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Laminar Natural Convection From a Vertical Plate With a Step Change in Wall Temperature

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Introduction

The study of natural convection heat transfer from a vertical flat plate in a quiescent medium has attracted a great deal of interest from many investigators in the past few decades. The plate with various thermal conditions that allow similarity transformations as well as those that are continuous and well defined have been examined. However, practical problems often involve wall conditions that are arbitrary and unknown a priori. To understand and solve problems involving general nonsimilar conditions at the wall, it is useful to investigate problems subjected to a step change in wall temperature. The problems impose a mathematical singularity and severe nonsimilar conditions at the wall.

An early attempt was made by Schetz (1963) to develop an approximate analytical model for the problems with discontinuous wall temperature conditions. Numerous investigations on the same problem were continued by using an experimental technique (Schetz and Eichhorn, 1964), numerical methods (Hayday et al., 1967), or by using series expansions (Kelleher, 1971; Kao, 1975).

In this paper, a new analytical model that can deal with a discontinuous wall temperature variation is presented. The method results in a set of approximate solutions for temperature and velocity distributions. The validity and accuracy of the model is demonstrated by comparisons with the results of the aforementioned investigators. The agreement is excellent and the results obtained with the solution of this work are remarkably close to existing numerical data of Hayday et al. (1967) and the perturbation series solution of Kao (1975).

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Analysis

Governing Equations. The natural convection problem under consideration deals with a two-dimensional vertical plate with a discontinuous temperature variation prescribed along the wall. The plate is located in an ambient fluid, which is maintained at uniform temperature. The conservation of mass, momentum, and energy for steady-state, laminar boundary layer flow yields the usual set of governing differential equations expressed as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+g\beta T \qquad (2)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \qquad (3)$$

where x and y are coordinates parallel and normal to the plate, respectively, u and v are corresponding components of the velocity, and T is the local temperature excess over the ambient fluid temperature. The properties α, ν, and β are the thermal diffusivity, kinematic viscosity, and thermal expansion coefficient of the fluid, and g is the gravitational acceleration. In addition to the boundary layer approximations, the assumption of constant fluid properties, except the density in the use of the Boussinesq approximation, is included, and the dynamic pressure work and viscous dissipation terms are neglected in the above equations.

The boundary conditions associated with the foregoing equations are

$$\begin{aligned} &\text{at } y=0,\quad u=v=0,\quad T=T_{w_0} \text{ for } x\le x_0\\ &\qquad\qquad\qquad\qquad\quad T=T_{w_1} \text{ for } x>x_0\\ &\text{as } y\to\infty,\quad u\to 0,\quad T\to 0\\ &\text{at } x=0,\quad u=0,\quad T=0 \end{aligned} \qquad (4)$$

where T_{w_0} and T_{w_1} are constants; $T_{w_0} > 0$ and $T_{w_1} \ge 0$.

Pseudotransient Equations. An approximate method is sought to solve the above set of equations by introducing a pseudotransient coordinate, t. The downstream location x is viewed as $u_c \times t$, where u_c is defined as a characteristic streamwise velocity. With the use of u_c, the original x-y plane is transformed into the t-y plane. An assumption is made such that diffusion is dominant across the boundary layer in the y direction at given time. This implies that the temperature and velocity profiles would take forms of those due to a transient conduction heat transfer into a half space. Subsequently, the convective derivatives in the x-y plane, appearing on the left side of equations (2) and (3), are replaced by transient derivatives, resulting in the following pseudotransient equations:

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}+g\beta T \qquad (5)$$

$$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial y^2} \qquad (6)$$

The boundary conditions that are compatible with those prescribed by equation (4) are

$$\begin{aligned} &\text{at } y=0,\quad u=0,\quad T=T_{w_0} \text{ for } t\le t_0\\ &\qquad\qquad\qquad\qquad T=T_{w_1} \text{ for } t>t_0\\ &\text{as } y\to\infty,\quad u\to 0,\quad T\to 0 \end{aligned} \qquad (7)$$

where t is currently an unknown function of x, and t_0 is the corresponding time of the discontinuity at x_0.

