DIMENSIONLESS SHAPE FACTORS AND DIFFUSION LENGTHS OF THREE-DIMENSIONAL BODIES

M. M. Yovanovich
Professor and Director, Fellow ASME
Microelectronics Heat Transfer Laboratory
Department of Mechanical Engineering
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

ABSTRACT

A novel method of nondimensionalization of shape factors using the square root of the total surface area is presented. A diffusion length which is related to the shape factor is proposed for steady conduction from isothermal complex convex bodies. The dimensionless shape factors and diffusion lengths are found for many body shapes which include the ellipsoid and related bodies such as the sphere, oblate and prolate spheroids, circular and elliptic disks. Many other body shapes with different aspect ratios are also presented. These include single and double cones, hemisphere, circular and elliptical toroids, the lens, two-tangent spheres, two intersecting spheres, solid spherical cap, right circular cylinder, circular annulus and the rectangular annulus. Correlation equations of numerical results are presented for body shapes which do not have analytical solutions. All dimensionless shape factors and dimensionless diffusion lengths are shown to be weakly dependent on the body shape and aspect ratio.

NOMENCLATURE

A	=	surface area of the body, m^2
AR	=	aspect ratio
a, b	=	two radii with $a \leq b$
$\sqrt{A}$	=	characteristic length of the body proposed by Yovanovich (1987a), m
a, b, c	=	semiaxes of ellipsoids, $a \geq b \geq c$
C	=	capacitance: charge per unit potential
$C^\star$	=	capacity of body, m
C_1, C_2, C_3	=	correlation coefficients
D	=	cylinder diameter, m
$E(\phi, \kappa)$	=	incomplete elliptic integral of second kind
e	=	eccentricity
$F(\phi, \kappa)$	=	incomplete elliptic integral of the first kind
H	=	single and double cone heights, m
$I(\beta, \gamma)$	=	ellipsoid integral
$I_0(t)$	=	modified Bessel function of first kind
$K(\kappa)$	=	complete elliptic integral of the first kind
k	=	thermal conductivity, $W/m{\cdot}K$
$K_0(t)$	=	modified Bessel function of second kind
L	=	cylinder and rectangle lengths, m
$\mathcal{L}$	=	characteristic body length, m
$\vec{n}$	=	outward surface normal vector
$P_{n-1/2}$	=	Toroidal function
Q	=	total heat flow rate, W
$Q^\star_{\sqrt{A}}$	=	dimensionless heat flow rate
$Q_{n-1/2}$	=	Toroidal function
Q_e	=	total charge, coulomb
S	=	shape factor, m
$S^\star_{\sqrt{A}}$	=	dimensinoless shape factor
$S^\star_a$	=	dimensionless shape factor
T	=	temperature, K
V	=	potential, *volts*
W	=	rectangle width, m

Subscripts

a	=	arbitrary characteristic length, m
$\sqrt{A}$	=	based on $\sqrt{A}$, as the characteristic length, m
e	=	electric quantity
$\mathcal{L}$	=	based on $\mathcal{L}$, as the characteristic length, m
∞	=	at a remote point from the body

Superscripts

0	=	on the body surface
$*$	=	dimensionless quantity

Greek Symbols

β	=	ellipsoid aspect ratio, b/a
Γ	=	Gamma function
γ	=	ellipsoid aspect ratio, c/a
κ	=	modulus of incomplete elliptic integral
Δ	=	diffusion length, m
ϵ	=	radii ratio of circular annulus; permittivity
Ψ	=	Polygamma function
ϕ	=	amplitude of incomplete elliptic integral
ϕ, ϕ_e	=	dimensionless temperature and electric potential
v	=	space variable
θ	=	temperature rise above ambient, K
ρ	=	radius or radii ratio, m

Miscellaneous

$\mathcal{A}$	=	area integral
$\mathcal{W}$	=	width of circular and rectangular annulus

INTRODUCTION

Shape factors of isothermal three-dimensional bodies of complex shapes and small-to-large aspect ratios are of considerable interest for applications in the nuclear, aerospace, microelectronic and telecommunication industries. The shape factor S also has applications in such diverse areas as antenna design, electron optics, electrostatics, fluid mechanics and plasma dynamics (Greenspan, (1966)).

In electrostatics, for example, the capacitance C is the total charge Q_e required to raise the electric potential ϕ of a body to a value V, and the relationship between them is (e.g. Greenspan (1966), Jeans (1963), Kellogg (1953), Mason and Weaver (1929), Morse and Feshbach (1953), Smythe (1968), Stratton (1941), Van Bladel (1985), and Weber (1965))

$$C = \frac{Q_e}{V} = \iint_A -\epsilon \frac{\partial \phi_e}{\partial n} dA \tag{1}$$

where ϵ is the permittivity of the surrounding space, ϕ_e is the dimensionless electric potential, n is the outward-directed normal on the surface, and A is the total area of the body.

Mathematicians prefer to deal with the capacity $C^\star$ of a body which they (e.g. Szego (1945) and Polya and Szego (1951)), defined as

$$C^\star = \frac{1}{4\pi} \iint_A -\frac{\partial \phi_e}{\partial n} dA \tag{2}$$

The conduction shape factor S is defined as

$$S = \frac{Q}{k(T_0 - T_\infty)} = \iint_A -\frac{\partial \phi}{\partial n} dA \tag{3}$$

where k is the thermal conductivity, T_0 is the temperature of the isothermal body, T_∞ is the temperature of points remote from the body, and ϕ is the dimensionless temperature.

The relationships between the shape factor S, the capacitance C and the capacity $C^\star$ are

$$S = \frac{C}{\epsilon} = 4\pi C^\star \tag{4}$$

The three parameters have units of length.

Analytical solutions are available for a small number of geometries such as the family of geometries related to the ellipsoid (e.g. sphere, oblate and prolate spheroids, elliptical and circular disks). Precise numerical values of S for other axisymmetric convex bodies have been obtained by various numerical methods such as the method proposed by Greenspan (1966) and the method proposed by Wang and Yovanovich (1994).

Chow and Yovanovich (1982) showed, by analytical and numerical methods, that the capacitance is a slowly changing function of the conductor shape and its aspect ratio provided the total area of the conductor is held constant.

