ANALYTIC MODELS TO COMPUTE TRANSIENT THERMAL STRESSES IN BIMATERIAL SYSTEMS

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Abstract

A general model to predict transient thermal stresses in bimaterial systems subjected to arbitrary time dependent thermal boundary conditions is developed. The model involves two components, one to compute the transient thermal response in a bimaterial, and another to compute the stresses which arise as a result of these changes in temperature. The solution to the governing equations for the temperature distribution in each layer is obtained using the Laplace transform method. A numerical inversion scheme is employed to obtain the solution in the time domain. This solution is then used to compute the thermal loading terms for a quasi-static thermal stress analysis. The model predicts the axial stress in each layer along with the normal and shear stresses acting at the interface at any time and for any type of thermal input. The model not only accounts for the stresses which result from differential expansion between the two layers, but also the stresses which arise as a result of differential bending between the two layers. It is the latter effect which leads to greater delamination stresses. The entire model was developed using the computer algebra system (CAS) *Maple.*

Nomenclature

A_i	=	Cross-sectional area $\equiv H_i W$, m^2	m	=	Roots of Eq.(26)
B_1, B_2	=	Constants	M	=	Bending moment, Nm
C_1, C_2	=	Constants	N	=	Number of terms in a series
$D_1 \ldots C_6$	=	Constants	p	=	Real part of complex root of Eq.(26)
CAS	=	Computer Algebra System	P	=	Axial force, N
E_i	=	Young's modulus, GPa	q	=	Complex part of complex root of Eq.(26)
G_i	=	Shear modulus $\equiv E/2(1+\nu)$, GPa	r	=	Real root of Eq.(26)
h_i	=	Film coefficient, W/m^2K	s	=	Laplace transform variable
H_i	=	Thickness, m	t	=	Time, s
I_i	=	Moment of inertia $\equiv \frac{1}{12}(W H_i^3)$, m^4	T	=	Temperature, K
k_i	=	Thermal conductivity, W/mK	W	=	Width of bimaterial system, m
K	=	Constant	x	=	Length coordinate
L	=	Length of bimaterial system, m	y	=	Thickness coordinate

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Greek Symbols

α_i	=	Thermal diffusivity, m^2/s
β_i	=	Coefficient of thermal expansion, $(m/m)/K$
$\Delta_1, \Delta_2, \Delta_3$	=	Thermal loading terms
ϵ_i	=	Thermal strain
σ	=	Normal stress, MPa
τ	=	Shear stress, MPa
ϕ	=	Arbitrary function
Φ	=	Interfacial transverse compliance, GPa/m
Ψ	=	Interfacial shear compliance, GPa/m
ρ	=	Radius of curvature, m

Subscripts and Superscripts

a	=	Denotes adhesive properties
c	=	Denotes contact properties
i	=	Denotes the i^{th} layer
o	=	Denotes initial value
t	=	Denotes transition
xx	=	Denotes axial normal stresses
T	=	Denotes thermal effects

Math Symbols

$\overline{f}$	=	Denotes the Laplace transform of a function, $\equiv \mathcal{L}(f)$
j	=	Complex number, $\equiv \sqrt{-1}$
$H(\cdot)$	=	Heaviside step function
$\Re(\cdot)$	=	Real part of complex number

Introduction

Bimaterial systems may be found in numerous engineering applications. Some of the more common applications include laminated beams or micro-electronic chips. Most bimaterial systems are constructed by bonding together two similar or dissimilar materials with an adhesive. Other systems may be formed when a material is deposited onto a substrate by means of special deposition techniques. The latter method usually does not involve the use of adhesives, instead, it relies on the self adhering properties of the film material. Interfacial fracture, specifically delamination in bimaterials is the result of a loss of adhesion between two layers having different or similar mechanical and thermal properties, which are subjected to thermal and/or mechanical loadings. Thermal loadings are the result of uniform temperature changes or steady and transient thermal gradients.

Thermal stresses arise because each material expands or contracts at different rates; but because of mechanical restraint, adhesion between layers and internal restraint, shear and transverse normal (peel) stresses are induced at the interface of the system along with axial stresses in each layer. A designer may optimize the properties to minimize the stresses at some elevated temperature or for some steady thermal gradient, however, under transient conditions these stresses cannot be avoided. Even material combinations with the same mechanical properties and coefficient of thermal expansion will experience thermally induced interfacial stress when subjected to a thermal gradient. This is an important consideration in the design of any bonded component.

There has been considerable research into the interfacial failure of microelectronic chip/substrate connections and multilayered components[1,2]. Current research in the field of microelectronics packaging has produced a number of simple models which consider both uniform temperature change and applied loadings when computing stresses[3−9]. These models are all simple extensions of the application of a mechanics of materials approach to a bimetallic beam which was first developed by Timoshenko[10]. These approaches assume a constant material temperature and do not consider thermal gradients.

In many bimaterial applications the system undergoes stress/strain reversals due to transient temperature fluctuations. The maximum temperature in the system may not occur when the environment temperature is at its maximum. In addition, the differential strain at the interface may be much larger during the initial stages of heating if heat diffuses slowly into the substrate, or if there is substantial contact resistance at the interface due to a lack of adhesion or debonding between the adherends. Computing stresses based upon steady state temperature distributions, may often underestimate the maximum (or minimum) stresses. It is therefore necessary to determine the temperature profile as a function of time, and to compute the stress distribution or maximum stress as a function of time. In order to achieve this result, a combined transient thermo-mechanical model is needed. Unfortunately, no such model exists, and many designers must resort to numerical methods such as finite element analysis. Even when resorting to finite element analysis, the designer is often plagued with meshing problems near the edges where stresses become large. Finally, complex thermal boundary conditions often cannot be modelled with conventional computer codes.