The solutions to the above transient equations with the specified boundary conditions are obtained by means of either Laplace transforms (Schetz and Eichhorn, 1962; Menold and Yang, 1962), or similarity methods (Lee, 1988). The resulting solutions for $t > t_0$ are

$$T = T_{w_0} \operatorname{erfc} \eta_0 + (T_{w_1} - T_{w_0}) \operatorname{erfc} \eta_1 \tag{8}$$

$$u = 2g\beta[T_{w_0}\, t\, f_u(\eta_0) + (T_{w_1} - T_{w_0})(t - t_0) f_u(\eta_1)] \tag{9}$$

where

$$f_u(\eta) = \begin{cases} \eta \text{ ierfc } \eta & \text{for } \Pr = 1 \\ \dfrac{2}{1 - \Pr}\left(\text{i}^2\text{erfc } \eta - \text{i}^2\text{erfc } \dfrac{\eta}{\sqrt{\Pr}}\right) & \text{for } \Pr \neq 1 \end{cases} \tag{10}$$

η_0 and η_1 are similarity variables given as

$$\eta_0 = \frac{y}{2\sqrt{\alpha t}} \tag{11}$$

$$\eta_1 = \frac{y}{2\sqrt{\alpha(t - t_0)}} \tag{12}$$

Pr is the Prandtl number, and erfc, ierfc, and i^{2}erfc are the complementary error functions. These solutions are exact for transient natural convection heat transfer from an infinitely long plate experiencing wall temperature variations given by equation (7). The solutions for $t \leq t_0$ can also be obtained by simply discarding the second term of each equation.

The problem is now reduced to finding a proper characteristic velocity or velocities, u_c, over the boundary layer. Determination of u_c will, in turn, convert the solutions in the t-y plane back into the x-y plane.

t-x Transformations for x ≤ x₀. The parameters from the steady-state solutions for a uniform temperature plate, and those appearing in the above pseudotransient solutions are compared as follows:

Reference Velocity Group:

$$C_u \sqrt{4g\beta T_{w_0} x} = 2g\beta T_{w_0} t \tag{13}$$

Similarity Variable:

$$C_\eta \left(\frac{\text{Gr}_x}{4}\right)^{1/4} \frac{y}{x} = \frac{y}{2\sqrt{\alpha t}} \tag{14}$$

where $\text{Gr}_x = g\beta T_{w_0} x^3/\nu^2$, and C_u and C_η are dimensionless proportionality constants, functions only of the Prandtl number. The parameters on the left side of the above comparisons are from the similarity analysis of Sparrow and Gregg (1958).

By rearranging equations (13) and (14) for t, one finds, respectively,

$$t = C_u \frac{x}{\sqrt{g\beta T_{w_0} x}} \tag{15}$$

$$t = \frac{\Pr}{2C_\eta^2} \frac{x}{\sqrt{g\beta T_{w_0} x}} \tag{16}$$

Equations (15) and (16) are the t - x transformations sought for $x \leq x_0$. Although they express different relationships between t and x, they both exhibit an identical functional form such that $u_c = \sqrt{g\beta T_{w_0} x}/C$, where

$$C = C_u \tag{17}$$

or

$$C = \frac{\Pr}{2C_\eta^2} \tag{18}$$

The former equation defines the time associated with the reference velocity group and the latter defines the time associated with the similarity variable. The constants C_u and C_η are determined by employing the integral method using the velocity and temperature profiles given by the first terms of equations (8) and (9), with t substituted by equations (15) and (16). It follows that

$$C_u = \left(G_m + \frac{2\sqrt{\Pr}}{1 + \sqrt{\Pr}} G_e\right)^{-1/2} \tag{19}$$

$$C_\eta = \sqrt{\Pr C_u G_e} \tag{20}$$

where G_m and G_e are tabulated in Table 1.

t-x Transformations for x > x₀. The similarity characteristic of the existing boundary layer, which was initiated at the leading edge, will no longer be maintained at the downstream location at $x > x_0$, as a secondary thermal boundary layer is established from the wall at $x = x_0$. This secondary boundary layer grows quickly into and will eventually engulf the existing one as x approaches infinity, where the solutions become similar once again based on T_{w_1}. Not only would the characteristic streamwise velocity over the existing boundary layer hence be altered, but another distinct characteristic velocity would also evolve over the secondary boundary layer. In the transient solutions, equations (8) and (9), t appearing in the first terms is related to the original boundary layer and $t - t_0$ appearing in the second terms is related to the secondary boundary layer. The t-x transformations are thus defined by

$$t = \frac{x}{u_c} = C\phi \frac{x}{\sqrt{g\beta T_{w_0} x}} \tag{21}$$

$$t - t_0 = \frac{x - x_0}{u_c} = C\psi \frac{x - x_0}{\sqrt{g\beta T_{w_0} x}} \tag{22}$$

where C represents, as previously defined by equations (17) and (18), two different coefficients depending on whether t and $t - t_0$ are associated with the velocity group or the similarity variables. The function ϕ in equation (21) modifies the existing characteristic velocity that would have been attained at the location of interest if the thermal condition at the wall were maintained at T_{w_0}. The other function, ψ in equation (22), represents the ratio of the existing characteristic velocity to the new one that characterizes the flow velocity within the secondary boundary layer initiated at $x = x_0$. These functions are dimensionless and dependent on Pr, θ_{w_1}, and ξ, where $\theta_{w_1} = T_{w_1}/T_{w_0}$ and $\xi = x/x_0$.

