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General Expression for Laminar Forced and Natural Convection Heat Transfer from Isothermal Flat Plates for all Prandtl Numbers

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GENERAL EXPRESSION FOR LAMINAR FORCED AND NATURAL CONVECTION HEAT TRANSFER FROM ISOTHERMAL FLAT PLATES FOR ALL PRANDTL NUMBERS

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ABSTRACT

The conventional correlation equations for forced and natural convection boundary-layer flows over isothermal flat plates are shown to collapse into a single simple expression which directly relates the dimensionless wall temperature excess to a new Prandtl number function which is applicable for the full range of the Prandtl number from zero to infinity.

The dimensionless wall temperature excess is defined with respect to the unknown area-mean wall heat flux, the thermal conductivity of the fluid, the given wall temperature excess and a new length scale which is related to the plate length, the fluid thermal diffusivity and the maximum fluid velocity within the respective boundary layers.

The conventional forced and natural convection dimensionless groups: Nusselt, Reynolds, Peclet, Grashof, Rayleigh and Boussinesq numbers do not appear in the proposed simple general correlation equation.

The new formulation allows the forced and natural convection heat transfer results to appear on the same graph as dimensionless temperature excess versus Prandlt number functions. This shows that the two physical phenomena are closely related; this is not apparent from the conventional correlation equations.

NOMENCLATURE

Bo_L - Boussinesq number, $g\beta\theta_0 L^3/\alpha^2$
C_0 - constant at $Pr \rightarrow 0$
Gr_L - Grashof number, $g\beta\theta_0 L^3/\nu^2$
g - gravitational acceleration, m/s^2
h - heat transfer coefficient, W/m^2K
k - thermal conductivity, W/mK
L - plate length, m
Pe_L - Peclet number, LU_∞/α
Pr - Prandtl number, ν/α
$\bar{q}_0$ - area-mean wall heat flux, W/m^2
Q - heat flow rate, W
Re_L - Reynolds number, LU_∞/ν
Ra_L - Rayleigh number, $g\beta\theta_0 L^3/(\alpha\nu)$
T_0 - wall temperature, $°C$
T_∞ - free stream temperature, $°C$
U_∞ - freestream fluid velocity, m/s
U_{max} - maximum fluid velocity, m/s

Greek Symbols

α - thermal diffusivity $\equiv k_f/(\rho \cdot c_p)$, m^2/s
δ - hydrodynamic boundary-layer thickness, m
δ_T - thermal boundary-layer thickness, m
Δ - conduction thickness, m
ν - kinematic viscosity, m^2/s
$\Theta^\star$ - dimensionless temperature excess, $\dfrac{k\theta_0}{\bar{q}_0}\Big/\sqrt{\dfrac{L\alpha}{U_{max}}}$
θ_0 - wall temperature excess, $T_0 - T_\infty$

Subscripts

f - fluid
fc - forced convection
∞ - freestream value or value at infinite limit
L - based on plate length
nc - natural convection
0 - wall-based value or value at zero limit
T - thermal
max - maximum

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INTRODUCTION

Historically the forced and natural convection models, theory and correlation equations are kept separate, and even today most heat transfer texts[1–4,9,10,12] and handbooks[11] present the respective material in separate chapters. On rare occasions a researcher may use the forced convection results to develop an approximate correlation equation for natural convection. The recently published heat transfer text by Mills[14] presents forced and natural convection fundamentals and correlation equations in one chapter; however, the respective material is placed in separate sections.

The forced and natural convection results are presented in such a way that the close relationship between the two solutions are obscured by the various dimensionless parameters (numbers) used to present the final results. The actual physics of the flow and the true heat transfer characteristics are hidden behind the numerous dimensionless groups which have been introduced into the correlation equations over many decades.

Laminar forced and natural convection heat transfer from isothermal flat plates are characterized by means of relationships between several dimensionless numbers: Nusselt, Reynolds, Prandtl, Peclet, Grashof, Rayleigh and Boussinesq[1–12].

The Reynolds and Grashof numbers are hydrodynamic parameters which appear in the forced-convection and buoyancy-induced momentum equations respectively, while the Prandtl number is a fluid property parameter.

