ASME

Department of Mechanical Engineering,
University of Waterloo,
Waterloo, Ontario, Canada N2L 3G1

The problem of natural convection through a channel formed by isothermal, parallel plates forms a cornerstone of our understanding of a class of natural convection flows. Following the pioneering study of Elenbaas, it is widely accepted that there is a fully developed régime, at low Rayleigh number, in which the Nusselt number becomes directly proportional to the Rayleigh number. This paper gives a detailed analysis of heat transfer in this régime. It is concluded that the previous numerical studies, which appeared to confirm this asymptote, used inappropriate boundary conditions, and that the asymptotic behavior should, in fact, not be expected except under very special conditions.

Introduction

Heat release in a cavity that has openings to cooler surroundings induces a natural convection flow. There are many examples of such flows (e.g., chimneys, parallel plate fins, computer boards, etc.), but probably the most extensively studied and well understood is the natural convection flow induced by parallel heated isothermal plates. A definition sketch for this problem appears in Fig. 1(A); the objective is to predict the total heat flow Q to the fluid from the surfaces of the plates that face each other.

Elenbaas (1942) did an exhaustive pioneering study of this problem. By analyzing a simplified set of equations, and by adjusting constants to fit experimental data, he proposed the following equation for the Nusselt number:

$$\mathrm{Nu}=\frac{\mathrm{Ra}^*}{3}\,[1-e^{-35/16\mathrm{Ra}^*}]^{3/4} \qquad \mathrm{Ra}^*=\mathrm{Ra}\,\frac{b}{L_c} \tag{1}$$

This relation is plotted in Fig. 2. One remarkable feature of the expression is that the channel aspect ratio, L_c/b, does not appear explicitly, but has been absorbed into a modified Rayleigh number, Ra^*, called the Elenbaas Rayleigh number. Equation (1) also yields the two asymptotes

$$\mathrm{Nu}\rightarrow\mathrm{Nu}_{bl}=0.60\mathrm{Ra}^{*1/4} \qquad \mathrm{Ra}^*\rightarrow\infty \tag{2A}$$

$$\mathrm{Nu}\rightarrow\mathrm{Nu}_{fd}=\frac{1}{3}\,\mathrm{Ra}^* \qquad \mathrm{Ra}^*\rightarrow 0 \tag{2B}$$

where Nu_{bl} is the Nusselt number for the boundary layer régime and Nu_{fd} is the Nusselt number for fully developed flow throughout the flow passage. Elenbaas provided an impressive collection of data that confirmed Eq. (1) over the range $10^{-1}<\mathrm{Ra}^*<10^5$.

Raithby and Hollands (1975) derived a different relation that also captures both limiting cases. Bar-Cohen and Rohsenow (1984) provided a comprehensive review of this and related problems. Bar-Cohen and Rohsenow (1984), and Raithby and Hollands (1985) correlated the heat transfer results using a blending function of the type proposed by Churchill and Usagi (1972). After modifying the coefficient in Eq. (2A)

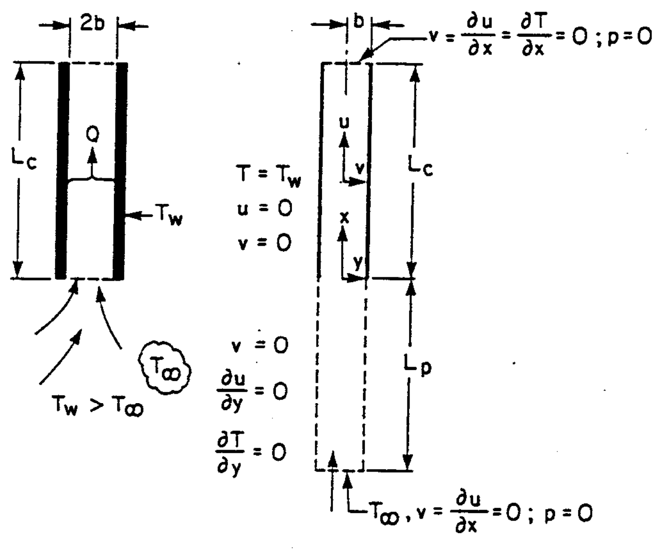

Fig. 1 Definition sketch for the generic parallel plate problem (A) and the specific problem analyzed in this paper (B)

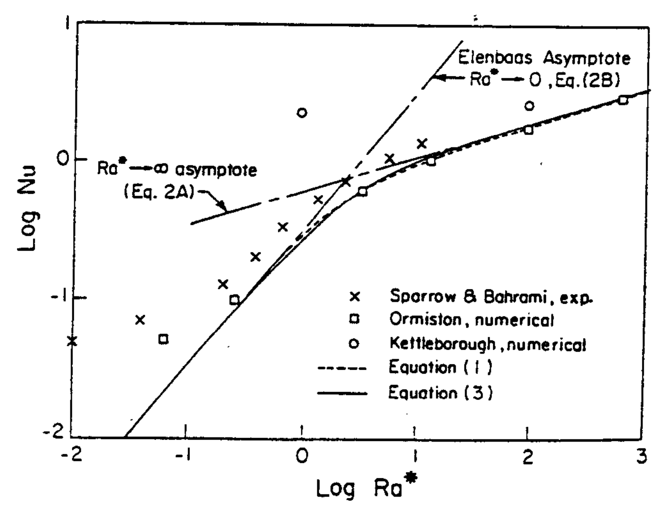

Fig. 2 Average heat transfer from a vertical parallel plate channel

Contributed by the Heat Transfer Division for publication in the JOURNAL OF HEAT TRANSFER. Manuscript received by the Heat Transfer Division March 7, 1990; revision received November 30, 1990. Keywords: Electronic Equipment, Natural Convection.

