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Approximate Analytical Solution of Forced Convection Heat Transfer From Isothermal Spheres for All Prandtl Numbers

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A new, simple and approximate analytical method based on linearization of the energy equation is proposed to develop solutions for forced convection heat transfer from isothermal spheres. Furthermore, heat transfer correlations from spheres are proposed in the range of Reynolds number, $0 \le Re_D \le 2 \times 10^4$, and all Prandtl numbers. This technique is performed as follows. The first step is to approximate the energy equation to the form of a transient heat conduction equation that has an existing solution. The second step is to evaluate the effective velocity through scaling analysis in the limit of $Pr \to \infty$ and $Pr \to 0$ and then resubstitute the effective velocity into the solution of the energy equation. Finally, a "blending method" is used to provide a general model for all Prandtl numbers. Comparison of the heat transfer correlations for Nu_D versus Re_D from the present study with the available correlations in the literature reveals very good agreement.

1 Introduction

Forced convection heat transfer from isothermal or isoflux external convex surfaces is an important problem for engineers. There are many engineering systems that are modeled using forced convection, such as electronic components on printed circuit boards placed in cabinets, hot-wire anemometers, and heat exchanger design. Steady laminar forced convection heat transfer from an isothermal sphere into a substantial amount of air or water has been investigated experimentally, theoretically, and numerically by many researchers for over 90 years. These researchers presented their area-averaged heat transfer results in the following general form:

$$\mathrm{Nu}_D = C_1 + C_D \,\mathrm{Re}_D^m \,\mathrm{Pr}^b \tag{1}$$

where C_1, C_D, m, and b are constants. The diffusive term, C_1, has been reported to lie in the range 1 to 3.2. Other investigators have reported values of zero or $1.2\,\mathrm{Pr}^{0.3}$ (Vliet and Leppert, 1961; Lochiel and Calderbank, 1964). The constant C_D depends on the range of Re_D. From the literature, it is seen that for $0.1 \le \mathrm{Re}_D \le 2 \times 10^5$, C_D is reported between 0.921 and 0.175, respectively. The exponents of Re_D and Pr varied from 0.47 to 0.62, and 0 to 0.42, respectively. The previous correlations, $\mathrm{Nu}_D = \mathrm{Nu}_D(\mathrm{Re}_D, \mathrm{Pr})$, are summarized in Table 1.

Most of the previous investigators agreed on the following:

1 In the diffusive limit, $\mathrm{Re}_D \to 0$, Nu_D is 2.0.
2 Exponent of Pr, b, is $\frac{1}{3}$.

However, the previous investigators disagreed on the value of C_D and the exponent of Reynolds number. On the other hand, Churchill (1977) proposed another function for Pr, $\mathrm{Pr}^{1/3}/[1 + (0.45/\mathrm{Pr})^{2/3}]^{1/4}$, instead of $\mathrm{Pr}^{1/3}$, which provides two limits: as $\mathrm{Pr} \to \infty$, $F(\mathrm{Pr}) \to \mathrm{Pr}^{1/3}$ and as $\mathrm{Pr} \to 0$, $F(\mathrm{Pr}) \to \mathrm{Pr}^{1/2}$. From the theoretical view, it has been proven by Lochiel and Calderbank (1964) using an analytical approach that $\mathrm{Sh} = \mathrm{Sh}(\mathrm{Re}^{1/2}\mathrm{Sc}^{1/3})$ for $\mathrm{Re} \gg 1$. Therefore, the analytical solution agrees with the experimental results, as seen in Table 1, for $\mathrm{Pr} \ge 1$. In addition, as $\mathrm{Pr} \to 0$, Sideman (1966) and Hsu (1963) (see Witte, 1968) have demonstrated through their analytical solutions that $F(\mathrm{Pr}) \to \mathrm{Pr}^{1/2}$. This also agrees with the experimental heat transfer correlations of Witte (1968) and Churchill (1977).

One of the objectives of this study is to develop a simple approximate analytical method based on linearization of the energy equation in order to develop an approximate analytical solution of the forced convection heat transfer from isothermal spheres. Another objective of the present study is to propose heat transfer correlations from spheres for the range of Reynolds number, $0 \le \mathrm{Re}_D \le 2 \times 10^4$, and all Prandtl numbers. One of the main goals of the present investigation is to explain why the previous studies appear to be in disagreement in regard to the exponent of Re_D and the constant C_D.

This paper is organized as follows: In the following section, the theoretical analysis is developed with appropriate assumptions; in the third section, results and discussions are presented; and conclusions are given in Section 4.

