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AIAA 92-0248
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30th Aerospace Sciences Meeting & Exhibit
January 6-9, 1992 / Reno, NV

APPROXIMATE ANALYTIC SOLUTION OF LAMINAR FORCED CONVECTION FROM AN ISOTHERMAL PLATE

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ABSTRACT

A method is presented to linearize the boundary-layer momentum and thermal energy equations. This is accomplished by the introduction of *effective* momentum and thermal-transport velocities which are related to the free-stream velocity through a constant parameter which in the case of the thermal energy equation is also dependent on the Prandtl number. Two methods are proposed for the estimation of the parameters: i) matching wall shear stress and heat flux, and ii) conserving momentum and enthalpy fluxes across the respective boundary-layers. The first method gives approximate solutions which are more accurate than those of the second method. The approximate analytic solutions are in closed form; they are simple and quite accurate when compared with the boundary-layer solutions of Blasius and Pohlhausen.

NOMENCLATURE

C, C^T	=	momentum and thermal parameters, Eqs. (3) and (22)
$f(Pr), F(Pr)$	=	Prandtl functions, Eqs. (23) and (33)
k	=	thermal conductivity
Nu_x	=	local Nusselt number, Eq. (32)
Pr	=	Prandtl number
q	=	heat flux
Re_x	=	local Reynolds number, $U_\infty x/\nu$
t	=	time scale
T	=	temperature
u, v	=	velocities in x and y directions
U_∞	=	free-stream velocity
u_e, u_e^T	=	effective momentum and thermal-transport velocities
u^*	=	dimensionless $u-$velocity, u/U_∞
x, y	=	coordinates parallel and normal to the wall

<u>Greek Symbols</u>

α	=	thermal diffusivity
δ, δ_T	=	momentum and thermal boundary-layer thicknesses
η, η_T	=	momentum and thermal similarity variables, Eqs. (13) and (25)
ν	=	kinematic viscosity
τ, τ^*	=	dimensional and dimensionless shear stresses, Eq.(15)
ϕ	=	dimensionless temperature, Eq. (24)
ψ	=	stream function, Eq. (36)

<u>Subscripts</u>

0	=	wall condition
∞	=	free-stream condition

INTRODUCTION

Since Blasius [1] presented his solution for steady, incompressible flow along a semi-infinite flat plate, many attempts have been made to find accurate approximate analytical solutions. Lord Rayleigh [2] first suggested that the Blasius flow could be approximated by the solution first given by Stokes [3] for an infinite plate which at $t = 0$ is suddenly moved in its own plane with a constant velocity U through a fluid initially at rest.

The solution for the Stokes flow is analogous to the solution of transient heat diffusion in a homogeneous semi-infinite solid which is initially at some constant temperature and whose free surface is suddenly raised to a fixed higher temperature.

Rayleigh proposed, as a first approximation, that the time t in the Stokes flow could be equated to the distance from the leading edge x and the plate velocity U, i.e., $t = x/U$. This gives for the approximate Blasius flow a velocity profile which gives for the dimensionless wall shear the constant $1/\sqrt{\pi} = 0.564$ which is significantly greater than the Blasius value of 0.332.

The Rayleigh approximation overestimates the convection effects; therefore, its prediction of the boundary-layer growth is too small, and its estimate of the wall shear is too large.

The time scale suggested by Rayleigh is too small, and it was observed that with $t = x/(0.346\,U)$ (see Rosenhead [4]) the Rayleigh approximation would give the Blasius wall shear exactly. No explanation or proof was given for this observation then, and it appears that it has not been pursued (see Churchill [5]). Other approximate analytic methods have been presented (see Schetz [6-8]) which yield less than satisfactory results when applied to the Blasius flow problem.

One objective of this paper is to pursue the observation noted above and to present the development of a novel approximate solution for Blasius flow which is based on a lin-

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earization of the inertia terms of the momentum equation by the introduction of the concept of an *effective* momentum-transport velocity u_e parallel to the plate. This *effective* momentum-transport velocity will be shown to be a constant fraction of the free-stream velocity U_∞.

A second objective of this paper is to introduce the concept of an *effective* enthalpy-transport velocity to find an approximate solution for the Pohlhausen [10] solution of the energy equation.