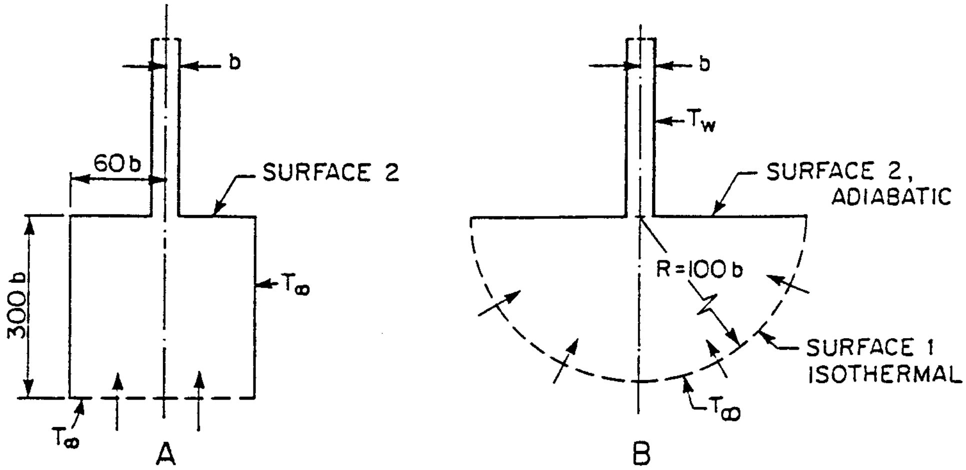

Fig. 3 Configurations assumed in numerical studies by Kettleborough (1972) and Nakamura et al. (1982), (A), and by Ormiston (1983), (B); solid and dotted lines represent impermeable and permeable boundaries, respectively

to agree with the measurements of Aung et al. (1972), the blended function is

$$Nu = (Nu_{fd}^m + Nu_{bl}^m)^{1/m}; \quad m = -1.9 \tag{3A}$$

where

$$Nu_{fd} = \frac{1}{3} Ra^* \tag{3B}$$

$$Nu_{bl} = 0.62 Ra^{*1/4} \tag{3C}$$

Equation (3) agrees closely with Eq. (1), as seen in Fig. 2.

Besides the experimental verification of Elenbaas (1942), Eqs. (1) and (3) are in close agreement with the parabolic solution reported by Bodia and Osterle (1962). Aihara (1973) carefully examined the effect of different inlet conditions for the parabolic problem and his predictions also agree closely with Eqs. (1) and (3).

Based on these and other confirmations, the current conclusion seems to be that natural convection heat transfer in an isothermal parallel-plate channel is a fully resolved problem, at least for the laminar flow régime.

There have been the disquieting pieces of evidence, however, that suggest major discrepancies with the Elenbaas' asymptote for fully developed heat transfer, Eq. (3B). Sparrow and Bahrami (1980), using the naphthalene sublimation technique, measured a higher transfer rate in this régime; their data, converted to equivalent heat transfer, are shown in Fig. 2. They argued that the large corrections Elenbaas used to account for radiation and edge effects had caused fortuitous agreement with the theoretical fully developed asymptote. During the sublimation process, however, heat must be absorbed by the solid naphthalene, and this must be drawn from both the plates and the gas. If a significant portion of the heat transfer comes from the gas, the buoyancy forces induced by the cooling can be shown to be of the same order as the buoyancy forces due to concentration changes. It can be argued, therefore, that this effect could account for the discrepancy between their measurements and Elenbaas' data.

Kettleborough (1972) and Nakamura et al. (1982) solved the full (elliptic) equations of motion for the geometry shown in Fig. 3(A). Kettleborough's predicted heat transfer was much higher than Elenbaas' curve (Fig. 2). The predictions of Nakamura et al. (1982) for the same geometry were in reasonable

Nomenclature

- b = half-width of channel, Fig. 1
- c_p = specific heat at constant pressure, J/kg K
- g = gravitational acceleration, m/s^2
- k = thermal conductivity, W/mK
- L_c = channel length, Fig. 1(B)
- L_c^* = L_c/b
- L_p = plenum length, Fig. 1(B)
- L_p^* = L_p/b
- $\dot{m}$ = mass flow rate through the channel, per unit depth = $\rho\bar{u}2b$
- Nu = Nusselt number = $[Q/2L_c(T_w - T_\infty)](b/k)$
- Nu_{bl} = Nusselt number for the laminar boundary layer régime, Eq. (3c)
- Nu_{COND} = Nu for conduction between the plates and the upstream plenum boundary
- Nu_{CONV} = Nu for the case $L_p^* \to \infty$
- Nu_{fd} = Nu for the fully developed régime: Eq. (3B) or (24)
- $\overline{Nu}$ = $Nu(L_c^*)^2$
- p = pressure, Pa
- p^* = $p/\rho u_r^2$
- Pe = Peclet number = $\rho b\bar{u}c_p/k$
- Pr = Prandtl number = $\mu c_p/k$
- Q = total heat transfer from both plates, per unit depth of channel
- Ra = Rayleigh number = $g\beta(T_w - T_\infty)b^3/\nu\alpha$
- Ra* = Elenbaas Rayleigh number = $Ra(b/L_c)$
- $\overline{Ra}$ = modified Rayleigh number = $Ra^*(L_c^*)^2$
- T = temperature, K
- T_w = temperature of the isothermal plates, K
- T_∞ = temperature of the ambient fluid and of the inlet of the plenum, K
- u, v = velocity components in the x and y directions, m/s
- u_r = reference velocity = $L_c^*k/\rho c_p b$
- $\bar{u}$ = average u velocity in the channel
- u^*, v^* = dimensionless velocities: $u^* = u/u_r$, $v^* = v/v_r$
- v_r = reference velocity = $k/\rho c_p b$
- x, y = coordinates, see Fig. 1(B)
- x^* = x/L_c
- y^* = y/b
- β = thermal expansion coefficient, 1/K
- θ = nondimensional temperature = $(T - T_\infty)/(T_w - T_\infty)$
- μ = dynamic viscosity, kg/ms
- ρ = fluid density, kg/m^3