Wang and Yovanovich (1994) showed that the dimensionless shape factor

$$S^\star_{\sqrt{A}} = \frac{S\mathcal{L}}{A} = \frac{\mathcal{L}}{A} \iint_A -\frac{\partial \phi}{\partial n} dA \tag{5}$$

where the characteristic scale length $\mathcal{L}$ was chosen to be $\sqrt{A}$ as recommended by Yovanovich (1987a), when applied to a range of convex bodies including disks and plates is a very weak function of body shapes and their aspect ratios.

One objective of this work is to report and demonstrate through the inclusion of additional accurate numerical results of Greenspan (1966) for complex body shapes such as a cube, an ellipsoid, circular and elliptical toroids, and a lens which is formed by the intersection of two spheres that $S^{\star}_{\sqrt{A}}$ is indeed a relatively weak function of the body shape and its aspect ratios.

A second objective is to introduce the geometric length Δ which is called the diffusion length, and to show that this physical length scale is closely related to the square root of the total body surface area.

A new dimensionless geometric parameter $\sqrt{A}/\Delta$ will be proposed as the appropriate parameter for determination of shape factors, and the capacitance or capacity of complex bodies.

CONDUCTION LENGTH

Problem Statement

The problem of interest is to obtain the dimensionless steady-state heat flow rate Q^* for an isolated convex body of arbitrary shape loosing heat to an isotropic medium whose conductivity k is constant. We seek the solution to the following mathematical problem:

$$\nabla^2\phi = 0 \qquad (6)$$

with the dimensionless potential $\phi = (T(\vec{r}) - T_\infty)/(T_0 - T_\infty)$, and the temperature of the isothermal body is T_0 and $T(\vec{r}) \rightarrow T_\infty$ as $\vec{r} \rightarrow \infty$.

The Dirichlet boundary conditions become $\phi = 1$ on the body where $\vec{r} = \vec{r_b}$ and $\phi \rightarrow 0$ as $\vec{r} \rightarrow \infty$.

Conduction Length and Dimensionless Heat Flow Rate

The total heat flow rate from the body into the surrounding medium is obtained by means of the surface integral

$$Q = k\theta_0 \iint_A -\frac{\partial\phi}{\partial n} dA \qquad (7)$$

where A is the total surface area of the body, n is the outward-directed normal to the body, and θ_0 is the temperature rise of the body above its surroundings.

The conduction length Δ is introduced in the following equation which assumes linear conduction through a "tube" of arbitrary, but constant cross-section A, the thermal conductivity is k, and the temperature drop is θ_0:

$$Q = kA\frac{\theta_0}{\Delta} \qquad (8)$$

Comparing the two equations for Q we obtain the following mathematical definition for the conduction length

$$\frac{1}{\Delta} = \frac{1}{A}\iint_A -\frac{\partial\phi}{\partial n} dA \qquad (9)$$

which reveals that Δ is the reciprocal of the area-mean value of the surface gradient $\partial/\partial n$.

The dimensionless total heat flow rate is defined as

$$Q^* = \frac{Q\mathcal{L}}{kA\theta_0} \qquad (10)$$

where $\mathcal{L}$ is an arbitrary characteristic length of the body.

Yovanovich (1977a,b, and 1987a,b,c) has shown that when $\mathcal{L} = \sqrt{A}$ is used to nondimensionalize conduction solutions, the results become weak functions of the body shape, its aspect ratios and its orientation. The dimensionless heat flow rate can be written as

$$Q^* = \frac{\sqrt{A}}{\Delta} \qquad (11)$$

The dimensionless heat flow rate is seen to be the ratio of two length scales; i.e. the square root of the total surface area of the body and the conduction length. Yovanovich (1987a,b,c) has demonstrated that this ratio has a minimum value of 3.192 when the body is a circular disk, a value of 3.388 for a cube, a value of $2\sqrt{\pi} = 3.545$ for a sphere, and a value of 4.040 for a prolate spheroid and a right circular cylinder when their aspect ratios are 8, and 4.080 for an elliptical disk when its aspect ratio is 8. This interesting property of the geometric-physical parameter $\sqrt{A}/\Delta$ will be examined in the subsequent sections.

SHAPE FACTORS OF ELLIPSOIDS: Integral Form

The capacity and/or the capacitance of isopotential ellipsoids are considered in several texts and Handbooks such as Flugge (1958), Jeans (1963), Kellogg (1953), Mason and Weaver (1929), Morse and Feshbach (1953), Smythe (1968) and Stratton (1941).

The capacity of an isopotential ellipsoid $a \geq b \geq c$ was given in integral form (Szego, 1945):

$$\frac{1}{C^\star} = \frac{1}{2}\int_0^\infty \frac{dv}{\sqrt{(a^2+v)(b^2+v)(c^2+v)}} \qquad (12)$$

where v is a space variable. This expression will be used to develop the dimensionless shape factor of isothermal ellipsoids. Since $S = 4\pi C^\star$, we can set the space variable $v = a^2 t$, where t is now a dimensionless variable. Next we normalize the two smaller axes: b, c of the ellipsoid with respect to the largest semiaxis a

such that $\beta = b/a$, and $\gamma = c/a$. This will lead to the following dimensionless integral:

$$\begin{aligned}\frac{8\pi a}{S} &= I(\beta,\gamma) \\ &= \int_0^\infty \frac{dt}{\sqrt{(1+t)(\beta^2+t)(\gamma^2+t)}} \qquad (13) \\ &\text{with} \qquad 0 \le \gamma \le \beta \le 1\end{aligned}$$

The ellipsoid integral can be expressed in terms of the incomplete elliptic integral of the first kind $F(\kappa, \phi)$ (Byrd and Friedman (1971), Magnus et al. (1966)):

$$I(\beta,\gamma) = \frac{2}{\sqrt{1-\gamma^2}} \cdot F\left(\sin^{-1}\sqrt{1-\gamma^2}, \sqrt{\frac{1-\beta^2}{1-\gamma^2}}\right) \qquad (14)$$

where κ and ϕ are the modulus and amplitude angle respectively.

The ellipsoid integral reduces to several special cases which will be examined next.

