Thermal Modeling of Isothermal Cuboids and Rectangular Heat Sinks Cooled by Natural Convection

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Abstract—**Thermally-induced buoyancy effects are not always sufficient to adequately cool high density microelectronic packages found in modern circuit boards. In many instances thermal enhancement techniques, such as heat sinks, must be used to increase the effective surface area for heat transfer and lower the thermal resistance between source and sink. The irregular surfaces of heat sinks present a formidable challenge for designers in determining the boundary conditions along the fluid–solid interface.**

A simple yet accurate method for calculating the thermal performance of rectangular heat sinks using a flat plate boundary layer model is presented. Several heat sink geometries are examined over a range of Rayleigh number between 10^3 and 10^{10}. The heat-transfer performance of the heat sinks, as given by the Nusselt number, is determined for each test based on the isothermal body temperature and the square root of the wetted surface area. Results obtained using a conjugate model, META, are compared against an analytically-based correlation and experimental data.

In addition to the rectangular heat sinks, isothermal cuboids of various sizes are modeled using META, where the cuboid is approximated as a thin uniformly-heated base plate with an attached extended surface. The cuboid results are compared with experimental data and an analytically based correlation.

NOMENCLATURE

A	Surface area, m^2.
A_f	Side wall area factor $\equiv A_{sw}/A_{base}$.
b	Fin thickness, m.
B_i	Biot number $\equiv ht/k_e$.
f_l	Function of Prandtl number, (9).
$F(\mathrm{Pr})$	"universal" Prandtl number function, (3).
g	Gravitational acceleration.
$G_{\sqrt{A}}$	Body gravity function, (4).
h	Heat transfer coefficient, W/m^{2}K.
H	Fin height (parallel to flow direction), m.
$\mathcal{I}$	Weighting function, (7).
k	Thermal conductivity, W/mK.
L	Fin length, m.
N	Number of discrete elements in flow direction.
N_f	Number of fins.
p	Pressure, atm.
P	Perimeter, m.
Pr	Prandtl number.
R_f	Thermal resistance of the fluid, °C/W.
$\mathrm{Ra}_{\sqrt{A}}$	Rayleigh number $\equiv g\beta \mathrm{Pr}\Delta T(\sqrt{A})^3/\nu^2$.
Ra_x^*	Modified local Rayleigh number $\equiv g\beta \mathrm{Pr} q x^4/(\nu^2 k)$.
Re_x	Local Reynolds number $\equiv Ux/\nu$.
q	Heat flux, W/m^2.
s	Inter fin spacing, m.
t	Base plate thickness, m.
T	Temperature, °C.
ΔT	Temperature excess $\equiv T_w - T_\infty$, °C.
U	Velocity in the primary flow direction, m/s.
W	Fin width, m.
x, y, z	Cartesian coordinates.
Greek Symbols	
α	Thermal diffusivity $\equiv k_f/(\rho \cdot c_p)$, m^2/s.
β	Thermal expansion coefficient, K^{-1}.
ν	Kinematic viscosity, m^2/s.
χ	Dimensionless position variable $\equiv \xi/x$.
ξ	Dummy variable in the x direction.
Subscripts	
$\sqrt{A}$	Characteristic length based on square root of wetted surface area.
base	Base.
e	Effective.
f	Fluid.
j	Joulean.
r	Radiation.
rsw	Side wall radiation.
sw	Side wall.
w	Wall.
Superscripts	
∞	Diffusive limit.

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I. INTRODUCTION

MICROELECTRONICS applications present a diverse mix of geometric configurations, thermophysical properties, and flow conditions which must be factored into thermal modeling tools used for design or reliability assessment. The geometries encountered in heat sink assemblies are difficult to model using analytical techniques because of the complex fluid flow around and between the various components of

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the heat sink. Most heat transfer analyses of heat sinks and extended surfaces are based on correlated Nusselt number versus Rayleigh number plots derived from empirical studies. Van de Pol and Tierney [1] presented a Nusselt versus Rayleigh relationship for natural convection cooling of vertical fins attached to a base plate, based on the experimental data of Welling and Wooldridge [2], obtained over a range $0.6 < \mathrm{Ra}^* < 100$. Jones and Smith [3] performed a similar study for rectangular fin assemblies facing upward and downward in relation to the gravity vector. Nusselt versus Rayleigh plots were presented over a range $2 \times 10^2 < \mathrm{Ra} < 6\times 10^5$. In both studies, the correlation equations were restricted to a fixed range of geometric and flow conditions, limiting their use as general purpose design tools.