2 Theoretical Analysis

Figure 1 shows an isothermal sphere of temperature T_S and diameter D, which is immersed in a steady, laminar, incompressible flow of a constant property fluid ($0 < \mathrm{Pr} < \infty$) at constant temperature T_∞ and uniform velocity V_∞. The energy equation inside the boundary layer is

$$v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} = \alpha\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)\right] \tag{2}$$

The terms on the left side of Eq. (2) will be approximated by a single equivalent term, i.e., $(\bar{v}_e/r)(\partial T/\partial \theta)$, where $\bar{v}_e$ is the average effective velocity, which will be determined later. This idea has been proposed by Oseen in order to linearize the inertia term for creeping flow, where Oseen assumed the convective term to be $V_\infty \nabla \cdot \mathbf{v}$, (for more details see Happel and Brenner, 1973). In addition, the effective velocity has been introduced by Yovanovich et al. (1992) and Jafarpur (1992); therefore, Eq. (2) becomes

$$\frac{\bar{v}_e}{r}\frac{\partial T}{\partial \theta} = \alpha\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)\right] \tag{3}$$

This equation is limited to the range $r \ge D/2$ and $0 \le \theta \le \pi$. Equation (3) will be transformed to transient heat conduction in

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order to find a suitable solution. Let us assume that the flow particles are moving with a constant velocity, $\bar{v}_e$, around the body. Therefore, the particles will take time, Δt, to travel a distant $r\Delta\theta$. Furthermore, for $\Delta\theta \to 0$ and $\Delta t \to 0$, one can obtain

$$\bar{v}_e = r\frac{\partial\theta}{\partial t} \quad \text{where} \quad D/2 \le r \le \delta + D/2 \tag{4}$$

This concept was also used by Sideman (1966) and Yovanovich et al. (1992). Therefore, the energy equation will also be reduced to the form of the transient heat conduction equation. Thus, by substituting Eq. (4) into Eq. (3) the energy equation can be written as follows:

$$\frac{\partial T^*}{\partial t} = \alpha\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T^*}{\partial r}\right)\right] \tag{5}$$

where $r \ge D/2$, $0 \le t \le \pi D/2\bar{v}_e$, and

$$T^* = (T - T_\infty)/(T_S - T_\infty).$$

The solution of Eq. (5) from Carslaw and Jaeger (1959) is

$$T^* = \frac{D}{2}\frac{1}{r}\,\text{erfc}\left(\frac{r - D/2}{2\sqrt{\alpha t}}\right)\Bigg|_{t=(\theta D)/(2\bar{v}_e)}$$

$$= \frac{D}{2}\frac{1}{r}\,\text{erfc}\left(\frac{r - D/2}{2\sqrt{(\alpha\theta D)/(2\bar{v}_e)}}\right) \tag{6}$$

The local Nusselt number

$$\text{Nu}_D(\theta) = \frac{q_S(\theta)D}{(T_S - T_\infty)k} \tag{7}$$

where $q_S(\theta) = -k(T_S - T_\infty)(\partial T^*/\partial r)|_{r=D/2}$.

Taking the derivative of Eq. (6) and substituting it into Eq. (7) gives

$$\text{Nu}_D(\theta) = 2 + \frac{1}{\sqrt{\pi}}\frac{\sqrt{D}}{\sqrt{(\alpha\theta)/(2\bar{v}_e)}} \quad \theta > 0 \tag{8}$$

The transient conduction solution provides an analytic solution for the local Nusselt number, which consists of the linear sum of the local boundary layer term and the constant term corresponding to the diffusive limit ($\text{Re}_D \to 0$). The area-averaged Nusselt number, $\text{Nu}_D = 1/A \iint_A \text{Nu}_D(\theta)dA$, is given by

Table 1 Correlation coefficients and parameters for heat and mass transfer from spheres

Author	C_1	C_D	m	b	$Pr(Sc)$	Re_D
Frössling (1938)†	2.0	0.55	1/2	1/3	0.6 - 2.7	2 - 1,000
Kramers (1946)	3.2	0.59	1/2	1/3	0.71	540 - 1,460
Kudryashev (1949)†	2.0	0.33	1/2	0.0	0.71	
Drake-Backer (1952)	2.0	0.459	0.55	0.333	0.71	0.1 - 200,000
Radusich (1956)†	2.83	0.6	1/2	1/3	0.71	
Griffth (1960)	2.0	0.6	1/2	1/3	0.7	
Yuge (1960)	2	0.551	1/2	1/3	0.715	10 - 1,800
	2	0.335	0.5664	1/3	0.715	1,800 - 150,000
Pasternak-Gauvin (1960)	0.0	0.692	0.514	1/3	0.71	500 - 5,000
Vliet-Leppert (1961)	$1.2Pr^{0.2}$	0.53	0.54	0.3	2-380	1 -300,000
Hsu (1964)*	0.0	0.921	0.5	0.5	$Pr \to 0$	
Rowe et al. (1965)	2	0.69	1/2	1/3	0.73	65 - 1,750
Lochiel-Calderbank (1964)	0.0	0.7	1/2	1/3		>> 1
Sideman (1966)	0.0	1.13	0.5	0.5	$Pr \to 0$	
Hughmark (1967)	2.0	0.6	1/2	1/3	< 250	26 - 450
	2.0	0.5	1/2	1/3	> 250	1 - 17
	2.0	0.4	1/2	1/3	> 250	17 - 450
	2.0	0.27	0.62	1/3	< 250	450-10,000
	2.0	0.175	0.62	0.42	> 250	450-10,000
Witte (1968)	2.0	0.386	0.5	0.5	Sodium	35,000 - 153,000
Raithby-Eckert (1968)	2.0	0.235	0.606	1/3	0.71	3,600 - 52,000
Churchill (1977)	2.0	0.5505	1/2	$F(Pr)$‡		
Clift et al. (1978)	1.0	0.757	0.47	1/3	0.7 - 0.73	100 - 4,000
	1.0	0.304	0.58	1/3	0.7 - 0.73	4,000 - 100,000
	1.0	0.724	0.48	1/3	$Pr > 188$	100 - 200
	1.0	0.425	0.55	1/3	$Sc > 1,100$	2,000 - 100,000
Yovanovich (1988)**	2.0	0.15	0.5	1/3	0.71	10 - 150,000