The Peclet and Boussinesq numbers appear in the solutions to the forced and natural convection energy equations rexpectively. They are frequently defined as products of the hydrodynamic parameters and the Prandtl number.

The Rayleigh number is the product of the Grashof and Prandtl numbers; it does not appear intrinsically in the energy equation or its solution; but, it is always used in the natural convection correlation equations.

The objectives of this study are: i) to *reveal* the common base of the forced and natural convection results for the asymptotic limits of large and small Prandtl numbers: $Pr \to \infty$ and $Pr \to 0$ respectively, and ii) to develop a general expression which directly relates the area-mean wall heat flux to the wall temperature excess, and it will be valid for the full range of Prandtl number: $0 \leq Pr < \infty$.

Also the general results will be presented in a form which will clearly show the important length scales and thermophysical parameters which control the advection-diffusion processes under forced and natural convection flows.

REVIEW OF FORCED AND NATURAL CONVECTION CORRELATIONS

Laminar forced and natural convection heat transfer from isothermal flat plates have been the subject of a very large number of analytical, experimental and numerical investigations.

The first analytical works of Blasius and Pohlhausen who solved the forced convection boundary-layer momentum and energy equations have led to numerous other analytical and experimental studies which are described in some detail in the many heat transfer texts which are currently available as well as some review articles.

The subject of laminar boundary-layer buoyancy-induced flow over an vertical isothermal flat plate has a longer history beginning with the approximate analytical work of Lorenz[9] which was followed some fifty years later by a more complete experimental and analytical investigation by Schmidt[14]. Subsequently more analytical, experimental and numerical results were reported, and they are found in various review articles, handbooks and text books.

The correlation equations for both forced and natural convection posses certain common features, i.e., the dimensionless area-mean wall heat flux is called the Nusselt number which is related to some independent flow parameter such as the Reynolds number in forced convection and the Grashof number in buoyancy-induced flow. For closure the respective correlation equations also introduce the important fluid parameter, Pr, either alone or in some combination with the flow parameters. For example, in forced convection the Peclet number, $Pe = RePr$ is frequently introduced into correlation equations, especially for heat transfer into very small Prandtl number fluids $Pr << 1$ such as liquid metals.

In natural convection the correlation equations are often presented in terms of the following products of the Grashof and Prandtl numbers: $GrPr$ and $GrPr^2$. These products arise naturally from scale analysis at the limits: $Pr \to \infty$ and $Pr \to 0$, and the are sometimes denoted as $GrPr = Ra$ and $GrPr^2 = RaPr$ or $GrPr^2 = Bo$.

The many correlation equations reported in the heat transfer texts for both forced and natural convection appear in different forms and are usually developed for a particular range of the Prandtl number. The correlation equations based on integral methods or experimental data are approximate and not applicable over the full range of Prandtl number.

LeFevre[13] was the first investigator to develop a general correlation equation valid over the full range of Prandtl number. It was based on the exact solutions obtained for the limits: $Pr \to 0$ and $Pr \to \infty$. Subsequently Ede[12] proposed a similar, slightly less accurate correlation equation; however he rounded the correlation coefficients to the nearest integers.

Churchill[7,8] proposed an alternate form of the correlation equation which is also based on the *blending* of the asymptotic analytic solutions. This form is most frequently quoted in the heat transfer texts; therefore it will be considered in this work.

Churchill[6,8] has also proposed a general correlation equation for forced convection. Since it is the most frequently quoted correlation equation in the heat transfer texts, it will also be considered in this work.

Forced Convection Correlation Equation

Churchill[7,8] has proposed the following accurate correlation equation for the area-mean Nusselt number which is valid over the full range of the Prandtl number from zero to infinity:

$$Nu_L = \frac{0.6774 Re_L^{1/2} Pr^{1/3}}{\left[1 + (0.0468/Pr)^{2/3}\right]^{1/4}} \qquad Pe_L > 100 \qquad (1)$$

Natural Convection Correlation Equation

Churchill[6,8] has proposed the following accurate correlation equation for the area-mean Nusselt number which is applicable over the full range of the Prandtl number from zero to infinity:

$$Nu_L = \frac{0.670 Ra_L^{1/4}}{\left[1 + (0.492/Pr)^{9/16}\right]^{4/9}} \qquad 10^4 < Gr_L < 10^9 \quad (2)$$

The two correlation equations appear to be quite different and therefore, they are assumed to be unrelated. We note that the product of the square root of the Reynolds number and Prandtl number to the power of one-third appears in the forced convection equation. On the other hand the fourth root of the Rayleigh number appears in the natural convection equation.