Some conjugate analytical solutions have been presented for rectangular heat sinks cooled by forced convection. Shvets and Didenko [4] developed a conjugate fin model where the heat transfer coefficient was constant along the length of the fin. Garg and Velusamy [5] used a boundary layer solution based on the Blasius equation to calculate a nonuniform boundary condition as the coupling condition in their iterative model. Similar models are not available for natural or mixed convection cooling.

The purpose of this paper is to demonstrate the ability of META to model natural convection cooling of isothermal cuboids and rectangular heat sinks over a wide range of Rayleigh number. META is an interactive simulation tool for modeling conjugate heat transfer with forced, natural or mixed convection. Development of META was initiated in 1984 for the purpose of analyzing the thermal behavior of flat circuit boards with flush mounted heat sources. Although the original intent of META was to model the thermal behavior of microelectronic circuit boards, many other heat transfer applications can now be modeled using the conjugate heat transfer model in META.

II. Model Summary

A. Analytical Correlations—Cuboid Model

An analytically-based correlation [6] is presented for natural convection cooling of isothermal cuboids of arbitrary size. The Nusselt number is based on the linear superposition of the diffusive and convective limits for an isothermal body, as shown in Fig. 1. The Nusselt number, where the characteristic length is the square root of the wetted surface area, can be written as in [7], [8]

$$\mathrm{Nu}_{\sqrt{A}} = \mathrm{Nu}^{\infty}_{\sqrt{A}} + F(\mathrm{Pr})G_{\sqrt{A}}\mathrm{Ra}^{1/4}_{\sqrt{A}} \tag{1}$$

where the diffusive limit is calculated by

$$\mathrm{Nu}^{\infty}_{\sqrt{A}} = \frac{3.192 + 1.868(L/H)^{0.76}}{\sqrt{1+1.189(L/H)}} \tag{2}$$

for a cuboid with dimensions $L \times W \times H$.

The "universal" Prandtl number function is

$$F(\mathrm{Pr}) = \frac{0.670}{[1+(0.5/\mathrm{Pr})^{9/16}]^{4/9}} \tag{3}$$

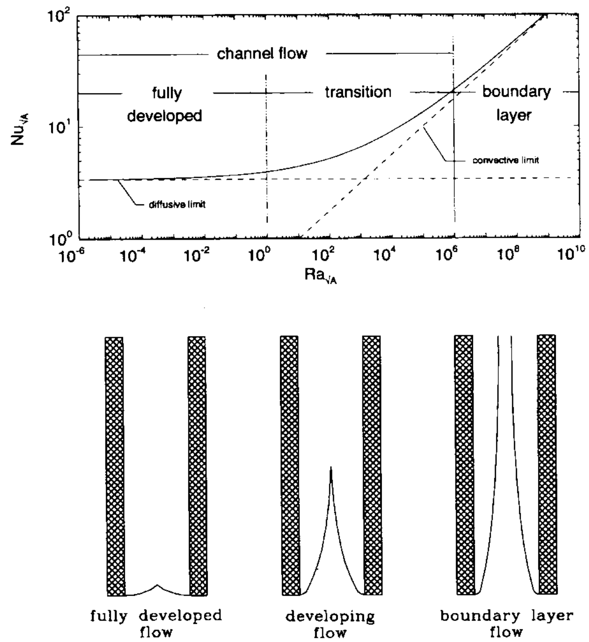


Fig. 1. Inter-fin spacing flow regimes encountered in rectangular fins.

with the body–gravity function [9], given as

$$G_{\sqrt{A}} = \left[\frac{1}{A}\iint_A \left(\frac{P\sin\theta}{\sqrt{A}}\right)^{1/3} dA\right]^{3/4} \tag{4}$$

where P is the local perimeter of the body, perpendicular to the flow direction, θ is the angle subtended by the gravity vector and the normal to the surface, and A is the total wetted surface area of the body.

The body gravity function, $G_{\sqrt{A}}$, for an arbitrary rectangular fin with base plate dimensions, $H \times W \times t$, and N_f fins with dimensions, $L \times b \times H$, can be written as

$$G_{\sqrt{A}} = 2^{\frac{1}{8}}\left[\frac{H(L\cdot N_f + t + W)^2}{(b\cdot L\cdot N_f + t\cdot W + H(L\cdot N_f + t + W))^{\frac{3}{2}}}\right]^{\frac{1}{4}}. \tag{5}$$

The base plate and fin height, H, are parallel to the gravity vector. The thickness of the base plate and the fin are t and b, respectively.