Sphere: $a = b = c$; and $\beta = \gamma = 1$

$$I(1,1) = \int_0^\infty \frac{dt}{(1+t)^{3/2}} = 2 \qquad (15)$$

which gives $S = 4\pi a$, a well-known result.

Circular Disk: $a = b, c = 0$; and $\beta = 1, \gamma = 0$

$$I(1,0) = \int_0^\infty \frac{dt}{(1+t)\sqrt{t}} = \pi \qquad (16)$$

which gives $S = 8a$, also a well-known result.

Elliptic Disks: $a \ge b$, $c = 0$; and $0 < \beta \le 1, \gamma = 0$

$$I(\beta,0) = \int_0^\infty \frac{dt}{\sqrt{(1+t)(\beta^2+t)t}} = 2K\left(\sqrt{1-\beta^2}\right) \qquad (17)$$

where $K(\kappa)$ is the complete elliptic integral of the first kind of modulus $\kappa = \sqrt{1-\beta^2}$. There are several methods available to compute accurately the complete elliptic integral (Abramowitz and Stegun (1965). Here are two simple approximations:

$$K\left(\sqrt{1-\beta^2}\right) \approx \frac{2\pi}{(1+\sqrt{\beta})^2} \qquad 0.2 < \beta \le 1 \qquad (18)$$

and

$$K\left(\sqrt{1-\beta^2}\right) \approx \ln\left(\frac{4}{\beta}\right) \qquad 0 < \beta < 0.2 \qquad (19)$$

Oblate Spheroids: $a = b > c$; and $\beta = 1, 0 \ge \gamma < 1$

$$I(1,\gamma) = \int_0^\infty \frac{dt}{(1+t)\sqrt{\gamma^2+1}} = \frac{2}{\sqrt{1-\gamma^2}}\cos^{-1}\gamma \qquad (20)$$

Prolate Spheroids: $a > b = c$; $\beta = \gamma \le 1$

$$\begin{aligned}I(\gamma,\gamma) &= \int_0^\infty \frac{dt}{(\gamma^2+1)\sqrt{1+t}} \\ &= \frac{1}{\sqrt{1-\gamma^2}}\ln\left[\frac{1+\sqrt{1-\gamma^2}}{1-\sqrt{1-\gamma^2}}\right] \qquad (21)\end{aligned}$$

The above results correspond to an important family of axisymmetric, convex geometries; they are presented in terms of the capacity in various mathematical texts and in terms of the capacitance in most electromagnetic texts.

The results presented in dimensional form or in nondimensional form as given above do not reveal an important property which this family of geometries and other geometries have when the appropriate physical characteristic scale length is used for the nondimensionalization.

The numerical values of $S_a^\star$ of oblate spheroids ($\beta = 1, 0 \le \gamma \le 1$), of prolate spheroids ($\beta = \gamma, 1 \ge \gamma \ge 0.10$), and of elliptic disks ($0 \le \beta \le 1, \gamma = 0$) are presented in Tables 1–3.

Table 1: Shape Factors and Diffusion Lengths of Oblate Spheroids

$\frac{a}{c}$	$S_a^\star$	$\frac{\sqrt{A}}{\Lambda}$
1	12.5664	3.54491
2	10.3923	3.52903
3	9.62476	3.49392
4	9.23085	3.45939
5	8.99090	3.42994
6	8.82932	3.40553
7	8.71308	3.38530
8	8.62546	3.36841
9	8.55700	3.35413
10	8.50206	3.34194
10^2	8.05085	3.21098
10^3	8.00509	3.19356
10^4	8.00051	3.19174

The dimensionless shape factor range for the oblate spheroids is $8 \le S_a^\star \le 4\pi$. The greatest and least values correspond to the sphere and the circular disk respectively. The radii of the disk and sphere are set to one unit.

The dimensionless shape factor range for the prolate spheroids is approximately $4.177 \leq S_a^\star \leq 4\pi$ for the aspect ratio range $1 \leq a/b \leq 10$. The major axis is $2a = 2$.

Table 2: Shape Factors and Diffusion Lengths of Prolate Spheroids

$\frac{a}{b}$	$S_a^\star$	$\frac{\sqrt{A}}{\Delta}$
1	12.5664	3.54491
2	8.26359	3.56613
3	6.72115	3.62769
4	5.89664	3.70638
5	5.37092	3.79053
6	5.00047	3.87533
7	4.72205	3.95878
8	4.50319	4.04005
9	4.32539	4.11883
10	4.17723	4.19508

Table 3: Shape Factors and Diffusion Lengths of Elliptic Disks

$\frac{a}{b}$	$S_a^\star$	$\frac{\sqrt{A}}{\Delta}$
1	8.00000	3.19154
2	5.82716	3.28763
3	4.96964	3.43397
4	4.48606	3.57936
5	4.16641	3.71670
6	3.93511	3.84541
7	3.75763	3.96618
8	3.61576	4.07995
9	3.49888	4.18755
10	3.40033	4.28974

The dimensionless shape factor range for the elliptic disks is approximately $3.4 \leq S_a^\star \leq 8$. The greatest value corresponds to a circular disk of unit radius, and the least value corresponds to an elliptical disk with a ten-to-one aspect ratio.

Clearly the overall range for all values presented in the three tables is quite large. The ratio of the largest and smallest values is approximately 3.7. These values correspond to the sphere and the large aspect elliptical disk respectively.

SURFACE AREA OF ELLIPSOIDS

The expression for the total surface area of ellipsoids is written as

$$\frac{A}{2\pi a^2} = \gamma^2 + \frac{\beta}{\sin\phi}\left[\gamma^2 F(\phi,\kappa) + (1-\gamma^2)E(\phi,\kappa)\right]$$

$$= \mathcal{A}(\beta,\gamma) \tag{22}$$

with

$$\phi = \cos^{-1}\gamma \qquad \text{and} \qquad \kappa = \left(\frac{1-(\frac{\gamma}{\beta})^2}{1-\gamma^2}\right)^{1/2}$$

The total surface area related to the semimajor axis is seen to be a function of the two aspect ratios: β and γ. The special functions $F(\phi,\kappa)$ and $E(\phi,\kappa)$ are incomplete elliptic integrals of the first and second kind respectively. They depend on the amplitude angle ϕ and the modulus κ. These special functions can be computed quickly and accurate by means of *Mathematica* (Wolfram, 1991). Their properties are given in Abramowitz and Stegun (1965).