The Prandtl number function which is appears in the denominator of each equation are totally different. The one common feature of the two equations is the numerical value which appears in the numerator of each equation. We note that they differ by only 1.1 %.

The apparent differences appear because of the *arbitrary* choice of dimensionless groups used to represent the hydrodynamics and the energy equations. Both equations were developed for the large Prandtl number limit as can be seen from the trends of the two denominators.

It will be shown in the subsequent sections that the apparent differences vanish when the appropriate dimensionless numbers are used in the development of the forced and natural convection asymptotes and correlation equations. The appropriate dimensionless numbers arise from the low Prandtl number limit.

ASYMPTOTES FOR FORCED AND NATURAL CONVECTION

Analytic solutions are available for the extreme values of the Prandtl number. Various analytic methods reported in several papers and heat transfer texts[1-12] have been used to determine the area-mean Nusselt number as a function of the dimensionless numbers discussed above. The asymptotic solutions to be used here will be presented below. The large Prandtl number results are appropriate for most fluids, while the small Prandtl results are limited to liquid metals. Therefore the large Prandtl number asymptotes will be consider first; however, the small Prandtl number dimensionless groups will be introduced into them.

Large Prandtl Number Asymptotes

The analytic solutions for laminar forced and natural convection heat transfer from isothermal flat plates for $Pr \rightarrow \infty$ are well-known[1-12] and are given in the following forms:

Forced Convection

$$Nu_L = 0.664\, Re_L^{1/2} Pr^{1/3} \qquad (3)$$

and

Natural Convection

$$Nu_L = 0.670\, Ra_L^{1/4} \qquad (4)$$

These apparently different solutions can be brought together by the introduction of the Peclet, Pe, and Boussinesq, Bo, numbers into the forced and natural convection asymptotes respectively.

As mentioned above, these important parameters appear in the dimensionless formulation of the energy equation when $Pr << 1$. Bejan[2] and more recently Mills[15] discuss the importance of the Boussinesq number in natural convection.

We now divide the forced convection result by $\sqrt{Pe_L}$ and the natural convection result by $Bo_L^{1/4}$; this gives the following interesting forms of the asymptotes:

$$\frac{Nu_L}{Pe_L^{1/2}} = \frac{0.664}{Pr^{1/6}} \qquad (5)$$

and

$$\frac{Nu_L}{Bo_L^{1/4}} = \frac{0.670}{Pr^{1/4}} \qquad (6)$$

Examination of the new forms of the large Prandtl number asymptotes reveals that the forced and natural convection flow parameters Pe_L and Bo_L are related: i.e., $Pe_L \equiv \sqrt{Bo_L}$.

A physical interpretation of the above results is that the ratio of the convection diffusion length δ_T to the conduction length Δ is equal to a constant divided by the Prandtl number to some power less than unity. This will become more apparent later.

Since the constants in the two asymptotes differ by approximately 1.1 %, we can set both to the value 2/3, and now the above results for forced and natural convection can be generalized in the following physical manner which does not require the Nusselt, Peclet and Boussinesq numbers:

$$\frac{\delta_T}{\Delta} = \frac{2}{3\,Pr^n} \qquad Pr \to \infty \tag{7}$$

where $n = 1/6$ for forced convection and $n = 1/4$ for natural convection. The Prandtl number remains in the general equation.

In the following section it will be shown that $Pe_L^{1/2}$ and $Bo_L^{1/4}$ can be replaced by system parameters such as the plate length, the thermal diffusivity of the fluid and the maximum velocity in the respective boundary layers.

In the following section we will seek simple relationships between the based on the maximum fluid boundary-layer velocity.

GENERAL RELATIONSHIP BASED ON MAXIMUM VELOCITY

In the solution to the energy equation, the forced convection flow parameter is

$$Pe_L = Re_L Pr = \frac{L}{\alpha} U_\infty \tag{8}$$

where the free stream velocity is the maximum boundary-layer velocity. Therefore, the Peclet number can be said to represent the maximum dimensionless velocity in the boundary layer.