† From Yovanovich (1988)

* From Witte (1968)

‡ $\dfrac{Pr^{1/3}}{[1 + (0.45/Pr)^{2/3}]^{1/4}}$

** $2.0 + (0.15\,Re_D^{0.5} + 0.273\,Re_D^{0.586})\,Pr^{1/3}$

$$\text{Nu}_D = 2 + 0.714\sqrt{\frac{D\bar{v}_e}{\alpha}} \tag{9}$$

The average effective velocity, $\bar{v}_e$, will be defined for the limiting cases of $\text{Pr} \gg 1$ and $\text{Pr} \ll 1$, then an interpolation function will be obtained to provide a relationship for all Prandtl numbers.

The present analysis has been based on the assumption that the flow does not separate at any point on the surface of the sphere. In fact, flow separation occurs in this type of problem at high Reynolds numbers. However, we will proceed with our analysis and compare it with the available experimental results which already have the separation effect in order to determine the capabilities of the present model.

2.1 $\bar{v}_e^\infty$ at $\text{Pr} \to \infty$. We will consider the high Prandtl number fluids first. Scaling analysis will be applied to the continuity,

Nomenclature

- A = surface area, m²
- b = exponent in Eq. (1)
- C_1, C_D = constants in Eq. (1)
- D = sphere diameter, m
- D_{AB} = binary mass diffusion coefficient, m²/s
- h = coefficient of convection heat transfer, W/m²K
- h_m = coefficient of convection mass transfer, m/s
- k = thermal conductivity, W/m K
- m = exponent in Eq. (1)
- n = exponent in Eqs. (33)–(38)
- Nu_D = area-averaged Nusselt number $= Dh/k$
- Pr = Prandtl number $= \nu/\alpha$
- Q = total heat flow rate, W
- q = heat flux, W/m²
- Re_D = Reynolds number $= DV_\infty/\nu$
- $\text{Re}_D(\theta)$ = local Reynolds number $= DV(\theta)/\nu$
- T = temperature, K
- T^* = nondimensional temperature $= (T - T_\infty)/(T_S - T_\infty)$
- t = time, s
- Sc = Schmidt number $= \nu/D_{AB}$
- Sh_D = Sherwood number $= h_m D/D_{AB}$
- V = local velocity at edge of thermal boundary layer, m/s
- $V(\theta)$ = local velocity at edge of hydrodynamic boundary layer, m/s
- V_∞ = free-stream velocity, m/s
- $\bar{v}_e$ = area-averaged effective velocity, m/s
- $v_e(\theta)$ = local effective velocity, m/s
- α = thermal diffusivity $= k/C_p\rho$, m²/s
- γ = constant in Eq. (26)
- δ = local thickness of HBL, m
- δ_T = local thickness of TBL, m
- δ_D^T = displacement thickness of TBL, m
- δ_M^T = momentum thickness of TBL, m
- η = nondimensional quantity $= y/\delta$
- ν = kinematic viscosity, m²/s
- ρ = density, kg/m³

Subscripts

- D = displacement
- e = effective
- M = momentum
- S = surface

Abbreviations

- CE = continuity equation
- EE = energy equation
- GEs = governing equations
- HBL = hydrodynamic boundary layer
- ME = momentum equation
- TBL = thermal boundary layer

Coordinates

- r, θ, ϕ = spherical coordinates
- x, y = local coordinates
- X, Y, Z = Cartesian coordinates

momentum, and energy equations to determine the area-averaged effective velocity. Consider that the hydrodynamic boundary layer, HBL, δ, is very thin, i.e., $D/2 + \delta \simeq D/2$ (see Fig. 1) where $Re_D \gg 1$ and also consider that the flow outside of the hydrodynamic boundary layer, HBL, is effectively inviscid. Thus, the local velocity at the edge of the HBL is equal to $v_\theta(D/2 + \delta) = V(\theta)$, where $V(\theta)$ is the solution to the inviscid flow problem, as shown in Fig. 1. The continuity equation inside the HBL is

$$\frac{2}{r} v_r + \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\cot\theta}{r} v_\theta = 0 \tag{10}$$