The relationship between the square root of the total surface area and the semimajor axis is

$$\frac{\sqrt{A}}{a} = \sqrt{2\pi\mathcal{A}(\beta,\gamma)} \tag{23}$$

SHAPE FACTOR AND DIFFUSION LENGTH OF ELLIPSOIDS

The dimensionless shape factor $S^\star_{\sqrt{A}}$ and the proposed dimensionlesss diffusion length $\sqrt{A}/\Delta$ of isothermal ellipsoids can be obtained from the shape factor integral $I(\beta,\gamma)$ and the relationship $\frac{\sqrt{A}}{a}$ given above. It is

$$S^\star_{\sqrt{A}} = \frac{\sqrt{A}}{\Delta} = \frac{4\sqrt{2\pi}}{I(\beta,\gamma)\sqrt{\mathcal{A}(\beta,\gamma)}} \tag{24}$$

The functions which appear in the above expression were computed quickly and accurately using *Mathematica*. The numerical values for oblate spheroids are presented in the third column of Table 1. The range of values has been significantly reduced. The ratio of the values for the sphere $a/c = 1$ and the circular disk $a/c \to \infty$ has been reduced from 1.57 to 1.11.

The numerical values for prolate spheroids are presented in the third column of Table 2. Here the reduction in the range is much greater. The ratio of the values for the sphere $a/b = 1$ and the long prolate spheroid $a/b = 10$ has been reduced from 3.0 to 1.18.

The numerical values for elliptic disks are presented in the third column of Table 3. Here, also, we observe that the reduction in the range is much greater. The ratio of the values for the circular disk $a/b = 1$ and the long elliptic disk $a/b = 10$ has been reduced from 2.35 to 1.34.

There is another benefit using $\mathcal{L} = \sqrt{A}$ as the characteristic body length. The differences between the values for the elliptic disks and the prolate spheroids is greatly reduced, becoming negligible for large aspect ratio. The largest difference of approximately 11 % occurs when the aspect ratio is one, i.e. when a sphere and a circular disk are compared.

The means that elliptic disks (zero thickness bodies) and prolate spheroids which have identical total surface areas and identical aspect ratios possess shape factors which are close in magnitude.

This important finding will be examined further in the subsequent sections where a wide range of body shapes will be considered.

The specific expressions for oblate and prolate spheroids are given here. They are useful for quick calculations of $\sqrt{A}/\Delta$ and they clearly show the trends for small and large aspect ratios, of axisymmetric, convex bodies.

Oblate Spheroids

$$S^{\star}_{\sqrt{A}} = \frac{\sqrt{A}}{\Delta} = \sqrt{\frac{\pi}{2}}\, 4e \cdot \left\{ \left[1 + \frac{u^2}{e}\frac{1}{2} \ln\left(\frac{1+e}{1-e}\right)\right]^{\frac{1}{2}} \cos^{-1} u \right\}^{-1} \quad (25)$$

(26)

Prolate Spheroids

$$S^{\star}_{\sqrt{A}} = \frac{\sqrt{A}}{\Delta} = \sqrt{\frac{\pi}{2}}\, 4e \cdot \left\{ \left[u^2 + \frac{u}{e}\sin^{-1} e\right]^{\frac{1}{2}} \frac{1}{2} \ln\left(\frac{1+e}{1-e}\right)\right\}^{-1} \quad (27)$$

where $u = c/b$ and $e = \sqrt{1-u^2}$, which is called the eccentricity. The aspect ratio is $AR = u$ for the oblate spheroids and $AR = 1/u$ for the prolate spheroids.

If the aspect ratio of the prolate spheroid is greater than 8, the solution approaches the asymptote:

$$S^{\star}_{\sqrt{A}} = \frac{4\sqrt{AR}}{\ln(2AR)}$$

OTHER BODY SHAPES

The shape factors and the diffusion lengths of several other body shapes will be considered next. They include two-tangent spheres, two intersecting spheres, solid spherical caps (Fig. 2), single and double solid cones, circular toroids (Fig. 1), an elliptic toroid, right circular cylinders, circular and rectangular annular plates.

Some of the body shapes to be considered have analytic solutions which will be presented. For those body shapes which do not possess analytic solutions, various numerical methods are used to find accurate numerical values which are correlated for applications.

Two-Tangent Spheres

The analytic solution for the capacitance of two tangent spheres (Jeans (1963), Smythe (1968), Weber (1965)) is converted to the dimensionless shape factor

$$S^{\star}_{\sqrt{A}} = \frac{\sqrt{A}}{\Delta} = \frac{-2\,\pi\, a\, b\left(1.15443 + \Psi(\frac{a}{a+b}) + \Psi(\frac{b}{a+b})\right)}{(a+b)\,\sqrt{\pi\,(a^2+b^2)}} \quad (28)$$

where a and b are the sphere radii respectively. The PolyGamma function $\Psi = \Gamma'/\Gamma$ and its properties are given in Abramowitz and Stegun (1972). They are accurately computed by means of *Mathematica.*

If $a = b$, the two-tangent spheres solution reduces to the bisphere solution: $S^{\star}_{\sqrt{A}} = 3.4749$, and if $a >> b$, the two-tangent spheres solution goes to the single sphere solution: $S^{\star}_{\sqrt{A}} = 3.5449$. The maximum difference is less than 2 %.