An approximate analytic solution of the energy boundary-layer equation for laminar forced convection heat transfer from an isothermal plate will be presented and compared with the solution of Pohlhausen [10].

The solutions of Blasius and Pohlhausen require similarity parameters to transform the momentum and energy boundary-layer equations into two *coupled* ordinary differential equations: a third-order equation for momentum and a second-order equation for energy (see Kays and Crawford [11], Burmeister [12], Schetz [13], Cebeci and Bradshaw [14], and Bejan [15]).

Numerical solutions of the coupled ordinary differential equations are required to obtain velocity and temperature profiles, and wall shear and wall heat flux.

In sharp contrast with the similarity solutions of Blasius and Pohlhausen the proposed method provides decoupled closed-form expressions for the velocity and temperature profiles. The wall shear and the wall heat flux can be determined easily and quite accurately.

LINEARIZATION AND SOLUTION OF BLASIUS FLOW

The laminar boundary-layer momentum equation is approximated as

$$u_e \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

where u_e denotes the effective boundary-layer transport velocity parallel to the plate. The definition of u_e, that is exact in the integral form of the momentum equation, can be obtained as

$$u_e = \frac{\frac{d}{dx}\int_0^\delta u(U_\infty - u)dy}{\frac{d}{dx}\int_0^\delta (U_\infty - u)dy} \tag{2}$$

where δ represents the boundary-layer thickness and U_∞ is the free-stream velocity. It can also be shown that u_e is constant and is linearly proportional to U_∞:

$$u_e = \frac{U_\infty}{C} \tag{3}$$

The constant C is defined by Eq. (3), and its numerical value can be determined by matching wall shear or momentum flux.

The linearized convective term on the left side of Eq. (1) can be transformed further by introducing a new variable t such that

$$dt = \frac{dx}{u_e} \tag{4}$$

or, since u_e is assumed to be constant, we obtain:

$$t = \frac{x}{u_e} \tag{5}$$

The parameter t can be viewed as the *effective residence time* of the fluid flow from the leading edge of the plate to the downstream location x.

With the introduction of Eq. (5) into Eq. (1) it becomes

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{6}$$

Introducing the normalized velocity

$$u^* = \frac{u}{U_\infty} \tag{7}$$

and the similarity parameter

$$\eta = \frac{y}{2\sqrt{\nu t}} \tag{8}$$

Eq. (1) reduces to the well-known second-order ordinary differential equation:

$$\frac{d^2u^*}{d\eta^2} + 2\eta\frac{du^*}{d\eta} = 0 \qquad 0 \le \eta < \infty \tag{9}$$

with Dirichlet boundary conditions:

$$\text{at} \quad \eta = 0, \quad u^* = 0 \tag{10}$$

$$\text{as} \quad \eta \to \infty, \quad u^* = 1 \tag{11}$$

The solution is

$$u^* = \text{erf}\ \eta \tag{12}$$

with the similarity parameter expressed in terms of the local Reynolds number Re_x and the as yet unknown *effective* velocity parameter C

$$\eta = \frac{1}{2\sqrt{C}}\ \frac{y}{x}\ Re_x^{1/2} \tag{13}$$

where

$$Re_x = \frac{U_\infty x}{\nu} \tag{14}$$

The dimensionless local wall shear stress (see Kays and Crawford [11], Burmeister [12], Schetz [13], Cebeci and Bradshaw [14], and Bejan [15]) is obtained directly from the solution:

$$\tau_0^* = \frac{\tau_0 x}{\mu U_\infty} = \frac{1}{\sqrt{\pi C}}\ Re_x^{1/2} \tag{15}$$

The value of C can be determined in the following two ways; one may compare Eq. (15) with the so-called *exact* solution of Blasius:

$$\tau_0^* = 0.332 \; Re_x^{1/2} \qquad (16)$$

and find

$$C = 2.887 \qquad (17)$$

which is in agreement with the observation reported in Rosenhead [4]. Or, from the definition of the *effective* velocity, Eq. (2), with the approximate solution given by Eq. (12), one may obtain another estimate of the required parameter:

$$C = 2.414 \qquad (18)$$

The dimensionless velocity profile given by Eq. (12) with C given by Eq. (17) yields the profile in such a way that the shear at the surface becomes identical to the Blasius solution, whereas Eq. (12) with C given by Eq. (18) conserves the momentum flux across the boundary layer.