B. META—Computer Simulations

All computer simulations presented in this paper were performed using META [10], a conjugate heat transfer model for modeling forced, natural or mixed convection from conductive, multilayered substrates with surface mounted raised heat sources. META incorporates an iterative procedure that uses the temperature of the fluid and the gradient calculated at the wall to compute a local convective boundary condition for a finite volume based solid-side solution. The solid-side temperature distribution is in turn used to calculate an improved temperature gradient along the wall for calculating fluid-side temperatures. The procedure is repeated until the fluid and solid body temperatures converge at all interface locations. The fluid-side solution is based on a laminar boundary layer model for flow over flat plates with arbitrarily located step changes in heat flux, as presented in Sparrow and Lin [11]. The temperature rise of the fluid adjacent to the wall at any location, x, in the flow direction can be written as

$$T_f(x) - T_\infty = \left\{ \frac{x}{0.454 \cdot k_f \mathrm{Pr}^{1/3} \mathrm{Re}_x^{1/2}} \right\} \sum_{i=1}^{N} [q_{w_i} \mathcal{I}_i] \quad (6)$$

where

$$\mathcal{I}_i = [1 - \chi_{(2i-1)}]^{1/3} - [1 - \chi_{(2i)}]^{1/3} \quad (7)$$

and $\chi_{(2i-1)}$ and $\chi_{(2i)}$ are the leading and trailing edge of each step change in heat flux between the leading edge of the plate and x.

As shown in (6), the fluid temperature rise and in turn the local heat transfer coefficient are only a function of the position variable in the x-direction. However, through the wall heat flux, q_{w_i}, two-dimensional effects over the planar surface $x \times y$, inherent in the solid-side solution, are also reflected in the fluid-side temperature calculation. Boundary layer effects resulting from flow over and around raised bodies attached to the flat plate are not accounted for in the calculation of temperature and the heat transfer coefficient within META.

The boundary layer solution given in (6) requires a knowledge of the flow velocity, $U(x)$, as reflected in the Reynolds number ($\mathrm{Re}_x = U(x) \cdot x/\nu$). The flow velocity, $U(x)$, is generally perceived to be a "forced" convection flow induced using an external source, such as a fan. However, (6) remains valid for natural and mixed convection if the velocity term, $U(x)$, is replaced with an effective velocity, equivalent to the local velocity induced through buoyancy or a combined velocity, resulting from buoyancy and an external source.

The calculation of an effective natural convection velocity is described in detail in [12]. In short, the natural convection effective velocity can be written as

$$U = \frac{(2.2 \cdot f_l)^2 \cdot (\mathrm{Ra}_x^*)^{2/5} \cdot \nu}{\mathrm{Pr}^{2/3} \cdot x} \quad (8)$$

where

$$f_l = \left(\frac{\mathrm{Pr}}{4 + 9 \cdot \sqrt{\mathrm{Pr}} + 10 \cdot \mathrm{Pr}} \right)^{1/5} \quad (9)$$

and

$$\mathrm{Ra}_x^* = \frac{g \beta q_w \mathrm{Pr} x^4}{\nu^2 k} \quad (10)$$

and Ra_x^* is the modified Rayleigh based on a flux specified boundary condition.

The solid-side model is based on a control-volume, finite-volume model as described in [13]. The substrate material on the solid-side is modeled as a single homogeneous material, with thermal conductivity k. For cases where multiple layers of various thermal conductivities are present a harmonic-mean effective thermal conductivity, k_e, can be used as described in [14].

Both the fluid- and solid-side models assume flat substrates of uniform cross section but raised bodies, which act as extended surfaces thereby enhancing heat transfer, can be accounted for as shown in [15]. The governing equation used in META to calculate solid-body temperatures for raised bodies attached to flat substrates is a modification of the Laplace equation, where for situations in which the thermal resistance of the fluid dominates, i.e., Bi < 0.1, the following equation must be solved

$$\frac{\partial^2 T_w}{\partial x^2} + \frac{\partial^2 T_w}{\partial y^2} + \frac{1}{k_e t}[q_j - q_w - q_r - (q_{\mathrm{sw}} + q_{\mathrm{rsw}}) \cdot A_f] = 0 \quad (11)$$

where

q_j	Applied equivalent Joulean heat flux.
q_w	Convective heat flux over all surfaces parallel to the plate.
q_{sw}	Convective heat flux over all surfaces perpendicular to the plate.
q_r	Radiative heat flux over all surfaces parallel to the plate.
q_{rsw}	Radiative heat flux over all surfaces perpendicular to the plate.
A_f	Area factor, $\equiv A_{\mathrm{sw}}/A_{\mathrm{base}}$.