Two Intersecting Spheres

The shape factor for two isothermal spheres intersecting orthogonally and having radii a and b respectively can be obtained from the closed-form capacitance solution which is based on the Kelvin transformation (Jeans (1963), Smythe (1968), Weber (1965)),

$$S = \frac{C}{\epsilon} = 4\pi\left[\frac{(a+b)(a^2+b^2)^{1/2} - ab}{(a^2+b^2)^{1/2}}\right] \quad (29)$$

The total surface area is

$$\frac{A}{4\pi a^2} = \frac{1}{2}\left[1 + \frac{1}{\sqrt{1+\rho^2}} + \rho^2\left(1 + \frac{\rho}{\sqrt{1+\rho^2}}\right)\right] \quad (30)$$

with the radii ratio $\rho = b/a \le 1$. The dimensionless shape factor is obtained from the above relationships:

$$S^{\star}_{\sqrt{A}} = \frac{\sqrt{A}}{\Delta} = 2\sqrt{\pi}\left[\frac{(1+\rho)(1+\rho^2)^{1/2} - \rho}{(1+\rho^2)^{1/2}}\right].$$

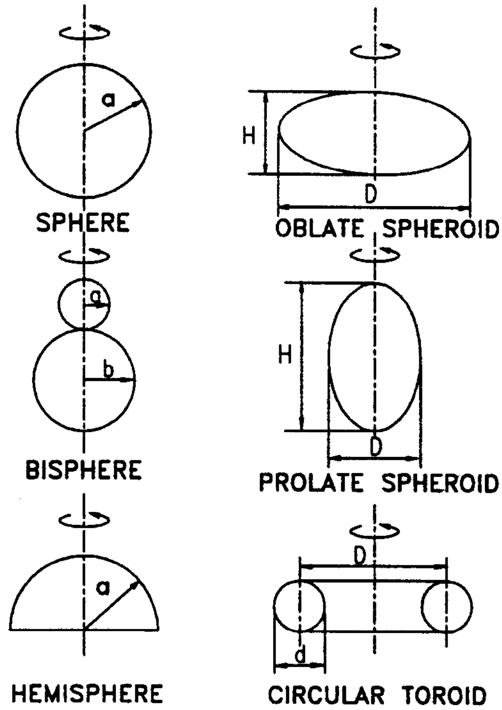

Figure 1: Body Shapes: Sphere, Bisphere, Hemisphere, Oblate and Prolate Spheroids, and Circular Toroid

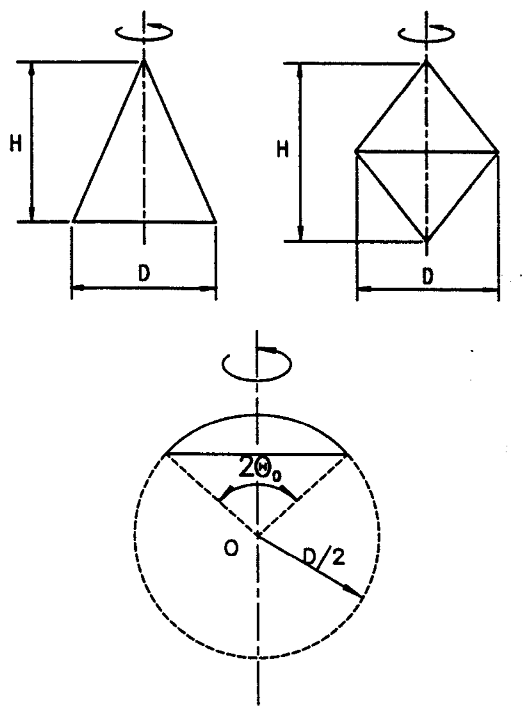

Figure 2: Body Shapes: Single and Double Cones, and Solid Spherical Cap

$$\left[1+\frac{1}{\sqrt{1+\rho^2}}+\rho^2\left(1+\frac{\rho}{\sqrt{1+\rho^2}}\right)\right]^{-1/2} \quad (31)$$

The dimensionless shape factor values vary slowly and are in close relationship with the shape factor of a sphere. The largest difference of approximately 1.1 % occurs at $\rho = 1$ and the general solution reduces to

$$S^{\star}_{\sqrt{A}} = \frac{\sqrt{A}}{\Delta} = 2\sqrt{\pi}\frac{(2-1/\sqrt{2})}{\sqrt{(1+1/\sqrt{2})}}$$

Solid Spherical Caps

The accurate numerical results of Wang (1993) for the solid spherical cap shown in Fig. 2 are correlated very accurately by means of the fourth-order polynomial:

$$\begin{aligned} S^{\star}_{\sqrt{A}} &= 3.1946 + 0.77247\,x - 0.66657\,x^2 \\ &\quad +0.51175\,x^3 - 0.26935\,x^4 \end{aligned} \quad (32)$$

where $x = \theta_0/180$ and $2\theta_0$ is the subtended angle which is related to the height-to-diameter ratio: $\tan\theta_0 = D/(2H)$.

When $\theta_0 = 1/2$ degree, $S^{\star}_{\sqrt{A}} = 3.1956$ in close agreement with the circular disk result. When $\theta_0 = 90$ degrees, the spherical cap becomes the solid hemisphere and $S^{\star}_{\sqrt{A}} = 3.4606$, and when $\theta_0 = 180$ degrees, the spherical cap becomes a sphere and $S^{\star}_{\sqrt{A}} = 3.5449$. Again we observe a relatively small variation of the dimensionless shape factor for a wide range of aspect ratios.

The single and double cones are shown in Fig. 2. They represent two interesting families of axisymmetric bodies.

Single Cone

The single cone numerical results of Wang (1993) are correlated accurately by means of the following two polynomials:
if $0.001 \le x = H/D \le 1$

$$\begin{aligned} S^{\star}_{\sqrt{A}} &= 3.19399 + 0.629823x - 0.933731x^2 \\ &\quad +0.862597x^3 - 0.312459x^4 \end{aligned} \quad (33)$$

and for the range: $1 \le x = H/D \le 8$

$$\begin{aligned} S^{\star}_{\sqrt{A}} &= 3.280967 + 1.61022(x/10) - 0.047366(x/10)^2 \\ &\quad - 0.30067(x/10)^3 + 2.99117\times10^{-3}(x/10)^4 \end{aligned} \quad (34)$$

Double Cone

The numerical data of Wang (1993) are accurately correlated by the following two fourth-order polynomials:
for the smaller range: $0.001 \le x = H/D \le 1$:

$$\begin{aligned} S^{\star}_{\sqrt{A}} &= 3.194264 + 0.626604x - 0.477791x^2 \\ &\quad + 0.0751056x^3 + 0.0531827x^4 \end{aligned} \quad (35)$$

and for the larger range: $1 \le x = H/D \le 10$:

$$\begin{aligned} S^{\star}_{\sqrt{A}} &= 3.41318 + 0.419048(x/10) + 2.02734(x/10)^2 \\ &\quad - 2.23961(x/10)^3 + 0.80661(x/10)^4 \end{aligned} \quad (36)$$