COMPARISON WITH BLASIUS SOLUTION AND EXPERIMENT

The approximate analytic solutions with the two values of C given by either Eq. (17) or Eq. (18) are compared with the experimental data of Liepmann [9] in Figure 1 for the range $0.85 \times 10^6 \leq Re_x \leq 1.24 \times 10^6$. Also shown in the figure is the Blasius solution and the agreement among the results is seen to be excellent with $C = 2.887$ and very good with $C = 2.414$ over the full range of the parameter $(y/x)\; Re_x^{1/2}$.

The value $C = 2.414$ gives a dimensionless wall shear $\tau_0^* = 0.363$ which is approximately 9.4% larger than the Blasius value.

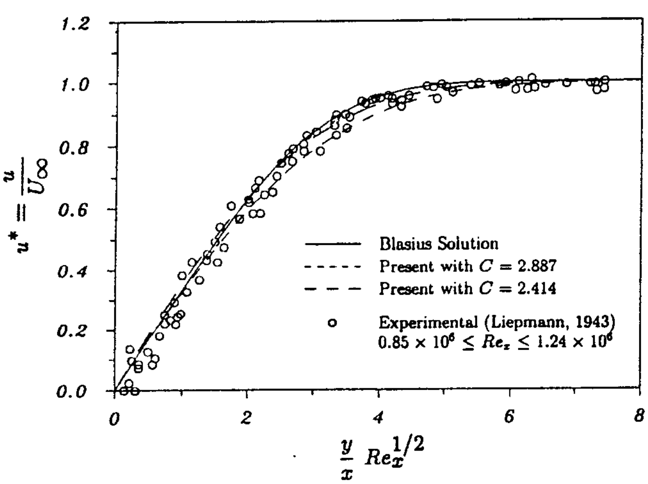

Figure 1 : Comparison of dimensionless velocity distributions.

LINEARIZATION AND SOLUTION OF ENERGY EQUATION

Following the method described above of linearizing the momentum equation, the energy boundary-layer equation is replaced by the linearized equation:

$$u_e^T \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2} \qquad (19)$$

in which the *effective* enthalpy-transport velocity parallel to the plate is denoted as u_e^T. This velocity is related to the constant free-stream velocity U_∞ through the Prandtl number, i.e., $u_e^T = U_\infty f(Pr)$ where $f(Pr)$ is some function of the Prandtl number to be determined by scaling analysis for small and large values of the Prandtl number.

$$u_e^T = \frac{\dfrac{d}{dx}\displaystyle\int_0^{\delta_T} u(T_\infty - T)dy}{\dfrac{d}{dx}\displaystyle\int_0^{\delta_T} (T_\infty - T)dy} \qquad (20)$$

where δ_T represents the thermal boundary-layer thickness and T_∞ is the free-stream temperature. It can also be shown through simple scaling analysis of the energy equation that the *effective* enthalpy-transport velocity u_e^T is constant and linearly proportional to the free-stream velocity U_∞:

Scaling analysis [15] is employed to obtain the relationships at the extreme values of Pr where it can be shown that as

$$Pr \to 0, \qquad u_e^T \to U_\infty \qquad (21)$$

and as

$$Pr \to \infty, \qquad u_e^T \to \frac{U_\infty}{C^T Pr^{1/3}} \qquad (22)$$

The interpolation method of Churchill and Usagi [16] is used to obtain the following *blended* expression which is valid for *all* values of Pr:

$$f(Pr) = \frac{1}{\left[1 + (C^T Pr^{1/3})^2\right]^{1/2}} \qquad 0 \leq Pr < \infty \qquad (23)$$

Hereafter the proposed solution procedure is simple and straight-forward. Introducing the dimensionless temperature

$$\phi = \frac{T - T_\infty}{T_0 - T_\infty} \qquad (24)$$

and the thermal energy equation similarity parameter η_T which can be shown to be the product of the momentum equation similarity parameter η and another function of the Prandtl number which accounts for the *effective* enthalpy-transport velocity defined for the energy equation boundary-layer,

$$\eta_T = \frac{1}{2}\frac{y}{x}\sqrt{Re_x}\sqrt{Pr\; f(Pr)} = \eta\sqrt{CPr\; f(Pr)} \qquad (25)$$