Using scaling analysis (the scaling analysis rules are stated by Bejan, 1985) on the continuity equation within the HBL gives the relationship:

$$\frac{4 v_r|_{D/2}}{D} + \frac{v_r|_{\delta+D/2} - v_r|_{D/2}}{\delta} + \frac{2}{D}\frac{v_\theta}{\theta} + \frac{2}{D}\frac{v_\theta}{\theta} \sim 0$$

with $v_\theta|_{\delta+D/2} = V(\theta)$, the inviscid flow solution and $v_r|_{D/2} = 0$, we obtain

$$v_r|_{\delta+D/2} \sim 2\frac{\delta}{D}\cdot\frac{V(\theta)}{\theta} \tag{11}$$

Applying scaling analysis on the continuity equation inside the thermal boundary layer, TBL, gives the relationship

$$v_r|_{\delta_T+D/2} \sim 2\frac{\delta_T^2}{D\delta}\cdot\frac{V(\theta)}{\theta} \tag{12}$$

where it is assumed that the ratio $V/V(\theta)$ is approximately equal to δ_T/δ, i.e., the flow has a linear velocity distribution as shown in Fig. 1.

The momentum equation in an axisymmetric flow along the body using the boundary layer theory has been discussed by White (1991). This equation can be converted to spherical coordinates (note that $x = r\theta$, i.e., $\partial x \simeq r\partial\theta$, and $y = r - D/2$, i.e., $\partial y = \partial r$), in steady-state form as follows:

$$v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} \simeq \frac{V(\theta)}{r}\frac{\partial V(\theta)}{\partial \theta} + \nu\frac{\partial^2 v_\theta}{\partial r^2} \tag{13}$$

Using scaling analysis on the momentum equation inside the HBL with Eq. (11) gives the following relationship:

$$\frac{2\delta V^2(\theta)}{D\delta\theta} + \frac{2V^2(\theta)}{D\theta} \simeq \frac{2V^2(\theta)}{D\theta} + \nu\frac{V(\theta)}{\delta^2} \tag{14}$$

Therefore, the local hydrodynamic boundary layer thickness is given by

$$\frac{\delta}{D} \sim \sqrt{\frac{\theta}{2\,Re_D(\theta)}} \tag{15}$$

where $Re_D(\theta) = DV(\theta)/\nu$. Applying scaling analysis on the energy equation, Eq. (2), and keeping in mind that $D/2 \gg \delta_T$ and $v_\theta|_{\delta_T} = V = [(\delta T/\delta)\cdot V(\theta)]$ we find that

$$\frac{2\delta_T^2}{D\delta}\cdot\frac{V(\theta)}{\theta}\cdot\frac{\Delta T}{\delta_T} + \frac{\delta_T V(\theta)\Delta T}{\delta\theta D/2} \sim \frac{\alpha\Delta T}{\delta_T^2} \tag{16}$$

The two convective terms on the left-hand side of Eq. (16) have the same magnitude, therefore we equate one of the convective terms to the diffusion term as follows:

$$\frac{\delta_T V(\theta)\Delta T}{\delta\theta D/2} \sim \frac{\alpha\Delta T}{\delta_T^2} \tag{17}$$

Therefore, the local dimensionless thermal boundary layer thickness is given by

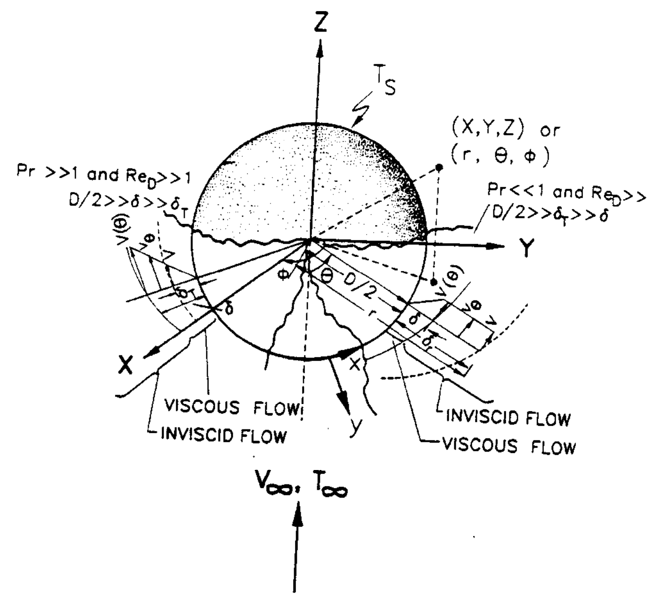

Fig. 1 Schematic diagram of the boundary layers over the sphere for $Pr \to \infty$ and $Pr \to 0$

$$\frac{\delta_T}{D} \sim Pr^{-1/3}\sqrt{\frac{\theta}{2\,Re_D(\theta)}} \tag{18}$$

Comparing Eq. (18) against Eq. (15) we find that

$$\frac{\delta_T}{\delta} \sim \frac{V}{V(\theta)} \sim \frac{1}{Pr^{1/3}} \tag{19}$$

This result will be used later to define $\bar{v}_e^\infty$.