Another important feature of META is its ability to simulate natural convection over a wide range of Rayleigh number. When air at standard pressure is used as the cooling fluid, the range of Rayleigh number, based on the square root of the wetted surface area, is limited to approximately one decade between 10^5 and 10^6 for body shapes of reasonable size. A method commonly used by experimentalists to extend the range of Rayleigh number is to vary the pressure of the cooling fluid, since $\mathrm{Ra} \propto p^2$. As a result, a wide range of Rayleigh number can be obtained for relatively small changes in the pressure of the cooling fluid. A similar method is used in the META simulation routine where a change in the fluid density is reflected in the kinematic viscosity and the thermal diffusivity, providing a convenient means of calculating heat transfer effects over a wide range of Rayleigh number. The results presented in the following case studies were simulated for a range of air pressure between 10^{-5} and 10.0 atm providing a range of $10^{-6} \leq \mathrm{Ra}_{\sqrt{A}} \leq 10^{10}$.

III. Case Studies

A. Rectangular Fin

Many high-powered electronic components cannot liberate sufficient quantities of heat given the high thermal impedance associated with air cooling. For situations where natural convection is a prerequisite, simple passive heat sinks can be attached to heat sources to enhance heat dissipation by increasing the available surface area exposed to the cooling fluid flow.

A conventional rectangular fin, as shown in Fig. 2, can be very effective as a means of lowering the thermal resistance between a heat source and the surrounding cooling fluid. The thermal resistance to the fluid sink can be characterized as $R_f = 1/(h \cdot A)$, where A is the surface area wetted by the cooling fluid. Given the narrow range of heat transfer coefficients available ($1 < h(\mathrm{W/m^2K}) < 10$) for buoyancy driven air cooling, the most effective means of lowering the

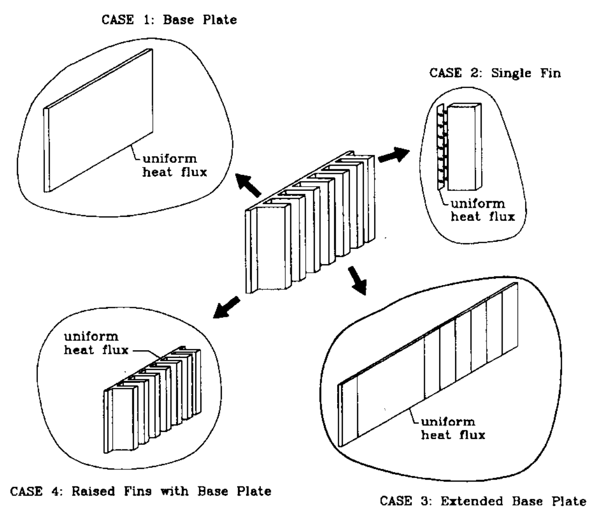


Fig. 2. Various methods of modeling natural convection cooling of a rectangular fin.

fluid-side thermal resistance is by increasing the area of the wetted surface.

The rectangular fin assembly shown in Fig. 2, can be modeled as a collection of individual components consisting of a series of fins and a uniformly heated base plate or as a composite structure where all components simultaneously interact. Four unique examples of rectangular fin simulations available in META are shown in Fig. 2. These examples consist of the following:

- *Base Plate*: the base plate from the rectangular fin is modeled as a flat plate with a uniform heat flux boundary condition over the base surface.
- *Single Fin*: a single fin from the fin assembly is modeled as a flat plate with a uniform heat flux boundary condition applied over one end.
- *Extended Base Plate*: the entire fin assembly is modeled as a base plate with an equivalent wetted surface area. A uniform heat flux boundary condition is applied over a surface equivalent to the area of the original base plate.
- *Raised Fins with Base Plate*: the entire fin assembly is modeled as a base plate with attached extended surfaces. A uniform heat flux boundary condition is applied over the base plate. This option allows for the addition of a contact resistance between the fins and the base plate.

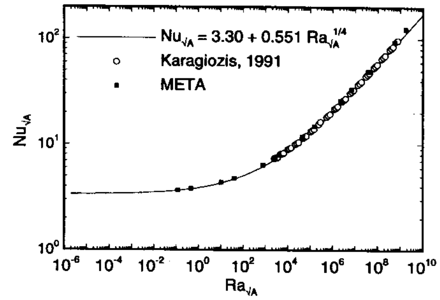
The wetted surfaces of a rectangular fin generally encounters two different types of flow which includes a conventional boundary layer flow over those surfaces where the formation of the boundary layer occurs unrestricted and a transitional region which leads to developing or fully-developed channel flow between individual fin sections. For a given surface area, the largest influence on the Rayleigh number ($\mathrm{Ra}_{\sqrt{A}} = g\beta \mathrm{Pr} \Delta T(\sqrt{A})^3/\nu^2$) is introduced through a change in the air pressure and in turn the kinematic viscosity, ν. Lowering the air pressure produces an increase in the kinematic viscosity and a decrease in the Rayleigh number. The rate of boundary layer growth for $\mathrm{Ra}_{\sqrt{A}} < 10^6$ tends to preclude boundary layer flow between fin sections as the boundary layers interact and a transitional or fully-developed flow regime is attained. The actual demarcation between boundary layer flow and channel flow is a function of the spacing between fins, however, for fin spacings in the range of 15 mm to 35 mm, a Rayleigh number of 10^6 appears to be the appropriate lower bound for developing flow conditions.