Following the transformation method above the linearized energy equation is transformed into the following second-order ordinary differential equation:

$$\frac{d^2\phi}{d\eta_T{}^2} + 2\eta_T \frac{d\phi}{d\eta_T} = 0 \qquad 0 \le \eta_T < \infty \tag{26}$$

with Dirichlet boundary conditions:

$$\eta_T = 0, \qquad \phi = 1 \tag{27}$$

and

$$\eta_T \to \infty, \qquad \phi \to 0 \tag{28}$$

The solution of the above equation is

$$\phi = \operatorname{erfc} \eta_T \tag{29}$$

The local wall heat flux is defined as

$$q_0 = -k(T_0 - T_\infty)\frac{d\phi}{d\eta_T}\frac{\partial \eta_T}{\partial y}\bigg|_{\eta_T=0} \tag{30}$$

leads to

$$q_0 = \frac{1}{\sqrt{\pi}} k(T_0 - T_\infty)\frac{1}{x}\sqrt{Re_x}\sqrt{Pr\, f(Pr)} \tag{31}$$

The dimensionless local wall heat flux or local Nusselt number is written in the form:

$$Nu_x \equiv \frac{q_0 x}{k(T_0 - T_\infty)} = Re_x^{1/2}\, F(Pr) \tag{32}$$

with the Prandtl number function:

$$F(Pr) \equiv \left[\frac{Pr}{\pi\left[1 + (C^T Pr^{1/3})^2\right]^{1/2}}\right]^{1/2} \qquad 0 \le Pr < \infty \tag{33}$$

The extreme values of $F(Pr)$ are

$$Pr \to 0, \qquad F(Pr) \to \frac{1}{\sqrt{\pi}}\sqrt{Pr} \tag{34}$$

and

$$Pr \to \infty, \qquad F(Pr) \to \frac{1}{\sqrt{\pi C^T}}\, Pr^{1/3} \tag{35}$$

Two methods can be used to find values of the thermal parameter C^T. The first method compares the approximate and the so-called exact dimensionless wall heat flux at the extreme value $Pr \to \infty$ which gives $C^T = 2.77$. The second method is based on an enthalpy flux balance across the thermal boundary layer; using the approximate solutions for the velocity and temperature distributions, Eqs. (12) and (29), and the definition of u_e^T, Eq. (20), gives the second estimate $C^T = 2.13$.

COMPARISON WITH SOLUTION OF POHLHAUSEN

The asymptotes given above for $F(Pr)$ are in complete agreement with the results of Pohlhausen [10]. Also, if $Pr = 1$, $F(Pr) = 0.329$ which is approximately 1 % below the numerical value given by Pohlhausen [10].

The approximate analytic solution is compared with the solution of Pohlhausen as presented in Kays and Crawford [11]:

$$Nu_x = \frac{Re_x^{1/2}}{\sqrt{2}\int_0^\infty exp\left(-Pr\int_0^{\eta_o}\psi\, d\eta_o\right) d\eta_o} \tag{36}$$

where η_o and ψ are the Blasius similarity variable and stream function, respectively.

The reciprocal of the complex integral appearing in the denominator of the Pohlhausen solution is the $F(Pr)$ obtained by the proposed method. Obviously, the $F(Pr)$ expression given by Eq. (33) is significantly simpler than that of Eq. (36).

In Figure 2 the temperature profiles for three values of Pr are plotted and compared with the Pohlhausen similarity solution; the agreement is seen to be very good everywhere and excellent near the wall.

The parameters and solutions of Blasius and Pohlhausen and those of the present approximate method are compared in Table 1. The complexity of the Blasius and Pohlhausen solutions relative to the present approximate solutions is clearly seen. The so-called exact solutions require extensive numerical integrations while the approximate solutions require the computation of the error function or its complement.