By substituting the above t-x transformations, equations (21) and (22), into equations (8) and (9), a set of approximate solutions to the original problem in the x-y plane can be obtained. After nondimensionalization, the resulting set of solutions for $x > x_0$ may be expressed as

$$\theta = \frac{T}{T_{w_0}} = \operatorname{erfc} \eta_0 + (\theta_{w_1} - 1) \operatorname{erfc} \eta_1 \tag{23}$$

Table 1 G_m and G_e

	$\Pr = 1$	$\Pr \neq 1$ ($p = \sqrt{\Pr}$)
G_m	$\dfrac{3\sqrt{2} - 4}{4}$	$\dfrac{(1 + p^2)^{5/2} - p^5 - (2\sqrt{2} - 1)(p^3 + p^2) - 1}{1.5\,p^2(1 - p^2)^2}$
G_e	$\dfrac{\sqrt{2} - 1}{4}$	$\dfrac{(1 + p^2)^{3/2} - p^3 - 2(\sqrt{2} - 1)p^2 - 1}{2\,p^2(1 - p^2)}$

Table 2 h_m and h_e

	Pr = 1	Pr ≠ 1 ($p = \sqrt{\mathrm{Pr}}$)
h_m	$\frac{4(1+\gamma^2)^{5/2} - 10\gamma^2(1+\gamma^2)^{1/2} - 4\gamma^5 - 4}{3\sqrt{2}-4}$	$\frac{(p^2+\gamma^2)^{5/2} + (1-p^2+\gamma^2)^{5/2} - p^2(1+p)(1+\gamma^2)^{3/2} - (1-p^2)(1-p^2)(1+\gamma^5)}{(1-p^2)^{5/2} - p^5 - (2\sqrt{2}-1)(p^5+p^4) - 1}$
h_e	$\frac{(1+\gamma^2)^{3/2} - \gamma^3 - 1}{\sqrt{2}-1}$	$\frac{(p^2+\gamma^2)^{3/2} + (1-p^2+\gamma^2)^{3/2} - 2p^2(1+\gamma^2)^{3/2} - (1+p-p^2)(1-p)(1+\gamma^3)}{(1+p^2)^{3/2} - p^3 - 2(\sqrt{2}-1)p^2 - 1}$

$$u^* = \frac{u}{2\sqrt{g\beta T_{w_0} x}} = C_u \phi [f_u(\eta_0) + (\theta_{w_1} - 1)\gamma f_u(\eta_1)] \quad (24)$$

where

$$\eta_0 = \frac{C_\eta}{\sqrt{\phi}} \left(\frac{\mathrm{Gr}_x}{4}\right)^{1/4} \frac{y}{x} \quad (25)$$

$$\eta_1 = \frac{\eta_0}{\gamma} \quad (26)$$

and γ is related to the modifying functions as follows:

$$\gamma = \sqrt{\left(1 - \frac{1}{\xi}\right) \frac{\psi}{\phi}} \quad (27)$$

In the above solutions, ϕ and γ are the only parameters that are required to complete the solutions. They are determined by solving the following set of first-order ordinary differential equations that are obtained again by employing the integral method:

$$\frac{d\Phi}{d\xi} = \frac{1}{H_m} \left[1 + (\theta_{w_1} - 1)\gamma - \frac{4}{5} \Phi \frac{dH_m}{d\gamma^3} \frac{d\gamma^3}{d\xi}\right] \quad (28)$$

$$\frac{d\gamma^3}{d\xi} = \frac{3}{4} \left[\frac{1 + (\theta_{w_1} - 1)/\gamma - H_e \frac{d\Phi}{d\xi}}{\Phi \frac{dH_e}{d\gamma^3}}\right] \quad (29)$$

with $\Phi = \phi^2\xi = 1$ and $\gamma^3 = 0$ at $\xi = 1$. Also, the dimensionless functions H_m and H_e are given by

$$H_m = 1 + (\theta_{w_1} - 1)^2\gamma^5 + (\theta_{w_1} - 1)h_m \quad (30)$$

$$H_e = 1 + (\theta_{w_1} - 1)^2\gamma^3 + (\theta_{w_1} - 1)h_e \quad (31)$$

where h_m and h_e are tabulated in Table 2. The above equations can be solved numerically for Φ and γ^3, hence ϕ and γ, with specified values of θ_{w_1} and Pr.