The local effective velocity, $v_e^\infty(\theta)$, for large Prandtl numbers fluids will be obtained from momentum flux balances through the thermal boundary layer. The momentum flux inside the thermal boundary layer is

$$\frac{\rho}{\delta_T}\int_0^{\delta_T} v_\theta(V - v_\theta)dy \tag{20}$$

On the other hand, if we determine the momentum flux by assuming that the flow has a uniform local effective velocity, $v_e^\infty(\theta)$ is constant in the y direction and variable in the x direction, we have

$$\frac{\rho}{\delta_T}\int_0^{\delta_T} v_e^\infty(\theta)(V - v_\theta)dy \tag{21}$$

Equating Eqs. (20) and (21) and solving for the local effective velocity, we obtain

$$v_e^\infty(\theta) = V\frac{\int_0^{\delta_T}\frac{v_\theta}{V}(V - v_\theta)dy}{\int_0^{\delta_T}(V - v_\theta)dy} \tag{22}$$

which can be expressed in terms of the momentum and displacement thicknesses as follows:

$$v_e^\infty(\theta) \sim V(\theta)\cdot\frac{\delta_T}{\delta}\cdot\frac{\delta_M^T}{\delta_D^T} \sim V(\theta)\cdot\frac{1}{Pr^{1/3}}\cdot\frac{\delta_M^T}{\delta_D^T} \tag{23}$$

For convenience of the subsequent analysis, we introduce the similarity parameter, $\eta = y/\delta_T$. This allows one to express the momentum and displacement thicknesses in the following forms:

$$\delta_M^T = \delta_T \int_0^1 \frac{v_\theta}{V}\left(1 - \frac{v_\theta}{V}\right) d\eta \tag{24}$$

$$\delta_D^T = \delta_T \int_0^1 \left(1 - \frac{v_\theta}{V}\right) d\eta \tag{25}$$

Clearly these important hydrodynamic thicknesses depend on the velocity distribution within the TBL. We may assume that v_θ is a power-law function of y in order to have a general form for the velocity profiles at different Reynolds numbers, i.e., $v_\theta/V = (y/\delta_T)^\gamma$ or $v_\theta/V = \eta^\gamma$ where $0 \le \gamma \le 1$. This form is chosen for the following reasons: It has been proposed by Kays and Crawford (1980) for the flat plate at high Re where $\gamma = \frac{1}{7}$ and it also agrees with the analytical solution where the velocity is linear at $\gamma = 1$. We introduce the power-law velocity distribution into Eq. (24) and Eq. (25) and integrate. One can obtain the relationship between the momentum and displacement thicknesses in terms of the power-law exponent:

$$\frac{\delta_M^T}{\delta_D^T} = \frac{1}{2\gamma + 1} \tag{26}$$

Therefore, the local effective velocity from Eq. (23) with Eq. (26) is

$$v_e^\infty(\theta) \sim \frac{V(\theta)}{(2\gamma + 1)\,\mathrm{Pr}^{1/3}} \tag{27}$$

The area-averaged effective velocity is defined as

$$\overline{v}_e^\infty \sim \frac{1}{(2\gamma + 1)\,\mathrm{Pr}^{1/3}} \cdot \frac{1}{A} \iint_A V(\theta) dA \tag{28}$$

Furthermore, the ideal flow solution can be used to represent the flow in the region outside of the boundary layer; therefore,

$$v_\theta |_{\delta + D/2} = V(\theta) = 1.5 V_\infty \sin\theta \tag{29}$$

After substitution of Eq. (29) into Eq. (28) we find that the area-averaged effective velocity as $\mathrm{Pr} \to \infty$ is given by

$$\overline{v}_e^\infty \sim \frac{1.178 V_\infty}{(2\gamma + 1)\,\mathrm{Pr}^{1/3}} \tag{30}$$

2.2 $\overline{v}_e^0$ at $\mathrm{Pr} \to 0$. Let us consider that the flow is inviscid, i.e., $\mathrm{Pr} \ll 1$, and $\mathrm{Re}_D \gg 1$. Therefore the HBL, δ, is very small; however, the TBL, δ_T, is very large relative to δ. Therefore at the edge of the TBL we have

$$v_\theta |_{(\delta + D/2)} = \frac{V_\infty}{2} \cdot \left[2 + \left(\frac{D}{2(\delta + D/2)}\right)^3\right] \sin\theta \tag{31}$$

However, Fig. 1 shows that $\delta \ll D/2$. So, $v_\theta|_{(\delta+D/2)} = V = 1.5 V_\infty \sin\theta$.