Circular Toroid

The circular toroid of ring diameter d and toroid diameter D as shown in Fig. 1 has an analytic solution which is presented in detail in Smith (1992). The solution is in the form of a series:

$$\begin{aligned} S^{\star}_{\sqrt{A}} &= \frac{\sqrt{A}}{\Delta} = \frac{4}{\pi}\sqrt{\xi - 1/\xi} \\ &\quad \cdot\left\{\frac{Q_{-1/2}(\xi)}{P_{-1/2}(\xi)} + 2\sum_{n=1}^{\infty}\frac{Q_{n-1/2}(\xi)}{P_{n-1/2}(\xi)}\right\} \end{aligned} \quad (37)$$

with $\xi = D/d > 1$. The functions which appear in the solution are Toroidal (or ring) functions whose properties are given in Abramowitz and Stegun (1965). The special functions appearing in the above solutions can be accurately computed using *Mathematica*.

The series converges very slowly for $D/d \to 1$.

For $D/d > 5$ the dimensionless shape factor for the toroid approaches the asymptote:

$$S^{\star}_{\sqrt{A}} = \frac{\sqrt{A}}{\Delta} = \frac{2\pi\sqrt{D/d}}{\ln(8D/d)} \quad (38)$$

Bouwkamp (1958) used a simple method based on the Kelvin tranformation to obtain the capacitance for the limiting case of a toroid without a hole, i.e. $D/d = 1$. The integral solution is cast in the form of the dimensionless shape factor:

$$S^{\star}_{\sqrt{A}} = \frac{\sqrt{A}}{\Delta} = \frac{8}{\pi}\int_0^{\infty}\frac{K_0(t)}{I_0(t)}\,dt = \frac{8}{\pi}\int_0^{\infty}\frac{1}{I_0^2(t)}\,dt \quad (39)$$

where $I_0(t)$ and $K_0(t)$ are modified Bessel functions of the first and second kind respectively. Numerical integration of either form by *Mathematica* gives the value $S^{\star}_{\sqrt{A}} = 3.482761$ which is in good agreement with the series solution which converges slowly to the value $S^\star_{\sqrt{A}} = 3.480$ for $D/d = 1.01$.

It is worth noting that the dimensionless shape factors for the toroid without a hole and the bisphere differ by less than 0.2 %.

Right Circular Cylinder

The capacitance for an isopotential right circular cylinder of length L and diameter D was given by Smythe (1956, 1962). It is used to develop the following expression for the dimensionless shape factor or dimensionless diffusion length:

$$S^\star_{\sqrt{A}} = \frac{\sqrt{A}}{\Delta} = \frac{3.1915\left(1 + 0.8688\,(L/D)^{0.76}\right)}{\sqrt{1+2L/D}} \tag{40}$$

where $0 \le L/D \le 8$.

This correlation equation has an accuracy of 0.2 % or better and it is in perfect agreement with the expression for the shape factor for a circular disk $L/D = 0$. Yovanovich (1987a) showed that in the range $0 \le L/D \le 1$, the values of $\sqrt{A}/\Delta$ for the right circular cylinder are in close agreement with the values for oblate spheroids which have identical total surface areas and nominally similar aspect ratios.

In the range $1 \le L/D \le 8$, the values of $\sqrt{A}/\Delta$ for the right circular cylinders and the prolate spheroids having identical total surface areas and nominally similar aspect ratios are in very close agreement (Yovanovich, 1987a). The largest difference which occurs when $L/D = 1$ is less than 1 %.

CIRCULAR ANNULUS

The capacitance of an isopotential circular annulus of inner radius a and outer radius b was presented by Smythe (1951). The two capacitance solutions are recast into the following two expressions for the dimensionless shape factor or the dimensionless difffusion length:

$$S^\star_{\sqrt{A}} = \frac{\sqrt{A}}{\Delta} = \pi\sqrt{2\pi}\sqrt{\frac{1+\epsilon}{1-\epsilon}}\cdot\frac{1}{\ln 16 + \ln\left[(1+\epsilon)/(1-\epsilon)\right]} \tag{41}$$

which is restricted to the range: $1.000 < 1/\epsilon < 1.1$; and

$$\begin{aligned} S^\star_{\sqrt{A}} &= \frac{\sqrt{A}}{\Delta} = \frac{8}{\pi}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{1-\epsilon^2}} \\ &\times\left[\cos^{-1}\epsilon + \sqrt{1-\epsilon^2}\tanh^{-1}\epsilon\right] \\ &\times\left[1 + 0.0143\epsilon^{-1}\tan^3(1.28\epsilon)\right] \end{aligned} \tag{42}$$

which is valid in the range: $0 < \epsilon < 1/1.1$. The parameter $0 \le \epsilon = a/b < 1$ is related to the parameter $2\mathcal{W}/D$:

$$\epsilon = 1 - \frac{2\mathcal{W}}{D} \tag{43}$$

and $\mathcal{W} = b - a$ is the annulus thickness.

RECTANGULAR ANNULUS

Here we consider the rectangular annulus which is defined by its outer length L and outer width W. The width $\mathcal{W}$ of the annular area is uniform. The interior open region has dimensions $L - 2\mathcal{W}$ by $W - 2\mathcal{W}$.

The dimensionless shape factor for the isothermal rectangular annulus is derived from the correlation equation of Schneider (1978) who obtained accurate numerical values of the thermal constriction resistance of doubly-connected rectangular contact areas by means of the Boundary Integral Equation Method:

$$\begin{aligned} S^\star_{\sqrt{A}} &= \frac{\sqrt{A}}{\Delta} \\ &= \frac{\sqrt{2}}{\left[C_1\left(\frac{\mathcal{W}}{2W}\right)^{C_2} + C_3\right]} \\ &\cdot\left[\frac{\mathcal{W}}{W} + \frac{L}{W} - 2\left(\frac{\mathcal{W}}{W}\right)^2\right]^{-1/2} \end{aligned} \tag{44}$$

with the recommended correlation coefficients:

$$C_1 = -0.00232\frac{L}{W} + 0.03128 \tag{45}$$

$$C_2 = 0.2927\left(\frac{L}{W} + 0.7463\right)^{-1} + 0.4316 \tag{46}$$

$$C_3 = 0.6786\left(\frac{L}{W} + 0.8145\right)^{-1} + 0.0346 \tag{47}$$

The correlation equation is restricted to the ranges: $1 \le L/W \le 4$ and $0.01 \le \mathcal{W}/W \le 0.5$ with a maximum error of 1.45 % between the correlation predictions and the numerical values. Selected values of the shape factor are given in Table 4.