In the solution to the natural convection energy equation the important parameter is

$$Bo_L = Ra_L Pr = \frac{g\beta(T_0 - T_\infty)L^3}{\alpha\nu} \cdot \frac{\nu}{\alpha} = \frac{g\beta(T_0 - T_\infty)L^3}{\alpha^2} \tag{9}$$

Therefore the new natural convection flow parameter can be written as

$$\sqrt{Bo_L} = \frac{L}{\alpha}\sqrt{g\beta(T_0 - T_\infty)L} \tag{10}$$

which is said to be the maximum dimensionless velocity scale in the buoyancy-induced boundary layer:

$$\sqrt{Bo_L} = \frac{L}{\alpha} U_{max} \tag{11}$$

The close physical relationship between Pe_L and $\sqrt{Bo_L}$ is now evident.

The above relationships lead to the following general asymptote applicable to both forced and natural convection:

$$\frac{Nu_L}{\sqrt{\frac{L}{\alpha} U_{max}}} = \frac{1}{\frac{3}{2} Pr^n} \qquad Pr \to \infty \tag{12}$$

The Nusselt number can be expressed as

$$Nu_L = \frac{\bar{h}L}{k} = \frac{\bar{q}_0 L}{k(T_0 - T_\infty)} = \bar{q}_0^\star \tag{13}$$

where $\bar{q}_0^\star$ is the dimensionless area-mean wall flux. Letting $U_{max}^\star = \frac{L}{\alpha} U_{max}$ provides us with another general expression applicable to the forced and natural asymptotes:

$$\frac{\bar{q}_0^\star}{\sqrt{U_{max}^\star}} = \frac{1}{\frac{3}{2} Pr^n} \qquad Pr \to \infty \tag{14}$$

The assertion that the left-hand side of the above equation is the ratio of two length scales can now be easily demonstrated. From the last equation we have

$$\frac{\sqrt{\frac{L\alpha}{U_{max}}}}{\frac{k(T_0 - T_\infty)}{\bar{q}_0}} \tag{15}$$

The numerator of the above general relationship is the thermal diffusion length scale for convection heat transfer, i.e., $\delta_T = \sqrt{\frac{L\alpha}{U_{max}}}$ and the denominator is the length scale for conduction, i.e., $\Delta = \frac{k(T_0 - T_\infty)}{\bar{q}_0}$.

Small Prandtl Number Asymptotes

The small Prandtl number $Pr \to 0$ asymptotes are presented in the following forms:

Forced Convection

$$Nu_L = \frac{2}{\sqrt{\pi}} Re_L^{1/2} Pr^{1/2} \tag{16}$$

and

Natural Convection

$$Nu_L = 0.8005 Ra_L^{1/4} Pr^{1/4} \tag{17}$$

The apparent differences between these asymptotic solutions can be removed by introducing the Peclet and Boussinesq numbers. With these dimensionless flow parameters the above relationships can be written as

$$\frac{Nu_L}{Pe_L^{1/2}} = \frac{2}{\sqrt{\pi}} \tag{18}$$

and

$$\frac{Nu_L}{Bo_L^{1/4}} = \sqrt{\frac{2}{\pi}} \tag{19}$$

The constant in Eq. (17) has been replaced by the constant $\sqrt{2/\pi} = 0.7979$ with negligible error. The constants in the forced and natural convection relationships given above differ by a factor of $\sqrt{2}$ or approximately 41 %.

By means of the two length scales introduced above the two asymptotic solutions can be replaced by a single equation similar to Eq. (7):

$$\frac{\delta_T}{\Delta} = C_0 \qquad Pr \to 0 \tag{20}$$

with $C_0 = 2/\sqrt{\pi}$ for forced convection and $C_0 = \sqrt{2/\pi}$ for natural convection.