Therefore, the local velocity at arbitrary θ will be considered uniform across the TBL. As a result of that $v_e^0(\theta) = V$ as shown in Fig. 1 (V is the local maximum velocity at the edge of the TBL and $v_e^0(\theta)$ is the local effective velocity at $\mathrm{Pr} \ll 1$). The area-mean effective velocity is

$$\overline{v}_e^0 = \frac{1}{A} \iint_A V dA = 1.178 V_\infty \tag{32}$$

2.3 $\overline{v}_e$ for all Pr. At this point the effective velocity has been found for the two limiting cases where $\mathrm{Pr} \gg 1$ and $\mathrm{Pr} \ll 1$. In order to develop an expression for $\overline{v}_e$ valid for any Prandtl number, the Churchill and Usagi (1972) blending technique will be used. The effective velocity can be determined in different ways such as:

$$(\overline{v}_e)^n = (\overline{v}_e^0)^n + (\overline{v}_e^\infty)^n \quad n \ge 1 \tag{33}$$

or

$$\overline{v}_e = \sqrt{\overline{v}_e^0 \cdot \overline{v}_e^\infty} \tag{34}$$

or

$$\frac{1}{(\overline{v}_e)^n} = \frac{1}{(\overline{v}_e^0)^n} + \frac{1}{(\overline{v}_e^\infty)^n} \quad n \ge 1 \tag{35}$$

The first two forms are not applicable at $\mathrm{Pr} \ll 1$ because $\overline{v}_e \to \infty$. However, the last form is consistent at both limits; at $\mathrm{Pr} \ll 1$, $\overline{v}_e \to \overline{v}_e^0$ and at $\mathrm{Pr} \gg 1$, $\overline{v}_e \to \overline{v}_e^\infty$. The area-averaged effective velocity can now be expressed in the following form:

$$\overline{v}_e = \frac{\overline{v}_e^\infty}{\left[1 + \left(\frac{\overline{v}_e^\infty}{\overline{v}_e^0}\right)^n\right]^{1/n}} \tag{36}$$

Substituting $\overline{v}_e^0$ and $\overline{v}_e^\infty$ into Eq. (36) gives the effective velocity valid for all Prandtl numbers in terms of the power-law parameter γ and the blending parameter n,

$$\frac{\overline{v}_e}{V_\infty} = \frac{1.178/[(2\gamma + 1)\,\mathrm{Pr}^{1/3}]}{(1 + [1/(2\gamma + 1)\,\mathrm{Pr}^{1/3}]^n)^{1/n}} \quad 0 < \mathrm{Pr} < \infty \tag{37}$$

where $0 \le \gamma \le 1$. The constant n will be determined in the following section.

3 Results and Discussion

In order to determine the equation of Nu_D, one must substitute Eq. (37) into Eq. (9). The area-averaged Nusselt number, Nu_D, becomes

$$\mathrm{Nu}_D = 2 + \frac{0.775}{\sqrt{2\gamma + 1}} \mathrm{Re}_D^{1/2} \frac{\mathrm{Pr}^{1/3}}{\left[1 + \left(\frac{1.0}{(2\gamma + 1)^3\,\mathrm{Pr}}\right)^{n/3}\right]^{1/(2n)}} \tag{38}$$

By examining the two asymptotic values of the area-averaged Nusselt number for $\mathrm{Pr} \ll 1$ and $\mathrm{Pr} \gg 1$, from Eq. (38), we obtain

$$\text{for } \mathrm{Pr} \ll 1, \quad \mathrm{Nu}_D = 2 + 0.775\, \mathrm{Re}_D^{1/2}\, \mathrm{Pr}^{1/2} \tag{39}$$

and

$$\text{for } \mathrm{Pr} \gg 1, \quad \mathrm{Nu}_D = 2 + \frac{0.775}{\sqrt{2\gamma + 1}} \mathrm{Re}_D^{1/2}\, \mathrm{Pr}^{1/3} \tag{40}$$

independent of the value of the blending parameter n. The constant C_D as defined in Eq. (1) is $0.775/\sqrt{2\gamma + 1}$ in Eq. (38). It is equal to 0.447 at $\gamma = 1.0$, 0.633 at $\gamma = \frac{1}{4}$, and 0.683 at $\gamma = \frac{1}{7}$. In addition, it has been found that $n = 3$ gives the best fit by matching Eq. (38) with the available air data correlation equations in the literature (Yuge, 1960; Churchill, 1977; Yovanovich, 1988). Therefore, Eq. (38) can be written in the following forms for $\gamma = 1$ and $\frac{1}{7}$, respectively:

$$\mathrm{Nu}_D = 2 + 0.447\, \mathrm{Re}_D^{1/2} \frac{\mathrm{Pr}^{1/3}}{\left[1 + \left(\frac{0.037}{\mathrm{Pr}}\right)\right]^{1/6}} \quad (\gamma = 1) \tag{41}$$

$$\mathrm{Nu}_D = 2 + 0.683\, \mathrm{Re}_D^{1/2} \frac{\mathrm{Pr}^{1/3}}{\left[1 + \left(\frac{0.471}{\mathrm{Pr}}\right)\right]^{1/6}} \quad (\gamma = \tfrac{1}{7}) \tag{42}$$

Figure 2(a) shows the comparison between the present results (Eqs. (41) and (42)) and the correlations of Frössling (1938), Drake and Backer (1952), and Yuge (1960) for air (Pr = 0.71). It can be seen that at the low range of Re_D, $10^{-1} \le \mathrm{Re}_D \le 10$, the values of the Nusselt number correlations are in very good agreement with Eq. (41). However, at the high range of Re_D, 10