It is clearly seen that the dimensionless diffusion length is a weak function of the two parameters: L/W and $\mathcal{W}/W$ over an interesting range of these parameters. The values are close to the circular and elliptic disk values.

The circular and square annular plate diffusion lengths can be compared over a wide range of the outer diameter D to thickness $\mathcal{W}$ ratio $D/\mathcal{W}$ provided we select the appropriate parameter for the square annulus.

Table 4: Dimensionless Diffusion Length of Rectangular Annulus

$\frac{L}{W}$	$\frac{\mathcal{W}}{W}$	$\frac{\sqrt{A}}{\Delta}$
1	0.1	3.2918
	0.2	3.2132
	0.3	3.1978
	0.4	3.2429
	0.5	3.3575
2	0.1	3.4893
	0.2	3.4270
	0.3	3.4044
	0.4	3.4177
	0.5	3.4681
3	0.1	3.7001
	0.2	3.6389
	0.3	3.6106
	0.4	3.6102
	0.5	3.6370
4	0.1	3.8774
	0.2	3.8153
	0.3	3.7829
	0.4	3.7744
	0.5	3.7884

Setting

$$\frac{D}{\mathcal{W}} = \frac{W - \mathcal{W}}{\mathcal{W}}$$

allows one to compare the dimensionless diffusion lengths as seen in Table 5. For the circular annulus range $1 \leq D/\mathcal{W} \leq 99$ the corresponding square annulus parameter range is $0.5 \geq \mathcal{W}/W \geq 0.01$, the $\sqrt{A}/\Delta$ values are close. The variation is small in the range $1 \leq D/\mathcal{W} \leq 4$ and the two geometries have similar values which are close to the other body shapes. For thin annular regions $4 < D/\mathcal{W} \leq 99$ the values change more rapidly and they are closer to the long prolate spheroid values or long elliptic cylinder results or similar to the large open circular toroids results.

NUMERICAL RESULTS OF GREENSPAN

Greenspan (1966) obtained accurate numerical values of the capacitance of five complex body shapes by means of a novel transformation followed by the application of the finite difference method to the tranformed domain.

He gave results for a unit cube, an ellipsoid ($a = 9, b = 4, c = 1$), a lens formed by the intersection of two spheres having different radii, a circular toroid and an elliptical toroid. The capacity $C^\star$ reported by him are given in Table 6 where it is seen that the smallest value corresponds to the cube and the largest value corresponds to the circular toroid. The range of calculated values $0.661 \leq C^\star \leq 4.89$ is seen to be quite large.

Table 5: Dimensionless Diffusion Length: Square and Circular Annulus

$\frac{\mathcal{W}}{W}$	$\frac{\sqrt{A}}{\Delta}$	$\frac{D}{\mathcal{W}}$	$\frac{\sqrt{A}}{\Delta}$
.01	10.0	99	10.6
.05	6.20	19	6.01
.10	4.86	9	4.76
.20	3.85	4	3.84
.30	3.44	2.33	3.45
.40	3.27	1.50	3.26
.50	3.23	1.0	3.19

The corresponding total surface areas and the proposed dimensionless shape factors and dimensionless diffusion lengths are also given in Table 5.

The dimensionless values are found to lie in the greatly reduced range $3.391 \leq \sqrt{A}/\Delta \leq 3.609$. The maximum and minimum values of $\sqrt{A}/\Delta$ are reported for the circular toroid and the unit cube respectively. The difference is approximately 8.5 %. The dimensionless diffusion length for the cube is approximately 4.5 % lower than the value for the sphere, and the dimensionless diffusion length for the circular toroid is approximately 3.8 % greater than the value for the sphere.

Table 6: Dimensionless Shape Factors: Cube, Lens, Ellipsoid, Circular and Elliptical Toroids of Greenspan (1966)

Body Shape	$C^\star$	A	$S^\star_{\sqrt{A}} = \frac{\sqrt{A}}{\Delta}$
Cube	0.661	6	3.391
Lens	2.29	63.577	3.609
Ellipsoid	4.26	245.8	3.415
Circular Toroid	4.89	279.16	3.678
Elliptic Toroid	1.703	36.451	3.546

CONCLUDING REMARKS

A new dimensionless diffusion length which is related to the well-known dimensionless shape factor has been proposed. This body length scale is closely related to the square root of the total surface area of convex singly connected geometries such as ellipsiods which reduce to spheres, oblate and prolate spheroids, elliptical and circular disks. The dimensionless diffusion length is found to be a weak function of the body shape and aspect ratio. This property was tested using the shape factor of other body shapes such as a right circular cylinder which has dimensionless diffusion lengths close to those of oblate spheroids when the cylinder length-to-diameter ratio is between zero and one, and the values are close to those of prolate spheroids when the aspect ratios are similar and greater than one.

Other analytic solutions are given for the solid hemisphere, two-tangent spheres, two intersecting spheres and the circular toroid. In all of these cases the dimensionless diffusion length was found to be a weak function of the body shape and its aspect ratio.

Accurate correlation equations are presented for the solid single cone, the double cone and the solid spherical cap. The solid spherical cap ranged from a thin cap which is similar to a circular disk, the solid hemisphere, and finally the sphere. For these body shapes the dimensionless diffusion length was found to be closely related to the other body shapes, and also to be weakly dependent on its shape and its aspect ratio.

The dimensionless shape factors and diffusion lengths of circular and rectangular annular plates were derived from published capacitance, capacity and constriction resistance results. Accurate correlation equations are presented for these two doubly connected bodies. The dimensionless diffusion length for these bodies are closely related to each other and closely related to the singly connected convex bodies.