An interesting and important observation can be drawn from equation (22). Although t_0, the corresponding pseudotime at fixed x_0, is constant in the t-y plane, it is no longer possible to isolate the fixed t_0 in the x-y plane. This makes the term $t - t_0$ a unique variable, and, therefore, the parameter t in equation (22) is not the same t defined by equation (21). The facts that there are two distinct time-spatial transformations, and the existing characteristic velocity had to be modified by ϕ, are all parts of the conditions that are required if the nonlinearity of the problem was to be reflected through the transformation functions for $x > x_0$.

Results and Discussion

The solutions given in equations (23) and (24) are evaluated for a wide range of Prandtl numbers with various wall temperature ratios θ_{w_1}. Clearly, the solutions to a problem in which the entire wall is at uniform temperature T_{w_0} can also be found from the solutions by discarding the second terms and setting $\phi = 1$. Due to the limited spaces, however, only the cases with a step change in air (Pr = 0.72) will be presented herein to demonstrate the accuracy of the present model. Further comparisons of the results with others can be found from Lee (1988).

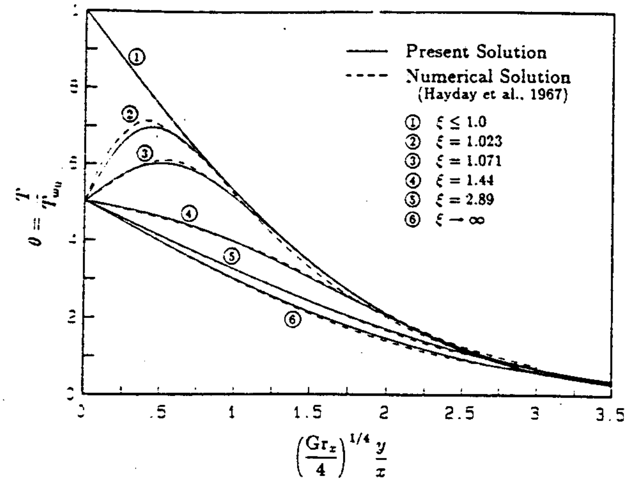

Fig. 1 Comparison of dimensionless temperature field development with a step change in wall temperature: $\theta_{w_1} = 0.503$, Pr = 0.72

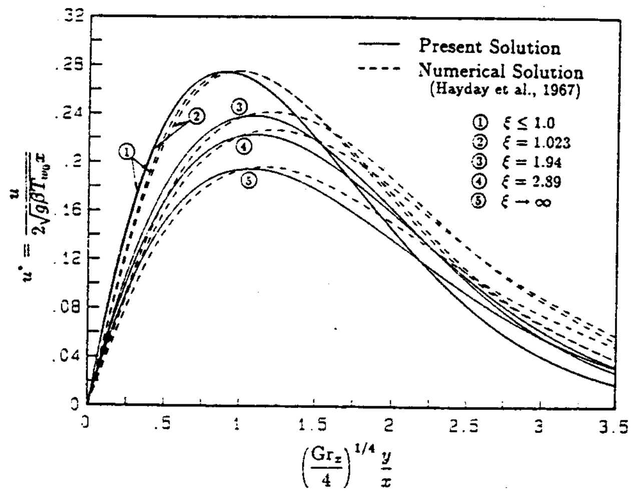

Fig. 2 Comparison of dimensionless velocity field development with a step change in wall temperature: $\theta_{w_1} = 0.503$, Pr = 0.72

Figures 1 and 2 depict the temperature and velocity distributions responding to step changes in wall temperature. Also presented in the figures are the numerical data of Hayday et al. (1967), who solved the boundary layer equations using finite difference methods. Excellent agreement is observed in the temperature distributions, and some discrepancies are found in the velocity distributions over the outer region of the boundary layers. However, the discrepancies are shown to have minimal effects on the surface heat transfer as demonstrated in the following comparison.