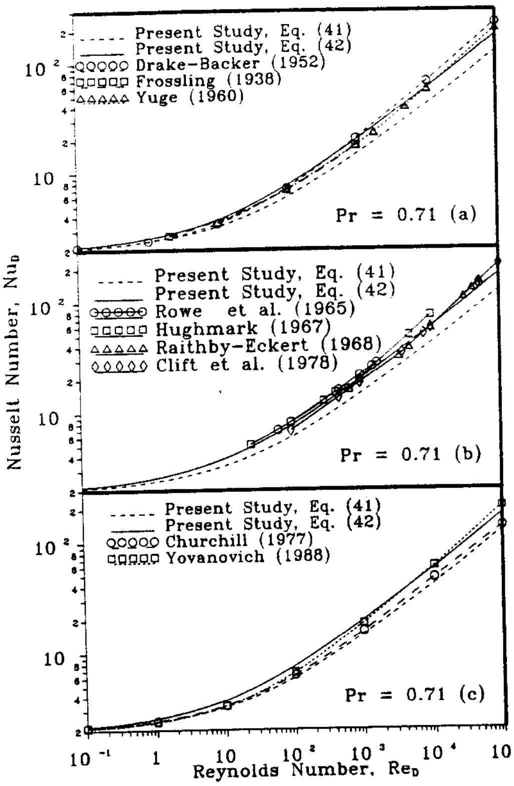

Fig. 2 Comparison between the present analytical model and the previous studies for air

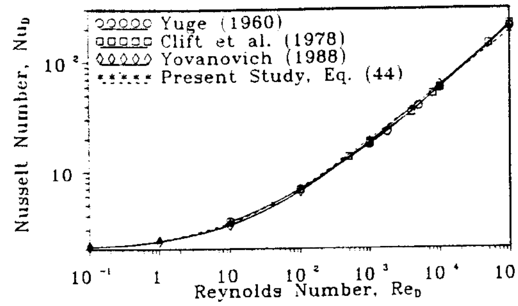

Fig. 3 Comparison between the present general analytical model and the previous studies for air

$\leq \mathrm{Re}_D \leq 10^4$, the experimental correlations of Frössling (1938), Drake and Backer (1952), and Yuge (1960) approach Eq. (42), where the velocity profile has the power $\frac{1}{7}$.

In addition, Fig. 2(*b*) shows the same comparison between Rowe et al. (1965), Hughmark (1967), Raithby and Eckert (1968), and Clift et al. (1978) with the present study. The correlations of Raithby and Eckert (1968) and Clift et al. (1978) behave similarly to the correlations of Drake and Backer (1952) and Yuge (1960). On the other hand, the results of Rowe et al. (1965) are higher than the present study. Furthermore, Yovanovich and Vanoverbeke (1988) examined carefully the work of Rowe et al. (1965) and they concluded that Rowe et al. (1968) had not removed the effect of free convection from their data. In addition, the correlations of Hughmark (1967) were higher than the present study. However, from Table 1, it is seen that Hughmark (1967) presented two correlations for $\mathrm{Pr} < 250$. These two correlations cover the range of Re_D from 26 to 450 and 450 to 10^4, respectively. However, his correlation for the high range of Re_D predicted lower values of Nu_D at $\mathrm{Re}_D < 700$, if it is compared with his low range correlation. In addition, Eq. (42) is compared with the general equation of Pasternak and Gauvin (1960), which was developed for several body shapes and it was found that the maximum difference is 15 percent, which occurs at $\mathrm{Re}_D = 5000$. In addition, the turbulent intensity in the measurements of Pasternak and Gauvin (1960) was around 9–10 percent. Also, Eq. (42) is compared with the correlation of Kramers (1946) and it is found that the maximum difference of 2 percent occurs at $\mathrm{Re}_D = 540$.

Figure 2(*c*) shows the comparison between the present study and the models of Churchill (1977) and Yovanovich (1988) for air. It is observed that the Yovanovich (1988) and Churchill (1977) models are in very good agreement with Eq. (41) up to $\mathrm{Re}_D = 100$. After that the Yovanovich (1988) model approaches Eq. (42) and finally crosses it at $\mathrm{Re}_D = 10^4$. However, the Churchill model (1977) lies between Eqs. (41) and (42). We observe that the Churchill model behavior is similar to the present model if $\gamma = \frac{2}{3}$. On the other hand, if $\gamma \to 0$, Eq. (42) will be higher than the Yovanovich model up to $\mathrm{Re}_D = 10^5$, i.e., Yovanovich was able to model different velocity profiles in one equation.