agreement with Eq. (1) or (3) but only one predicted point fell in the fully developed régime. Ormiston (1983) predicted flow and heat transfer for the geometry shown in Fig. 3(B). He was able to show that counterflow heat conduction, out the inlet, caused his Nusselt numbers (see Fig. 2) to lie well above the fully developed prediction of Elenbaas for low values of Ra*, even when the semicircular inflow boundary (Fig. 3B) was far removed from the plates. These results exhibit the trends already seen in Sparrow and Bahrami's data, and the questions concerning the validity of both Elenbaas' data and the earlier analyses for low Rayleigh numbers become more difficult to dismiss.

Because the problem of natural convection through parallel plate channels is arguably the most important cornerstone problem of the open-cavity type, it is imperative that the flow and heat transfer at low Rayleigh numbers be understood. The purpose of this paper is to explore this low Rayleigh number régime.

Description of Problem

Because the purpose of this study is to enhance understanding, it is helpful to select the simplest geometric arrangement that still contains the relevant physics. The flow is therefore taken to be steady and two dimensional (no variation normal to the plane of Fig. 1A). If there is a large array of plates (such as a stack of circuit boards) at the same temperature, the flow and heat transfer in the channels in the central portion of the array will be the same, so it is sufficient to analyze one channel. Further assuming that the plates are thin, the problem reduces to that shown in Fig. 1(B). On the channel boundaries (solid lines) the velocity will be zero and the temperature fixed at T_w. In the plenum upstream of the channel, sheltering by neighboring plates will result in symmetry along the vertical dotted lines ($v=\partial u/\partial y=\partial T/\partial y=0$). The flow is assumed to enter the upstream plenum boundary at temperature T_∞. Fully developed conditions ($v=\partial u/\partial x=\partial T/\partial x=0$) are prescribed at the downstream boundary, but the limitations of these conditions are discussed later. Furthermore, the pressure at one point on the inlet and exit plane are set to the same value.

Equations of Motion and Their Numerical Solution. The two-dimensional equations of motion for steady flow of a Newtonian fluid with constant properties are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{4}$$

$$\frac{\partial}{\partial x}(\rho uu)+\frac{\partial}{\partial y}(\rho vu)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\rho\beta(T-T_\infty)g \tag{5}$$

$$\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(\rho vv)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{6}$$

$$\frac{\partial}{\partial x}(\rho c_p uT)+\frac{\partial}{\partial y}(\rho c_p vT)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{7}$$

These equations were discretized on a Cartesian mesh, with the flexibility to concentrate the mesh along the walls and near the channel inlet where gradients in the dependent variables are expected to be greatest. The computational method used (see Martin, 1988) was a variant of the methods described by Patankar (1980) and Van Doormaal and Raithby (1984).

It is necessary to add to the boundary condition specification already given that the pressure was specified as zero (relative to atmospheric pressure) at only the center node on the inlet and exit planes. To accommodate this pressure specification, one control-volume mass conservation equation was removed; this is permitted (in fact desirable—see Van Doormaal and Raithby, 1984) because the velocity boundary conditions make one control-volume continuity equation redundant on each of these planes.

An exhaustive study was undertaken by Martin (1988) to ensure that the code was free of errors and that the mesh was sufficiently fine to provide accurate predictions. For low Ra*, solutions were obtained on a sequence of grids, and the value of Nu at zero grid spacing was estimated by extrapolation. The Nu values reported in this study used grids for which Nu is within 1 percent of this asymptote. For Ra*→0, the Nu values obtained by the code were checked against results from a conduction analysis. At Ra* = 229, where the boundary layer assumptions are expected to be valid, the present code predictions were found to agree with Aihara's (1973) predictions to within 1.1 percent. Based on these and other studies, all solutions reported in this paper give heat transfer and mass through flow rates that are believed accurate to within about 2 percent, with the accuracy of most results well within this bound. These predictions will be presented in later sections.