Karagiozis [16] performed a series of experiments to determine the thermal performance of rectangular heat sinks cooled by natural convection. Karagiozis presented test data for a variety of fin geometries, ranging from a simple base plate, 150 mm $\times$ 170 mm $\times$ 9.5 mm to a fin assembly consisting of a base plate, 150 mm $\times$ 220 mm $\times$ 9.5 mm with five rectangular fins, 150 mm $\times$ 50 mm $\times$ 9.5 mm, attached in such a manner that minimized the contact resistance between the base plate and the fins. Tests were conducted in a vacuum chamber that allowed data collection over a range of Rayleigh number between 10^{-3} to 10^8.

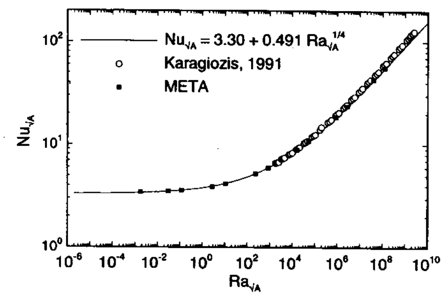
1) Example 1. Base Plate: An aluminum base plate, 150 mm $\times$ 220 mm $\times$ 9.5 mm with a uniformly-specified heat flux, is considered. The base plate is cooled by natural convection with the primary flow direction and the gravity vector parallel to the side measuring 150 mm. Karagiozis [16] presents empirical data for a slightly smaller base plate (150 mm $\times$ 170 mm $\times$ 9.5 mm) over a range of Rayleigh numbers between 10^2 and 10^9, where Rayleigh number is based on the plate length, L, as the characteristic length. The analytical model, given in (1)–(4), results in a 2% increase in the Nusselt number when the fin length is increased from 170 to 220 mm. For the purpose of this investigation a characteristic length based on the square root of the wetted surface area is used. Yovanovich [7], [8] has shown that $\sqrt{A}$ is the appropriate choice of characteristic length when comparing different body shapes suspended in various orientations because of its ability to provide a common reference when comparing the heat transfer response presented as $\mathrm{Nu}_{\sqrt{A}}$ versus $\mathrm{Ra}_{\sqrt{A}}$.

Thermal simulations for the base plate described above were performed using META, where a uniform heat flux was imposed over one side of the plate. META assumes that the edges of the plate are adiabatic, therefore to properly simulate Karagiozis' base plate, where the entire wetted surface area of the plate contributes to heat transfer, the edges of the plate are assumed to be folded out into the larger planar surface of the plate to produce a plate which is 229.5 mm $\times$ 159.5 mm ([width + thickness] $\times$ [length + thickness]). The base plate used for testing was made of polished aluminum, with a thermal conductivity of 200 W/(m·K) and an emissivity of 0.09. Since the results presented by Karagiozis were adjusted to remove radiation effects, META simulations do not account for radiative heat losses. Just as in the experimental testing procedure, the effects of the fluid properties can be adjusted for pressure effects by scaling the density of the air in proportion to the density of air at atmospheric pressure. In order to obtain a Rayleigh number over a range of 10^{-5} to 10^9, simulations were performed with a heat flow rate of 1 and 10 W for a range of pressure from 10^{-5} atm to 10 atm.

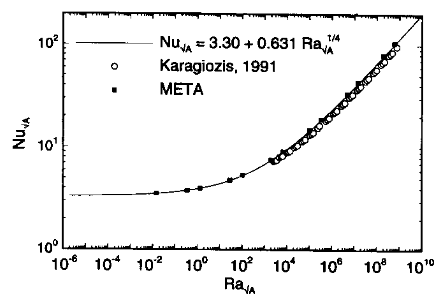
Fig. 3(a) shows a comparison of the experimental data of Karagiozis, 1991, the analytical cuboid model of Yovanovich and Jafarpur [6] and simulations obtained using META for the 150 mm $\times$ 220 mm $\times$ 9.5 mm base plate. The agreement



(a)



(b)



(c)

Fig. 3. Base plate heat transfer for $10^{-6} \leq Ra_{\sqrt{A}} \leq 10^{10}$ in plates of various widths: a) Base plate: 150 mm × 220 mm × 9.5 mm, b) Single fin: 150 mm × 50 mm × 9.5 mm, c) Base plate: 150 mm × 479.3 mm × 9.5 mm.

between the Nusselt number obtained from the three sources of data is within 8% over the full range of Rayleigh number examined, with the maximum discrepancy occurring at $Ra_{\sqrt{A}} = 10^9$. Although Karagiozis' data begins at 10^3, the META simulations and the Yovanovich cuboid model extend to $Ra_{\sqrt{A}} = 10^{-4}$, where the diffusive limit dominates and both models show good agreement. The Nusselt number calculated using META is based on an area weighted average of the predicted local heat transfer coefficient, the square root of the wetted surface area and the thermal conductivity of the fluid.