GENERAL INTERPOLATION FUNCTION

A general interpolation function for $0 < Pr < \infty$ can be obtained by inversion of the previous relationships and the introduction of the dimensionless temperature excess (or dimensionless thermal resistance):

$$\Theta^\star \equiv \frac{k\theta_0}{\bar{q}_0} \Big/ \sqrt{\frac{L\alpha}{U_{max}}} \tag{21}$$

which has the following limits:

$$\frac{1}{C_0} \leq \Theta^\star \leq \frac{3}{2} Pr^n \qquad 0 \leq Pr < \infty \tag{22}$$

Following the natural convection work of Churchill[8] and the forced convection work of Churchill and Ozoe[7] the above limits are *blended* by means of the interpolation parameter $m > 1$:

$$(\Theta^\star)^m = \left(\frac{1}{C_0}\right)^m + \left(\frac{3}{2} Pr^n\right)^m \tag{23}$$

The above interpolation function can be expressed in terms of either the small or large Prandtl number asymptotes. Following the convention established by Churchill[5-8] the large Prandtl number asymptote is selected to give the general interpolation function applicable to both forced and natural convection heat transfer:

$$\Theta^\star = \frac{3}{2} Pr^n \left[1 + \left(\frac{2}{3C_0 Pr^n}\right)^m\right]^{1/m} \qquad 0 \leq Pr < \infty \tag{24}$$

The value of the interpolation parameter does not effect the asymptotic values established above. The appropriate value of the interpolation parameter m is important for $Pr = 1$. The values recommended by Churchill[8], i.e., $m = 9/4$ for natural convection, and the value recommended by Churchill and Ozoe[6], i.e., $m = 4$ for forced convection are used here.

For convenience and further discussion the interpolation functions for the natural and forced convection are summarized below.

Natural Convection Interpolation Function

With $C_0 = \sqrt{2/\pi}, m = 9/4, n = 1/4$, Eq. (24) becomes

$$\Theta^\star_{nc} = \frac{3}{2} Pr^{1/4} \left[1 + \left(\sqrt{2\pi} \cdot \frac{1}{3Pr^{1/4}}\right)^{9/4}\right]^{4/9} \qquad 0 \leq Pr < \infty \tag{25}$$

and $1/\Theta^\star_{nc} = Nu_L/(Ra_L^{1/4} Pr^{1/4})$.

Forced Convection Interpolation Function

With $C_0 = 2/\sqrt{\pi}, m = 4, n = 1/6$, Eq. (24) becomes

$$\Theta^\star_{fc} = \frac{3}{2} Pr^{1/6} \left[1 + \left(\sqrt{\pi} \cdot \frac{1}{3Pr^{1/6}}\right)^{4}\right]^{1/4} \qquad 0 \leq Pr < \infty \tag{26}$$

and $1/\Theta^\star_{fc} = Nu_L/(Re_L^{1/2} Pr^{1/2})$.

We observe from the two preceding relationships that for very large Prandtl numbers $\Theta^\star_{nc}/\Theta^\star_{fc} \to Pr^{1/12}$

CONCLUDING REMARKS

A single, simple relationship between the dimensionless temperature excess and the Prandtl number was developed for forced and natural convection heat transfer from isothermal flat plates.

The dimensionless temperature excess is based on the given thermophysical parameters such as the temperature excess, fluid thermal conductivity and thermal diffusivity, plate length and the respective maximum velocities within the forced and buoyancy-induced flows. The unknown area-mean wall heat flux appears in the dimensionless temperature excess.

The independent parameter is a Prandtl number function which depends on an intrinsic parameter which appears in the scaling analyses performed at the very large Prandtl number limit; and a second parameter which arises from the development of a general interpolation function which is valid for the full range of the Prandtl number from zero to infinity.

The *blending* parameters for forced and natural convection respectively are based on the works of Churchill.

The proposed general expression does not dependent on the conventional dimensionless groups such as: Nusselt, Reynolds, Peclet, Grashof, Rayleigh, and Boussinesq numbers. The reciprocal of the dimensionless temperature excess (which could be called the dimensionless area-mean wall flux) is shown to be related to the conventional dimensionless groups as they appear in the solutions at the very low Prandtl number limits.

During the development of the proposed general expression, important relationships between the forced and natural convection parameters such as the Peclet and Boussinesq numbers are presented, as well as other relationships between the thermal boundary-layer thickness and the conduction thickness.

The general expression allows the forced and natural convection results to be plotted on the same graph because they now have a common base. This is not possible using the conventional correlation equations.

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