Dimensional Analysis. Introducing the nondimensional variables defined in the Nomenclature, Eqs. (4)–(7) become

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0 \tag{8}$$

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\mathrm{Pr}\left(\left(\frac{1}{L_c^*}\right)^2\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right)+\mathrm{Ra}^*\mathrm{Pr}\theta \tag{9}$$

$$u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}=-(L_c^*)^2\frac{\partial p^*}{\partial y^*}+\mathrm{Pr}\left(\left(\frac{1}{L_c^*}\right)^2\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{10}$$

$$u^*\frac{\partial \theta}{\partial x^*}+v^*\frac{\partial \theta}{\partial y^*}=\left(\frac{1}{L_c^*}\right)^2\frac{\partial^2 \theta}{\partial x^{*2}}+\frac{\partial^2 \theta}{\partial y^{*2}} \tag{11}$$

The boundary conditions are

$$y^*=\pm 1;\quad 0\le x^*\le 1:\quad u^*=v^*=0\quad \theta=1 \tag{12A}$$

$$y^*=\pm 1;\quad -\frac{L_p^*}{L_c^*}\le x^*\le 0:\quad v^*=\frac{\partial u^*}{\partial y^*}=\frac{\partial \theta}{\partial y^*}=0 \tag{12B}$$

$$x^*=-\frac{L_p^*}{L_c^*},\quad -1<y^*<1:\quad v^*=\frac{\partial u^*}{\partial x^*}=\theta=0;\quad p^*=0(1\text{point}) \tag{12C}$$

$$x^*=1,\quad -1<y^*<1:\quad v^*=\frac{\partial u^*}{\partial x^*}=\frac{\partial \theta}{\partial x^*}=0;\quad p^*=0(1\text{point}) \tag{12D}$$

If the parabolic approximation is made, all diffusion effects in the x direction are ignored ($\partial^2(\ )/\partial x^{*2}=0$), and the y-momentum equation is replaced by $p^*(x,y)=p^*(x)$. Because there is no diffusion in the x direction, there will be no change in the solution in the inlet region; $-(L_p/L_c)<x^*<0$, so the solution of Eqs. (8)–(12) becomes independent of L_p^* and L_c^*. This results in the following functional dependence of the nondimensional heat transfer and flow rate

$$\mathrm{Nu}=\mathrm{Nu}(\mathrm{Ra}^*,\mathrm{Pr}):\quad \mathrm{Pe}=\mathrm{Pe}(\mathrm{Ra}^*,\mathrm{Pr}) \tag{13}$$

These are the dependencies suggested, but not proved, by Elenbaas (1942).

At low Ra* (fully developed régime) one must expect that diffusion in the x direction will be comparable to convection, so that $\partial^2(\ )/\partial x^{*2}$ terms should not be dropped from Eqs. (8)–(12). If the inlet plenum is very long ($L_p^*\to\infty$), the following dependence is deduced:

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Ra}^*, \mathrm{Pr}, L_c^*): \quad \mathrm{Pe} = \mathrm{Pe}(\mathrm{Ra}^*, \mathrm{Pr}, L_c^*) \qquad (14)$$

For the fully developed régime, this dimensional analysis does not support Elenbaas' conclusion (that Eq. (13) is valid) except when L_c^* becomes asymptotically large.

At values of Ra^* that are small enough for x diffusion to be important, *and* if L_p^* is not asymptotically large, the solution to Eqs. (8)–(12) will also depend on the plenum length, $L_p^* = L_p/b$. The dependencies of the dimensionless average heat transfer and mass flow rates for the *general* problem in Fig. 1(B) are therefore

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Ra}^*, \mathrm{Pr}, L_c^*, L_p^*) \quad \mathrm{Pe} = \mathrm{Pe}(\mathrm{Ra}^*, \mathrm{Pr}, L_c^*, L_p^*) \qquad (15)$$

From dimensional analysis, it is therefore concluded that Elenbaas' asymptote, Eq. (3B), is not correct at low values of Ra^* unless $L_p^* \to \infty$ *and* $L_c^* \to \infty$. It remains to establish the actual Ra^* range over which the effects of L_c^* and L_p^* become important.

Fully Developed Asymptote $L_p^* \to \infty$

Before giving the results of a numerical study, some guidance (relating to the question of the behavior of Nu in the Ra^* domain over which x-diffusion is important) is obtained from an approximate analysis. The case of a long inlet plenum, $L_p^* \to \infty$, is considered first, where the functional relation (14) is expected to be valid.

When there was no flow through the channel in Fig. 1(B), the temperature on the $x=0$ plane would vary from T_w at $y = \pm b$ to a slightly smaller value near $y=0$. Any flow will cause this temperature to decrease toward T_∞. Specification of $T = T_w$ on $x=0$ is therefore an upper bound on the temperature on the inlet plane that becomes approximately correct as $\mathrm{Pe} \to 0$ (i.e., in the fully developed régime). In this régime, the dynamic pressure variations are small, so that the u velocity in the inlet plenum can be approximated as uniform at $\bar{u}$.

In the fully developed régime, the upper bound on the temperature distribution in the plenum is therefore described approximately by

$$\rho c_p \bar{u} \frac{dT}{dx} = k \frac{d^2T}{dx^2} \qquad (16)$$

$$x = 0 \qquad T = T_w$$

$$x = -L_p \approx -\infty \qquad T = T_\infty$$

The nondimensional plenum temperature distribution is therefore given by

$$\theta = \exp(\mathrm{Pe}\, L_c^*\, x^*) \qquad x^* \le 0$$
$$\theta = 1 \qquad x^* \ge 0 \qquad (17)$$

The total buoyancy force inducing flow through the channel and plenum, per unit depth (i.e., distance normal to the plane of Fig. 1) of channel and for $L_p^* \to \infty$, is

$$F_B = 2\rho g \beta (T_w - T_\infty) L_c b \int_{-\infty}^{1} \theta\, dx^* \qquad (18)$$

Equating this to the viscous drag force on the wall over the domain $0 \le x^* \le 1$, assuming the velocity profile to be parabolic, results in the velocity

$$\bar{u} = \frac{\rho g \beta (T_w - T_\infty) b^2}{3\mu} \left[1 + \frac{1}{L_c^*\, \mathrm{Pe}} \right] \qquad (19)$$