Other body shapes considered are the cube, an elliptic toroid and a lens. The proposed dimensionless diffusion length was found to be close to the value of a sphere.

The final conclusion is that bodies which have identical total surface areas and have similar aspect ratios will have closely related dimensionless shape factors and diffusion lengths. Therefore, it is possible to estimate quite accurately the shape factor and the heat transfer characteristics of bodies of arbitrary shape.

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REFERENCES

Abramowitz, M., and Stegun, I., 1965, *Handbook of Mathematical Functions*, Dover Publications, New York, pp. 589–600.

Bouwkamp, C.J., 1958, "A Simple Method of Calculating Electrostatic Capacity," *Physica*, **24**, pp. 538–542.

Byrd, P.F, and Friedman, M.D., 1971, *Handbook of Elliptic Integrals for Engineers and Scientists*, Second Edition, Springer-Verlag, New York.

Chow, Y.L., and Yovanovich, M.M., 1982, "The Shape Factor of the Capacitance of a Conductor," J. Appl. Phys., **53**(12), December, pp. 8470–8475.

Flugge, S., 1958, *Handbuch der Physik*, Vol. XVI, Springer, pp. 145–147.

Greenspan, D., 1966, "Resolution of Classical Capacity Problems by Means of a Digital Computer," *Canadian Journal of Physics*, Vol. 44, pp. 2605–2614.

Hahne, E., and Grigull, U., 1974, "A Shape Factor Scheme for Point Source Configurations," *Int. J. Heat Mass Transfer*, Vol. 17, pp. 267–272.

Hahne, E, and Grigull, U., 1975, " Formfaktor und Formwiderstand der Stationaren Mehrdimensionalen Warmeleitung," *Int. J. Heat Mass Transfer*, Vol. 18, pp. 751–766.

Jeans, J., 1963, *The Mathematical Theory of Electricity and Magnetism*, Cambridge University Press, pp. 244–249.

Kellogg, O.D., 1953, *Foundations of Potential Theory*, Dover Publications, New York, pp. 188–189.

Magnus, W., Oberhettinger, F., and Soni, R.P., 1966, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York.

Mason, M., and Weaver, W., 1929, *The Electromagnetic Field*, Dover Publications, New York, pp. 130–131.

Morse, P.M., and Feshbach, H., 1953, *Methods of Theoretical Physics*, Vol. 2, McGraw-Hill, New York, pp. 1308.

Schneider, G.E., 1978, "Thermal Resistance Due to Arbitrary Dirichlet Contacts on a Half-Space," *AIAA Prog. in Astro. and Aeronautics: Thermophysics and Thermal Control*, Vol. 65, pp. 103–119.

Smith, T., 1992, "Heat Transfer from Toroidal-Shaped Bodies," M.A.Sc. thesis, Department of Mechanical Engineering, University of Waterloo.

Smythe, W.R., 1951, "The Capacitance of a Circular Annulus," *American Journal of Physics*, Vol. 22, No. 8, pp. 1499–1501.

Smythe, W.R., 1956, "Charged Right Circular Cylinder," *Journal of Applied Physics*, Vol. 27, No. 8, pp. 917–920.

Smythe, W.R., 1962, "Charged Right Circular Cylinder," *Journal of Applied Physics*, Vol. 33, No. 10, pp. 2966–2967.

Smythe, W.R., 1968, *Static and Dynamic Electricity*, 3rd Edition, McGraw-Hill, New York, pp. 122–124.

Stratton, J.A., 1941, *Electromagnetic Theory*, McGraw-Hill, New York, pp. 207–211.

Szego, G., 1945, "On the Capacity of a Condenser," *American Mathematical Society*, Ser. 2, Vol. 51, pp. 325–352.

Van Bladel, J., 1985, *Electromagnetic Fields*, Hemisphere Publishing, New York, pp. 94–98.

Wang, C.S., 1993, "Surface Element Method Based on Ring Sources and Line Sources for Accurate Calculation of Shape Factors for Arbitrary Isothermal Axisymmetric Surfaces and Some Two Dimensional Problems," M.A.Sc. thesis, Department of Mechanical Engineering, University of Waterloo.

Wang, C.S., and Yovanovich, M.M., 1994, "Ring Source Method for Calculation of Shape Factors of Complex Axisymmetric Bodies," Submitted to the AIAA J. Thermophysics and Heat Transfer.

Weber, E., 1965, *Electromagnetic Theory, Static Fields and Their Mapping*, Dover Publications, New York, pp. 449–453.

Wolfram, S., 1991, *Mathematica, A System for Doing Mathematics by Computer*, Second Edition, Addison-Wesley Publishing, Redwood City, CA.

Yovanovich, M.M., 1974, "A General Expression for Predicting Conduction Shape Factors," *AIAA Prog. in Astro. and Aeronautics: Thermophysics and Spacecraft Thermal Control*, Vol. 35, MIT Press, Cambridge, MA, pp. 265–291.

Yovanovich, M.M., 1977a, "General Conduction Resistance for Spheroids, Cavities, Disks, Spheroidal and Cylindrical Shells," AIAA 77-742, AIAA 12th Thermophysics Conference, Albuquerque, New Mexico, June 27-29, 1977.

Yovanovich, M.M, and Burde, S.S., 1977b, "Centroidal and Area Average Resistances of Nonsymmetric, Singly Connected Contacts," *AIAA Journal*, Vol. 15, No. 10, pp. 1523–1525.

Yovanovich, M.M., 1987a, " New Nusselt and Sherwood Numbers for Arbitrary Isopotential Geometries at Near Zero Peclet and Rayleigh Numbers," AIAA 87-1643, AIAA 22nd Thermophysics Conference, Honolulu, Hawaii.

Yovanovich, M.M., 1987b, "Natural Convection from Isothermal Spheroids in the Conductive to Laminar Flow Regimes," AIAA 87-1587, AIAA 22nd Thermophysics Conference, Honolulu, Hawaii.

Yovanovich, M.M. 1987c, "On the Effect of Shape, Aspect Ratio and Orientation Upon Natural Convection from Isothermal Bodies of Complex Shape," ASME HTD-Vol. 82, pp. 121–129.