3.1 General Model. Figure 2 shows that the correlations of Yuge (1960), Drake and Backer (1952), Clift et al. (1978), and Yovanovich (1988) are in very good agreement with Eq. (41) up to $\mathrm{Re}_D = 100$ and after that their correlations approach Eq. (42) and finally, cross it at $\mathrm{Re}_D = 10^4$. This leads us to conclude that the previous studies fitted their data in various ranges of Re_D, which could have different velocity profiles because Eq. (41) was developed for a linear velocity profile and Eq. (42) was developed for the power-low velocity profile, $\gamma = \frac{1}{7}$. Therefore, if we correlate γ as a function of Re_D based on the trend of the previous studies between Eq. (41) and Eq. (42), a general model can be developed for the forced convection heat transfer from spheres. It is found that a simple form for γ as $f(\mathrm{Re}_D)$ based on the previous work is given by:

$$\gamma = 1.0/\mathrm{Re}_D^{0.25} \quad (\text{if } \gamma > 1 \text{ put } \gamma = 1) \tag{43}$$

From Eq. (38) with $n = 3$ we obtain the general equation:

$$\mathrm{Nu}_D = 2 + 0.775\,\mathrm{Re}_D^{1/2} \frac{\mathrm{Pr}^{1/3}/[\sqrt{2\gamma} + 1]}{\left[1 + \left(\dfrac{1.0}{(2\gamma + 1)^3\,\mathrm{Pr}}\right)\right]^{1/6}} \tag{44}$$

$\mathrm{Re}_D < 10^5$ and $0 < \mathrm{Pr} < \infty$ with Eq. (43).

The right-hand side of Eq. (44) can be written as follows: $\mathrm{Nu}_D^0 + C_D\,\mathrm{Re}_D^{0.5} F(\mathrm{Pr}, \gamma)$, where Nu_D^0 is 2 and C_D is 0.775 and $F(\mathrm{Pr}, \gamma)$ is $\mathrm{Pr}^{1/3}/[\sqrt{2\gamma} + 1]/[1 + (1.0/(2\gamma + 1)^3\,\mathrm{Pr})]^{1/6}$.

Figure 3 shows the comparison between the general model, Eq. (44), and the previous studies of Yuge (1960), Clift et al. (1978), and Yovanovich (1988) for air. In addition, very good agreement between the previous studies and Eq. (44) up to $\mathrm{Re}_D = 2 \times 10^4$ is shown in Fig. 3. After that the Yuge (1960), Clift et al. (1978), and Yovanovich (1988) correlations cross the pres-

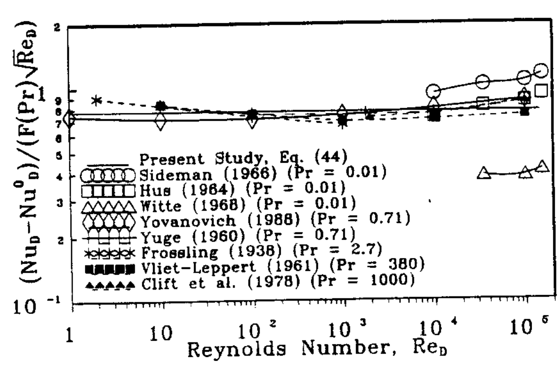

Fig. 4 Comparison between the present general analytical model and the previous studies for different Prandtl numbers

ent model, Eq. (44). The maximum differences between Yuge (1960), Clift et al. (1978), and Yovanovich (1988) correlations and Eq. (44), which occur at $Re_D = 10^5$, are 9.7, 13.9, and 10.8 percent, respectively.

Figure 4 shows the relationship between $[(Nu_D - Nu_D^0)/(F(Pr) Re_D^{0.5})]$ (which is C_D) and Re_D. Also, Fig. 4 presents the comparison between the present model and the previous studies for various Prandtl numbers. One observes that the maximum difference between the previous studies and the present model in the range $1 < Re_D < 10^5$ is approximately 11 percent, which generally occurs at $Re_D = 10^5$. On the other hand, there is almost a 33 percent difference between the present model and that of Sideman (1966). The main reason for this is that Sideman approximated the convective term of the energy equation by assuming $(V_\infty/r)/(\partial T/\partial\theta)$. Therefore, Sideman's model overpredicted the Nusselt number because the velocity in this model is very high. Figure 4 also shows that there is 50 percent difference between the present model and the experimental correlation (for sodium) of Witte (1968). Finally, it can be concluded from Fig. 4 that the constant C_D for the sphere is approximately 0.775. In addition, $F(Pr, \gamma)$ is quite acceptable over the range of $0 < Re_D < 10^5$.

4 Summary and Conclusions

An approximate analytical solution is developed for the area mean Nusselt number for forced convection from isothermal spheres valid for the range of Reynolds number, $0 \leq Re_D \leq 10^4$ and all Prandtl numbers. In addition, the present solution is found to be in very good agreement with many previous studies such as Frössling (1938), Kramers (1946), Yuge (1960), Raithby and Eckert (1968), Churchill (1977), and Yovanovich (1988). Furthermore, in the present study, it is concluded that the main reason for the differences in the exponent of Re_D and the constant C_D in the previous studies is due to their fitting data in various ranges of Re_D, which have different velocity profiles. This was observed very clearly from the trends of the previous studies, which are between the bounds of Eqs. (41) and (42), where the velocity profile in Eq. (41) is assumed linear and in Eq. (42) it is a power law with $\gamma = \frac{1}{7}$. Finally, this study has led to a design correlation, Eq. (44), which is very accurate for the range of Reynolds number $0 \leq Re_D < 10^5$ and all Prandtl numbers $0 < Pr < \infty$.

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