The second term in Eq. (19) is a correction to the Elenbaas equation for velocity. This is an implicit equation for $\bar{u}$, because Pe contains $\bar{u}$, which can be solved to yield

$$\mathrm{Pe} = \frac{\mathrm{Ra}}{6} \left[1 + \sqrt{1 + \frac{12}{L_c^*\, \mathrm{Ra}}} \right] \qquad (20)$$

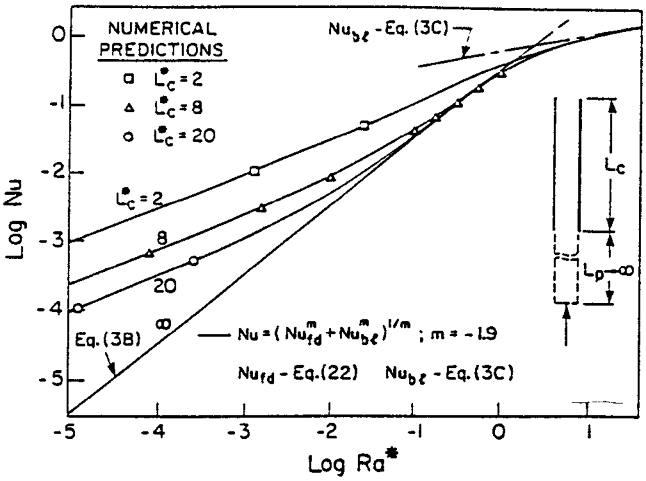

Fig. 4 Heat transfer predictions for various channel heights, L_c^*, and for a large plenum length, $L_p^* \to \infty$

Table 1 Values of $\mathrm{Nu}_{\mathrm{COND}}$

L_p/L_c	$L_c^* = 8$	$L_c^* = 10$
1	0.01481	0.009574
0.5	0.02814	0.01838
0.25	0.05120	0.03400
0.10	0.1007	0.06939

Performing an energy balance on the channel and plenum, the total heat flow from the plates, Q, is

$$Q = \dot{m} c_p (T_w - T_\infty) \qquad (21)$$

The average Nusselt number is therefore

$$\mathrm{Nu}_{fd} = \frac{\mathrm{Pe}}{L_c^*} = \frac{\mathrm{Ra}}{6 L_c^*} \left[1 + \sqrt{1 + \frac{12}{L_c^*\, \mathrm{Ra}}} \right] \qquad (22)$$

This equation is the appropriate fully developed asymptote for the problem in Fig. 1(B) for the special case: $L_p^* \to \infty$.

To obtain an equation for the Nusselt number that spans the fully developed and boundary layer régime, it is appropriate to use Eq. (3), but with Eq. (3B) replaced by Eq. (22). This equation is plotted in Fig. 4 for $\mathrm{Ra}^* < 50$. For $L_c^* \le 10$ (short channels) it is seen that there is no range in Ra^* over which the Elenbaas asymptote, Eq. (3B), is valid. As L_c^* increases, the range in Ra^* over which the Elenbaas asymptote is approached becomes wider. The $L_c^* \to \infty$ curve is the Elenbaas asymptote ($\mathrm{Nu}_{fd} \to \mathrm{Ra}^*/3$ in Eq. (22) as $L_c^* \to \infty$).

The symbols that appear in Fig. 4 are predictions based on the numerical solution of the full equations of motion (4)–(7). The agreement between these and the approximate solution just presented is seen to be excellent.

New Nusselt and Rayleigh Numbers for the Fully Developed Régime

If a new Nusselt and Rayleigh numbers are defined as follows:

$$\widetilde{\mathrm{Nu}}_{fd} = (L_c^*)^2 \mathrm{Nu}_{fd} \quad \widetilde{\mathrm{Ra}} = L_c^* \mathrm{Ra} = (L_c^*)^2 \mathrm{Ra}^* \qquad (23)$$

Eq. (22) becomes

$$\widetilde{\mathrm{Nu}}_{fd} = \frac{\widetilde{\mathrm{Ra}}}{6} \left[1 + \sqrt{1 + \frac{12}{\widetilde{\mathrm{Ra}}}} \right] \qquad (24)$$

This equation is significant because there is no separate dependence on L_c^*; that is, all predictions in the fully developed régime should fall on a single curve of $\widetilde{\mathrm{Nu}}$ versus $\widetilde{\mathrm{Ra}}$.