2) Example 2. Single Fin: In a manner similar to the base plate described above, a single fin can be modeled as a flat plate where the applied heat flux occurs over one end of the plate. A necessary condition for modeling a fin as an isolated plate is the assumption that the boundary layer formed around the fin remains in the developing region. Karagiozis does not present experimental data for a base plate the size of a single fin. Experimental data presented in Fig. 3(b) are for the 170-mm wide base plate described previously.

META is used to simulate a flat plate with dimensions 150 mm × 50 mm × 9.5 mm. As described above, META assumes that the edges of the plate are adiabatic, therefore to simulate a single fin where one edge of the fin is in contact with the base plate and therefore not wetted by the fluid, a plate 159.5 mm × 54.5 mm is used to simulate the fully wetted area of the fin. The power is supplied to the fin over a thin strip, 15 mm × 150 mm along one end of the plate. This heated strip is equivalent to the contact area between the fin and the base plate. In a manner similar to the base plate, a range of Rayleigh numbers between 10^{-5} to 10^9 are obtained by using heat flow rates of 1 and 10 W for a range of air pressure from 10^{-5} atm to 10 atm.

The comparison between the simulated data calculated in META, the experimental data of Karagiozis and the analytical model of Yovanovich is shown in Fig. 3(b). The data are seen to be between the cuboid model predictions and the META results.

3) Example 3. Extended Base Plate: More practical applications of heat sinks involve the use of a base plate with several fins extending from the base plate at right angles. If the fins have a high thermal conductivity, the entire fin assembly can approach an isothermal condition while the increased surface area enhances heat transfer. One means of simulating a rectangular fin assembly using a flat plate model is to approximate the fin assembly as a flat plate of equivalent surface area. Again, this implies that the boundary layers formed between the fins remain in the developing regime over the full height of the fin.

Karagiozis [15] examined a rectangular fin with a base plate 150 mm × 220 mm × 9.5 mm and five fins 150 mm × 50 mm × 9.5 mm. His tests were conducted over a range of Rayleigh between 10^2 and 10^9, where all tests below a Rayleigh number of 10^4 resulted in the boundary layer becoming fully developed in the 30-mm spacing between adjacent fins.

META simulations are presented in Fig. 3(c) for a base plate 150 mm × 479 mm × 9.5 mm, having an a total wetted surface area equivalent to that of the rectangular fin used by Karagiozis. The imposed heat flux is over an area 150 × 220 mm, representative of the base plate in the Karagiozis fin assembly. The cuboid model predictions and the META results are in excellent agreement and the data fall below the prediction by approximately 9% over the test range.

4) Example 4. Raised Fin with Base Plate: Although the fin assembly can be adequately modeled as a base plate of equivalent area, a more convenient means of simulating heat transfer in a rectangular fin using META is to model the assembly as a uniformly heated base plate with raised packages, where the packages are blocks of aluminum which serve to increase the effective surface area for heat transfer.

The development of the boundary layer over the surface of a fin goes through three distinct regimes as the boundary layer grows over the fin surfaces. The first regime, associated with large Rayleigh numbers, i.e., $Ra_{\sqrt{A}} > 10^6$, is the developing region where boundary layer growth occurs unrestricted. The total exposed surface area must be considered in the calculation of the characteristic length, namely $\sqrt{A}$. Low Rayleigh numbers, i.e., $Ra_{\sqrt{A}} < 1$ generally lead to situations where the

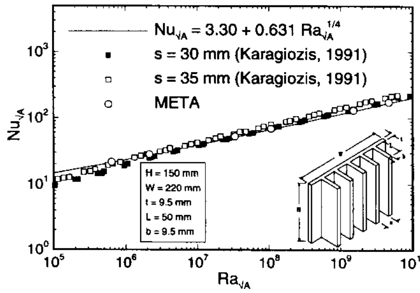


Fig. 4. Rectangular fin assembly heat transfer for $10^{-5} \leq \mathrm{Ra}_{\sqrt{A}} \leq 10^{10}$ with $L = 50$ mm and $b = 9.5$ mm.