From equation (23), the nondimensionalized local wall heat flux at the location $x > x_0$ may be obtained as

$$q^*_{w_1} = \frac{q_{w_1}}{q_{w_0}} = \frac{1}{\sqrt{\phi}} \left[1 + \frac{\theta_{w_1} - 1}{\gamma}\right] \quad (32)$$

where q_{w_0} is the local heat flux at the location of interest with the entire wall maintained at temperature T_{w_0}. The result obtained by using the above equation is plotted and compared with other data in Fig. 3. The values indicated by arrows are the asymptotic values at large ξ, and they are obtained from

$$\lim_{x \to \infty} q^*_{w_1} = \theta_{w_1}^{5/4} \quad (33)$$

The laminar regime is not likely to be maintained at far downstream locations in practice. Nevertheless, it is worthwhile stressing that the resulting solutions satisfy all the limiting

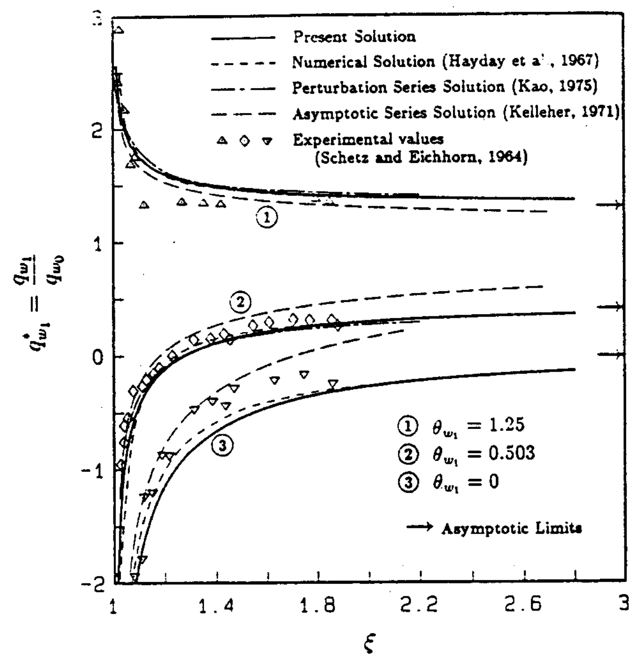

Fig. 3 Comparison of local wall heat-flux variation: Pr = 0.72

conditions at large ξ, as both ϕ and ψ can be shown to become independent of the Prandtl number and approach $\theta_{w_1}^{-1/2}$. Figure 3 exhibits an excellent agreement of the predictions especially with the fully numerical results of Hayday et al. (1967) and the perturbation series solution of Kao (1975). For the case of $\theta_{w_1} = 0$, Kao's solution is unavailable since his perturbation parameter becomes unbounded. Considering the limiting conditions at large ξ, inconsistent deviations are observed in the asymptotic series solution presented by Kelleher (1971).

As with most other approximate solutions, further studies would be needed to validate the use of the solutions for Prandtl numbers and conditions other than those presented. The current methodology is equally applicable for developing a model for problems with a step, as well as multistep changes in surface heat flux (Lee, 1988).

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Use of Vortex Generators and Ribs for Heat Transfer Enhancement at the Top Surface of a Uniformly Heated Horizontal Channel With Mixed Convection Flow

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Nomenclature

g = acceleration due to gravity
h = heat transfer coefficient
H = channel height or plate separation
k = thermal conductivity
L = channel length
Nu = Nusselt number = hH/k
Nu_{bot} = spanwise-averaged Nusselt number on bottom plate
Nu_{sm} = spanwise-averaged Nusselt number for smooth channel (no attachments)
Nu_{top} = spanwise-averaged Nusselt number on top plate
Pr = Prandtl number
q''_{conv} = convection heat flux
Ra* = modified Rayleigh number = $g\beta q''_{conv}H^4/k\nu\alpha$
Re = Reynolds number = $\overline{w}H/\nu$
$\overline{w}$ = average axial velocity
x, y, z = spanwise, vertical, and axial coordinates
z^* = nondimensional axial coordinate = z/HRePr
z_c = axial coordinate at which secondary flows become evident
α = thermal diffusivity
β = coefficient of thermal expansion
ν = kinematic viscosity

Introduction

Although secondary flows driven by buoyancy forces enhance heat transfer from the bottom surface of a heated, horizontal channel, heat transfer coefficients at the upper surface are known to remain near forced convection levels (Osborne and Incropera, 1985). In situations where performance is limited by the maximum local temperature, such as the cooling of electronic circuitry, enhanced heat transfer at one surface may be of little advantage if approximately equivalent enhancement does not exist at the opposite surface. Hence differences between top and bottom surface conditions may prevent a designer from taking full advantage of buoyancy-

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