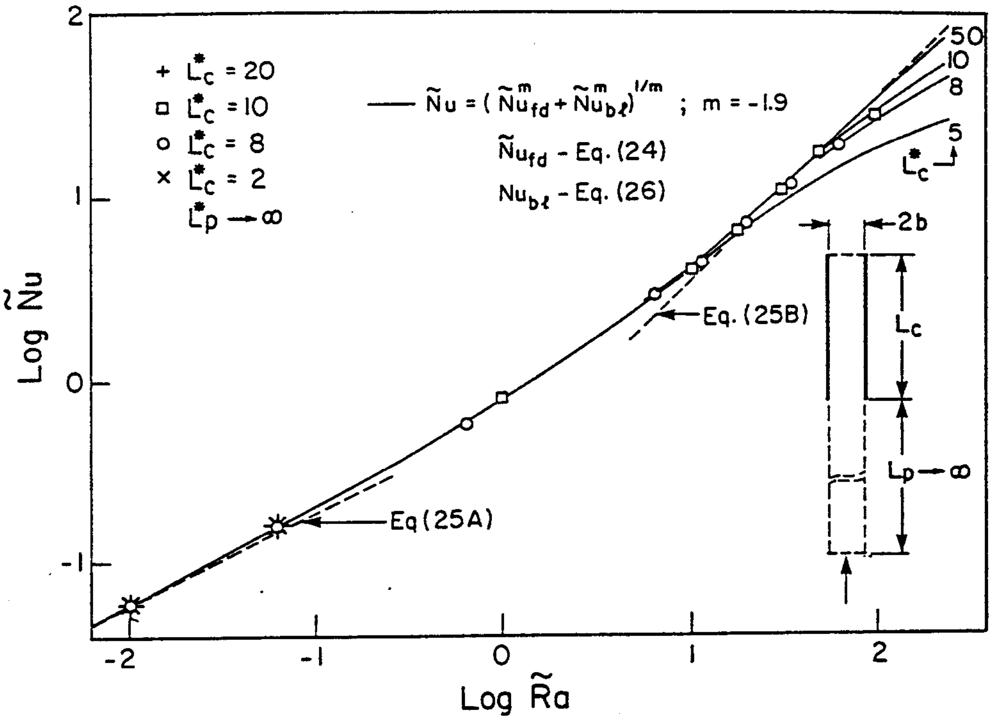

Fig. 5 New coordinates that collapse all predictions in the fully developed régime onto a single curve

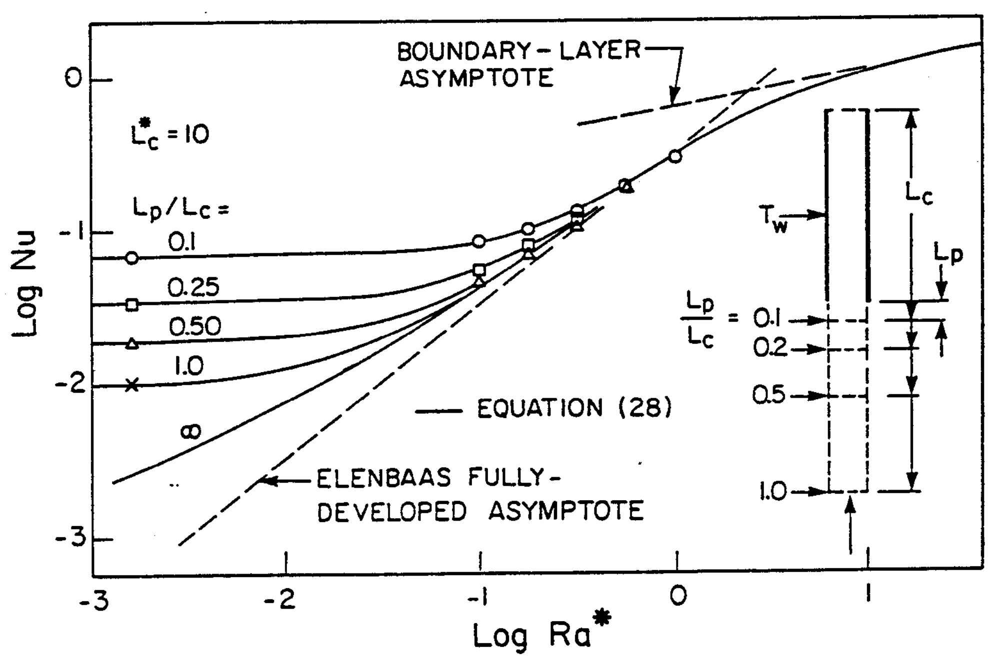

Fig. 6 Heat transfer for a fixed channel length, $L_c^* = 10$, and for various inlet plenum lengths, L_p^*

Equation (24) has two asymptotes

$$\tilde{Nu}_{fd} \to \sqrt{\frac{\tilde{Ra}}{3}} \quad \text{as} \quad \tilde{Ra} \to 0 \qquad (25A)$$

$$\tilde{Nu}_{fd} \to \frac{\tilde{Ra}}{3} \quad \text{as} \quad \tilde{Ra} \to \infty \qquad (25B)$$

The second of these is the Elenbaas equation, Eq. (3B). The first, Eq. (25A), represents a new asymptote that accounts for the effect of upstream conduction. Because of transition to the boundary layer régime, the range of Ra over which the Elenbaas asymptote is valid is limited; the previous section showed, in fact, that there is no Ra range over which it is valid for short channels ($L_c^* \leq 10$).

In order to obtain an equation in these coordinates that spans the fully developed and boundary layer régime, Eq. (3C) is transformed to

$$\tilde{Nu}_{bl} = 0.62 \tilde{Ra}^{1/4} (L_c^*)^{3/2} \qquad (26)$$

and Eqs. (24) and (26) are substituted into the blending relation

$$\tilde{Nu} = (\tilde{Nu}_{fd}^m + \tilde{Nu}_{bl})^{1/m} \quad m = -1.9 \qquad (27)$$

Equation (27), using Eq. (24) for $\tilde{Nu}_{fd}$ and Eq. (26) for $\tilde{Nu}_{bl}$, is plotted in Fig. 5. Also shown are the two fully developed asymptotes given by Eqs. (25A) and (25B).

Numerical predictions obtained for $L_p^* \to \infty$ are plotted in $\tilde{\text{Nu}}$-$\tilde{\text{Ra}}$ coordinates in Fig. 5. This figure confirms that these coordinates indeed collapse all the predictions, for different L_c^*, onto a single curve in the fully developed régime. The boundary layer asymptote (in these coordinates) now depends on the value of L_c^*; Eq. (27) predicts the correct $\tilde{\text{Ra}}$ at which results break away from the fully developed régime toward the boundary layer régime.