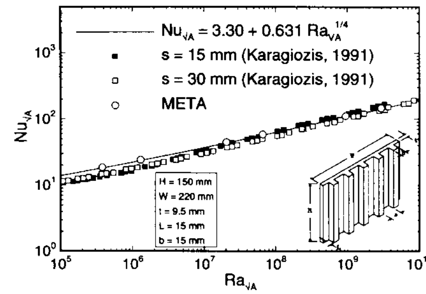


Fig. 5. Rectangular fin assembly heat transfer for $10^{-5} \leq \mathrm{Ra}_{\sqrt{A}} \leq 10^{10}$ with $L = 15$ mm and $b = 15$ mm.

boundary layers between adjacent fin sections interact almost immediately, resulting in a fully-developed flow over the entire height of the fin. In these instances, the exposed surface area is equivalent to the outside dimensions of the fin assembly. The transitional region, $1 \leq \mathrm{Ra}_{\sqrt{A}} \leq 10^6$, results in a mixed condition where the effective surface area is neither the total exposed surface area nor the surface area associated with the outside dimensions of the fin assembly. The fully-developed and transitional regions will not be addressed in this paper.

The fin assembly presented in Fig. 4 consists of a base plate 150 mm × 220 mm × 9.5 mm with five attached fins 150 mm × 50 mm × 9.5 mm. Karagiozis [16] tested this fin assembly for fin spacings of 30 mm and 35 mm over a range of Rayleigh number from $10^5 \leq \mathrm{Ra}_{\sqrt{A}} \leq 10^{10}$. The data are in excellent agreement with the heat sink model of Yovanovich, obtained from (1)–(5), over the range $10^5 \leq \mathrm{Ra}_{\sqrt{A}} \leq 10^{10}$. Below $\mathrm{Ra}_{\sqrt{A}} = 10^6$, the boundary layers forming over the fin sections begin to interact, marking the onset of fully-developed flow. Since the correlation of Yovanovich only considers situations with developing boundary layer flow, a deviation between the model and the experimental data is expected below $\mathrm{Ra}_{\sqrt{A}} = 10^6$. Also shown in Fig. 4 are the simulated results obtained using META. A heat input of 1 W was uniformly distributed over the base plate for a range of atmospheric pressure between 0.05 atm and 10 atm. This produced results for a range of Rayleigh number between $6 \times 10^5 \leq \mathrm{R}_{\sqrt{A}} \leq 5\times 10^9$. The simulated results from META are in excellent agreement with the heat sink model of Yovanovich over the full range of Rayleigh number. Although results from META are not presented below $\mathrm{Ra}_{\sqrt{A}} = 10^6$, a deviation between experimental data and META results is expected, since META does not compensate for transitional or fully-developed flow between fin sections.

Fig. 5 presents a similar comparison of experimental data, the Yovanovich heat sink model and data obtained from META. The fin assembly shown in Fig. 5 consists of a base plate 150 mm × 220 mm \times 9.5 mm with five attached fins 150 mm × 15 mm × 15 mm. As in the previous example, the agreement between the Yovanovich heat sink model and the META results is very good, however, the experimental data of Karagiozis fall below the predictions at $\mathrm{Ra}_{\sqrt{A}} \leq 10^7$. In addition the experimental data points for a fin spacing of 30 mm should result in a marginally higher Nusselt number than a fin spacing of 15 mm, but as shown in Fig. 5 this trend is reversed. Since the experimental data have been obtained from the open literature, with no explanation for this trend, further insight into the differences between the experimental data and the predictions of the Yovanovich heat sink model and META cannot be addressed at this time.

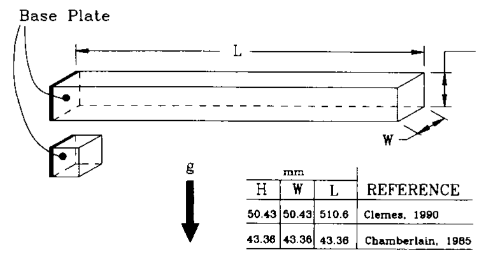


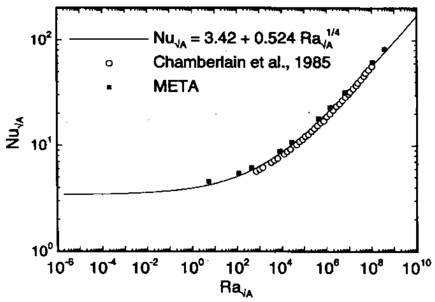
Fig. 6. Cuboid orientation and meta simulation Geometry.