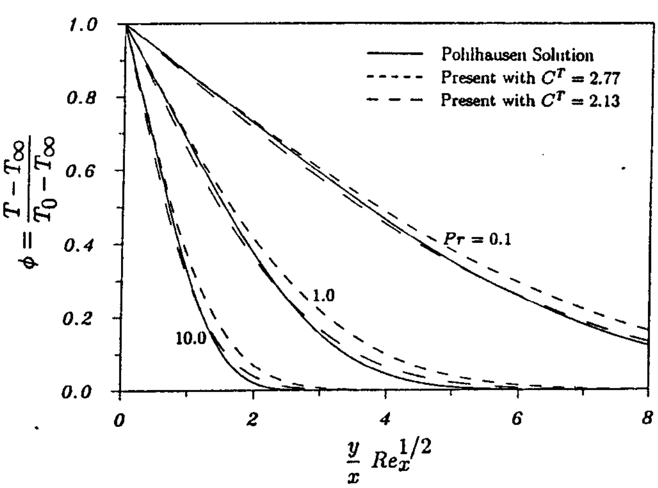

Figure 2 : Comparison of dimensionless temperature distributions for various Prandtl numbers.

Table 1 : Comparison of Blasius and Pohlhausen methods and present method.

	Blasius and Pohlhausen	Present
Similarity Variables	$\eta_o = \frac{y}{x}\sqrt{\frac{Re_x}{2}}$	$\eta = \frac{1}{\sqrt{2C}}\eta_o$ $\eta_T = \sqrt{\frac{Pr\, f(Pr)}{2}}\eta_o$
ODE's	$\frac{d^3\psi}{d\eta_o^3} + \psi\frac{d^2\psi}{d\eta_o^2} = 0$ $\frac{d^2\phi}{d\eta_o^2} + Pr\,\psi\frac{d\phi}{d\eta_o} = 0$	$\frac{d^2u^*}{d\eta^2} + 2\eta\frac{du^*}{d\eta} = 0$ $\frac{d^2\phi}{d\eta_T^2} + 2\eta_T\frac{d\phi}{d\eta_T} = 0$
u^*	$\frac{d\psi}{d\eta_o}$	$\mathrm{erf}\ \eta$
$\frac{v}{U_\infty}\sqrt{Re_x}$	$\frac{1}{\sqrt{2}}\left(\eta_o\frac{d\psi}{d\eta_o} - \psi\right)$	$\sqrt{\frac{C}{\pi}}\left(1 - e^{-\eta^2}\right)$
$\frac{\partial u^*}{\partial y}\frac{x}{\sqrt{Re_x}}$	$\frac{1}{\sqrt{2}}\frac{d^2\psi}{d\eta_o^2}$	$\frac{1}{\sqrt{\pi C}}\,e^{-\eta^2}$
ϕ	$\frac{\int_{\eta_o}^{\infty} exp\left(-Pr\int_0^{\eta_o}\psi\, d\eta_o\right)d\eta_o}{\int_0^{\infty} exp\left(-Pr\int_0^{\eta_o}\psi\, d\eta_o\right)d\eta_o}$	$\mathrm{erfc}\ \eta_T$
$\frac{\partial\phi}{\partial y}\frac{x}{\sqrt{Re_x}}$	$-\frac{exp\left(-Pr\int_0^{\eta_o}\psi\, d\eta_o\right)}{\sqrt{2}\int_0^{\infty} exp\left(-Pr\int_0^{\eta_o}\psi\, d\eta_o\right)d\eta_o}$	$-\sqrt{\frac{Pr\, f(Pr)}{\pi}}\,e^{-\eta_T^2}$

CONCLUDING REMARKS

A novel method is proposed of linearizing the boundary-layer momentum and energy equations. This is accomplished by the introduction of *effective* momentum-transport and enthalpy-transport velocities which are assumed to be constant across the momentum and thermal boundary-layers respectively. Introducing time scales which are related to the distance from the leading edge to the local position and the *effective* velocities, permits one to transform the linearized momentum and energy equations into identical parabolic equations whose solutions can be expressed in closed form in terms of the error function or its complement.

The approximate analytic solutions require a single parameter which relates the *effective* momentum and thermal-transport velocities to the free-stream velocity. Two methods are proposed for estimating the parameter: i) matching the wall shear or heat flux with the exact solutions, or ii) conserving the momentum and enthalpy fluxes across the respective boundary-layers. The first method provides a value of the parameter which gives approximate solutions which are in excellent agreement with the exact solutions. The second method provides a value of the parameter which overestimates the wall shear stress and wall heat flux, but gives good agreement with the dimensionless velocity and temperature distributions.

ACKNOWLEDGMENT

This work was supported by the Natural Sciences and Engineering Research Council of Canada under Operating Grant A7455.

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