Effect of Upstream Conduction to Plenum Boundary (Finite L_p^*)

In all the results just presented, as Ra $\to$ 0 the Nusselt number approaches zero. In any real situation this will not occur because there will always be some heat conduction to surrounding surfaces. For the boundary conditions shown in Fig. 1(B), as Ra $\to$ 0, the Nusselt number will approach that for conduction from the plates at T_w to the upstream boundary at T_∞. As $L_p^* \to \infty$, this limiting heat transfer approaches zero (the case reported in the previous sections). Note that the boundary conditions allow no conduction through the $x = L_c$ boundary in Fig. 1(B), a point still to be addressed.

In this section, the Nusselt number for the $L_p^* \to \infty$ case is given the new notation, Nu_{CONV} (i.e., Nu_{CONV} is given by Eq. (3A), using Eq. (3C) for Nu_{bl} and Eq. (22) for Nu_{fd}). The Nusselt number for conduction to the plenum inlet boundary, at $x = -L_p$, is denoted by Nu_{COND}.

The solution to the pure conduction problem, given by Gibbs (1958), requires elliptic integrals and related functions that can be easily calculated using theta functions, as outlined by Lemczyk and Yovanovich (1988). The values of Nu_{COND} for the conduction limit are given in Table 1. Note that as $L_p \to \infty$, $\text{Nu}_{\text{COND}} \to 1/(L_p^* L_c^*)$

The Nusselt number, including conduction, can again be approximated by the blending equation

$$\text{Nu} = (\text{Nu}_{\text{COND}}^n + \text{Nu}_{\text{CONV}}^n)^{1/n} \quad n = 1.9 \tag{28}$$

Equation (28) is plotted in Fig. 6 for four values of L_p/L_c between 0.1 and 1.0, all for $L_c^* = 10$. Heat conduction to the upstream boundary is seen to have a pronounced effect on the heat transfer from the plates, even when the inlet channel has a length equal to the plate length ($L_p/L_c = 1.0$).

The results of numerical predictions are also shown in Fig. 6. For the limited range of parameters examined, there is good agreement with Eq. (28).

Discussion

For the geometrically simple case shown in Fig. 1(B), upstream conduction can have an important effect on the heat transfer from the plates. For $L_p \to \infty$, the upstream conduction preheats the fluid before it enters the channel causing the effective height of the buoyant column to increase; this, in turn, induces a higher throughflow and therefore a higher heat transfer from the plates.

For a short inlet plenum, heat transfer is further augmented by conduction to the upstream boundary. The combined effects of higher effective height of the buoyant column and of heat conduction to the boundary can result in heat transfer rates that are very much larger than those predicted by the low Rayleigh number Elenbaas asymptote.

For other shapes of inlet channel, upstream conduction will play a different, but important, role. For the geometry shown in Fig. 3(B), upstream conduction is much more important than for the geometry of this study (Fig. 1B) for a given distance between the channel entrance and the upstream boundary. This is because the area for the flow increases with distance upstream, which has the dual effect of decreasing the velocity that moves counter to the upstream heat conduction and of decreasing the resistance to conduction. Even for $R/b = 100$, the results of Ormiston (1983), shown in Fig. 2, indicate a very significant contribution to the low-Ra heat transfer.

In reality, Surface 2 in Fig. 3(B) could not be maintained adiabatic for gas flows. The heat flow path from the channel to Surface 2 is much shorter than the path to Surface 1, so that heat transfer from the plates to this surface would likely be very important at low Ra. This would further increase the discrepancy between observed heat transfer and the Elenbaas asymptote. It is, therefore, concluded that Elenbaas' low Rayleigh number asymptote will not be observed except under extremely special conditions.

Since the parabolic solutions (e.g., Bodoia and Osterle, 1962) have been one of the key pieces of supporting evidence for the Elenbaas asymptote, one must now enquire as to why these solutions are also incorrect. The answer seems to be that the parabolic flow assumptions are invalid at low Ra. Certainly axial diffusion is important at low Ra, and clearly the specification of $T = T_\infty$ at $x^* = 0$ is incorrect. In fact, if axial diffusion were retained with this boundary condition specification, asymptotically large heat transfer would be found for Ra $\to$ 0 because of the thermal short circuit where the inlet plane intersects the plates. It is only because of a partial cancellation of simplifying approximations that the parabolic solutions yield reasonable results at all.

The remaining questions relate to the disagreement among different sets of experimental data. It now seems that Sparrow and Bahrami (1980) measured high (equivalent) heat transfer at low Ra partly because of the cooling that accompanied the sublimation and partly because upstream diffusion at the inlet both augmented buoyancy and transferred mass to fluid that never passed through the channel. It also appears that Elenbaas' data yielded reasonable agreement with his low-Ra asymptote partly because his L_c^* were large, but also because the large corrections to the data (see Sparrow and Bahrami, 1980) resulted in fortuitous agreement.

The present study has analyzed the heat transfer from the parallel plates for two-dimensional flow, for one channel in an infinite stack of channels, and for zero heat conduction from the top of the channel. For the analysis and discussion in this paper, the heat transfer is expected to be much different for one, or only a few, channels because the upstream diffusion will be different. (The present analysis should not, therefore, be expected to agree with the Sparrow and Bahrami data.) For low Ra, neither the depth of the plates nor the number of channels would be sufficient to prevent three-dimensional conduction effects at the inlet. Furthermore, the fully developed boundary condition on temperature at the duct outlet ($(\partial T/\partial x) = 0$ at $x = L_c$) often becomes inappropriate at low Ra, especially if there are nearby cool surfaces to which the plates can conduct. While the problem analyzed in the present study has been educational, the conclusion of this study is that these predictions will only be valid under restrictive conditions. In general, the low-Ra heat transfer will depend on the particular inlet and outlet configuration used.