B. Isothermal Cuboid

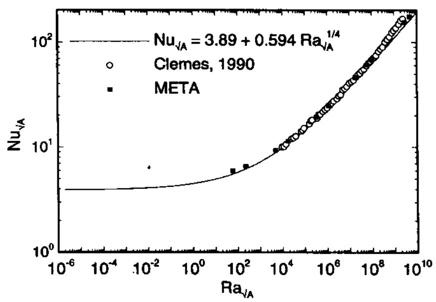
As described in the fin example above, META can be used to model raised rectangular packages attached to heat conducting substrates. An extension of this feature allows an isolated, isothermal cuboid suspended in still air to be modeled by simulating a raised body in perfect contact with a thin base plate of the same plan dimensions as one side of the cuboid as shown in Fig. 6.

Two examples of natural convection cooling from isothermal cuboids will be studied. The first example is a 43.36 mm square aluminum cube tested experimentally by Chamberlain *et al.* [17]. Tests were performed over a range of Rayleigh number from 10^3 to 10^8. Chamberlain's data are compared against the cuboid model of Yovanovich and Jafarpur [6] and against simulated results obtained using META. The META simulations were made using a thin wafer (43.36 mm × 43.36 mm × 0.2 mm) as a base plate with an extended surface (43.36 mm × 43.36 mm × 43.16 mm) in perfect contact with one side of the base plate, as shown in Fig. 6. A uniformly imposed heat flux was applied to the base plate and the mean heat-transfer coefficient over the cube was calculated. A wide range of Rayleigh numbers was obtained by varying the fluid pressure over a range of 10^{-4} atm to 10 atm. Fig. 7 shows excellent agreement between the experimental data, the cuboid model and META.

A more severe case for the extended surface model used in META is the example of a square bar, as shown in Fig. 6. As in the example above, the base is simulated as a thin base plate (50.43 mm × 50.43 mm × 0.2 mm) with an extended



(a)



(b)

Fig. 7. Heat transfer from cuboids with aspect ratios of 1:1 and 10:1 for $10^{-5} \leq Ra_{\sqrt{A}} \leq 10^{10}$.

surface more than 10 times greater than the length or width of the base plate (50.43 mm × 50.43 mm × 510.4 mm). Clemes [18] presented empirical data for a square bar over a range of Rayleigh from 10^4 to 10^9. Fig. 7 shows the comparison between the data of Clemes, the analytical correlation of Yovanovich [6] and the simulated results of META.

IV. Conclusion

The excellent agreement obtained in comparisons between experimental data, the analytical cuboid and heat sink models of Yovanovich and the conjugate model, META, indicate that a boundary layer model can be used for complex flow over three dimensional bodies if the appropriate characteristic length is used in determining the dimensionless groups, Nusselt and Rayleigh numbers. As demonstrated in Yovanovich [7], [8] and again in the comparisons presented in this paper, the appropriate characteristic length for most three dimensional bodies is the square root of the wetted surface area. It should be clearly noted, as shown in Table I, that the boundary layer equations are only applicable if the boundary layer growth occurs unrestricted over the full length of the body, especially over the length of the U-shaped channels formed in rectangular fins. If the boundary layers become fully-developed in these U-shaped channels, the models used in META will over estimate the rate of heat transfer as reflected in the Nusselt number.

Although experimental data are not available for validating the diffusive limit, obtained at low values of Rayleigh number, the analytical models of Yovanovich show excellent agreement with the predictions of META for flat plates, cuboids of various sizes and combinations of plates and cuboids.

TABLE I

1. fully developed flow,	$<< s$	Flow within the inter fin spacing is fully developed. The fin is modeled as a cuboid with dimensions $H \times W \times (t + L)$.
2. boundary layer flow,	$\delta_H \leq \frac{s}{2}$	Boundary layer growth is unrestricted over the length of the heat sink. Use boundary layer flow over a flat plate considering the total surface area.
3. developing flow,	$\delta_H > \frac{s}{2}$	The boundary layers within the inter fin spacing interfere. The current model cannot be used for this case. A future publication will address this problem.

META can be used to model heat sinks as multiply connected plates, maintaining the overall surface area of the heat sink, or as a base plate with attached cuboids acting as the extended surfaces of the fin. The latter approach offers some advantages in that it preserves the shape and physical structure of the actual heat sink and it allows a contact resistance to be used between the fins and the base plate, if it is applicable to the heat sink being modeled.

A wide range of Rayleigh numbers can be modeled by scaling the density and the kinematic viscosity of the cooling fluid in proportion to the relative atmospheric pressure. In this study a range of air pressure between 10^{-5} and 10.0 atm provided the range $10^{-6} \leq Ra_{\sqrt{A}} \leq 10^{10}$.

Acknowledgment

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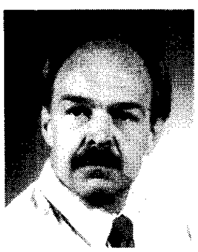

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