Summary and Conclusions

The problem of flow and heat transfer by natural convection through a channel formed by parallel isothermal plates is central to our understanding of open-cavity flows. Despite some scattered experimental and numerical evidence to the contrary, there is wide acceptance that the Elenbaas fully developed régime exists at low Rayleigh numbers, in which the Nusselt number becomes directly proportional to the Rayleigh number.

This paper has presented a numerical study, a simple analysis, and correlation equations for natural convection through an array of large parallel, vertical, heated plates. The Elenbaas fully developed régime is shown to exist only as a very special limiting case. For the problem studied, new asymptotic (fully developed) relations are presented for this particular problem.

From the analysis presented, it becomes clear that the heat transfer at low Rayleigh number will depend on the shape and boundary conditions of the inlet and outlet plenums. In particular, Elenbaas' asymptotic expression for low Rayleigh numbers will not be valid except under extremely special circumstances.

Acknowledgments

This work was partially supported by the Manufacturing Research Corporation of Ontario (MRCO) and partially by an operating grant to the second author from the Natural Sciences and Engineering Research Council (NSERC) of Canada.

References

Aihara, T., 1973, "Effects of Inlet Boundary Conditions on Numerical Solutions of Free Convection Between Vertical Parallel Plates," *Rep. Inst. High Speed Mech.*, Vol. 28, No. 258.

Aung, W., Fletcher, L. S., and Sernas, V., 1972, "Developing Laminar Free Convection Between Flat Plates With Asymmetric Heating," *Int. J. Heat Mass Transfer*, Vol. 15, pp. 2293–2308.

Bar-Cohen, A., and Rohsenow, W. M., 1984, "Thermally Optimum Spacing of Vertical, Natural Convection Cooled, Parallel Plates," ASME JOURNAL OF HEAT TRANSFER, Vol. 106, pp. 116–123.

Bodoia, J. R., and Osterle, J. F., 1962, "The Development of Free Convection Between Heated Vertical Plates," ASME JOURNAL OF HEAT TRANSFER, Vol. 84, pp. 40–44.

Churchill, S. W., and Usagi, R., 1972, "A General Expression for the Correlation of Rates of Transfer and Other Phenomena," *AIChE J.*, Vol. 18, pp. 1121–1128.

Elenbass, W., 1942, "Heat Dissipation of Parallel Plates by Free Convection," *Physica*, Vol. 9, pp. 1–28.

Gibbs, W. J., 1958, *Conformal Transformations in Electrical Engineering*, The British Thompson-Houston Co. Ltd.

Kettleborough, C. F., 1972, "Transient Laminar Free Convection Between Heated Vertical Parallel Plates Including Entrance Effects," *Int. J. Heat Mass Transfer*, Vol. 15, pp. 883–896.

Lemczyk, T. F., and Yovanovich, M. M., 1988, "Efficient Evaluation of Incomplete Elliptic Integrals and Functions," *Comp. Math. Applic.*, Vol. 16, No. 9, pp. 747–757.

Martin, L. C., 1988, "A Numerical Study of Natural Convection Cooling of Parallel Plates at Low Rayleigh Number," M.A.Sc. Thesis, Dept. of Mechanical Engineering, University of Waterloo, Ontario, Canada.

Nakamura, H., Asako, Y., and Naitou, T., 1982, "Heat Transfer by Free Convection Between Two Parallel Flat Plates," *Numerical Heat Transfer*, Vol. 5, pp. 95–106.

Ormiston, S. J., 1983, "A Numerical Study of Two Dimensional Natural Convection in a Trombe Wall System Including Vent and Room Effects," M.A.Sc. Thesis, Dept. of Mechanical Engineering, University of Waterloo, Ontario, Canada.

Patankar, S. V., 1980, *Numerical Heat Transfer and Fluid Flow*, McGraw-Hill, New York.

Raithby, G. D., and Hollands, K. G. T., 1975, "A General Method of Obtaining Approximate Solutions to Laminar and Turbulent Free Convection Problems," *Advances in Heat Transfer*, T. F. Irvine, Jr., and J. P. Hartnett, eds., Vol. 11, Academic Press, New York, pp. 265–315.

Raithby, G. D., and Hollands, K. G. T., 1985, "Natural Convection," *Handbook of Heat Transfer*, W. M. Rohsenow, J. P. Hartnett, and E. N. Granic, eds., Chap. 6, McGraw-Hill.

Sparrow, E. M., and Bahrami, P. A., 1980, "Experiments on Natural Convection From Vertical Parallel Plates With Either Open or Closed Edges," ASME JOURNAL OF HEAT TRANSFER, Vol. 102, pp. 221–227.

Shewen, E. C., 1986, "A Peltier-Effect Technique for Natural Convection Heat Flux Measurement Applied to the Rectangular Open Cavity," PhD Thesis, Dept. Mechanical Engineering, University of Waterloo, Waterloo, Ontario, Canada.

Van Doormal, J. P., and Raithby, G. D., 1984, "Enhancements of the SIMPLE Method for Predicting Incompressible Fluid Flows," *Numerical Heat Transfer*, Vol. 7, pp. 147–163.