This paper presents a new solution for predicting transient thermal stresses in a bimaterial by extending a model from the literature[5−7] to account for thermal bending effects produced by the transient thermal gradients.

Thermal Model

In this section the solution for transient heat conduction in a bimaterial is presented. Classic thermoelastic analysis allows the temperature and stress distributions to be computed separately if the mechanical deformation of the system does not alter the temperature field[11,12].

Since it is the through thickness temperature distribution of a bimaterial which produces the greatest interfacial stresses, a one dimensional transient conduction model will be assumed. A two dimensional transient temperature distribution may be solved, however, the effects of an axial temperature gradient are small if the axial dimension is large relative to the total thickness, i.e. $2L >> (H_1 + H_2)$.

One dimensional steady state heat conduction in layered materials has been dealt with in all basic heat transfer texts. The mathematical formulation of transient heat flow in layered materials has been dealt with in a number of advanced texts on heat conduction[13,14]; however, solutions are generally complex for systems with non-homogeneous boundary conditions, or systems with more than two layers. Often the mathematical formulation of heat transfer in composite materials is used only as a guide for numerical modelling. For the special case of a bimaterial, various analytic solution methods are available; such as separation of variables or the Laplace transform method. Of these methods, the best suited for solving transient heat conduction problems with non-homogeneous and/or time dependent boundary conditions is the Laplace transform method.

The one dimensional temperature profile of the bimaterial system shown in Fig. 1 is given by the solution of

$$\frac{\partial T_i}{\partial y_i} = \frac{1}{\alpha_i}\frac{\partial T_i}{\partial t} \tag{1}$$

for $i = 1, 2$ and subject to the boundary conditions presented in Table 1. The origin is taken at the center of each layer. In addition to the boundary conditions given in Table 1, an initial condition for each layer is required. For simplicity, the initial temperature in each layer will be taken to be uniform

$$T_i(y_i, 0) = T_{i,o} \tag{2}$$

however, an initial temperature distribution in each layer may also be prescribed.

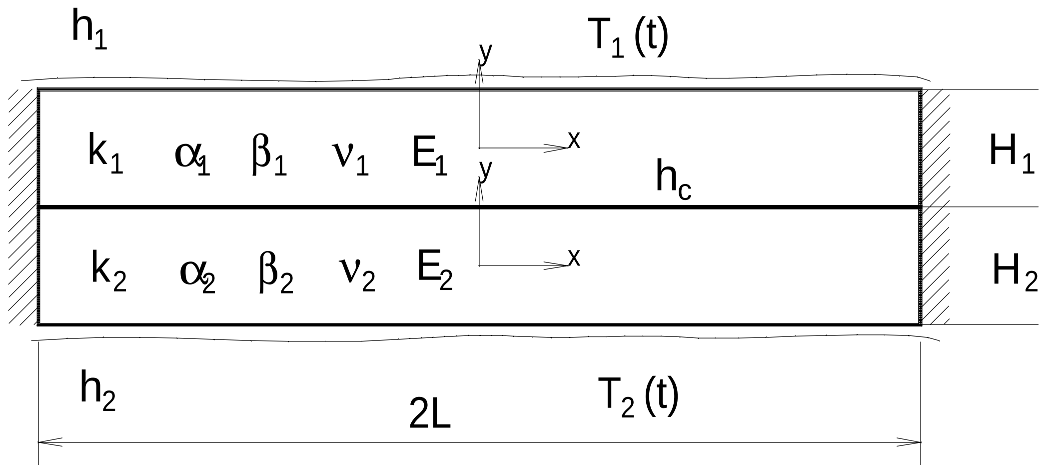

Fig. 1: Bimaterial System

Table 1: Thermal Boundary Conditions for a Bimaterial System

Top Face	$-\left.\dfrac{\partial T_1}{\partial y_1}\right\vert_{y_1=H_1/2} = \dfrac{h_1}{k_1}\{T_{1\infty}(t) - T_1(H_1/2,t)\}$
Interface	$-\left.\dfrac{\partial T_1}{\partial y_1}\right\vert_{y_1=-H_1/2} = \dfrac{h_c}{k_1}\{T_1(-H_1/2,t) - T_2(H_2/2,t)\}$ $k_1 \left.\dfrac{\partial T_1}{\partial y_1}\right\vert_{y_1=-H_1/2} = k_2 \left.\dfrac{\partial T_2}{\partial y_2}\right\vert_{y_2=H_2/2}$
Bottom Face	$-\left.\dfrac{\partial T_2}{\partial y_2}\right\vert_{y_2=-H_2/2} = \dfrac{h_2}{k_2}\{T_2(-H_2/2,t) - T_{2\infty}(t)\}$

The solution to this system of equations is obtained using the Laplace transform method[13–17]. The general solution after taking the Laplace transform of Eq.(1) is

$$\overline{T}(y_i,s) = B_i \cosh(\sqrt{s/\alpha_i}\, y_i) + C_i \sinh(\sqrt{s/\alpha_i}\, y_i) + \frac{T_{i,o}}{s} \tag{3}$$

The boundary conditions in Table 1 must also be transformed. After transformation, the boundary conditions retain their form in Table 1, however, the variable t is replaced by the transform variable s, and T is replaced by $\overline{T}$ which denotes transformation. Now applying the boundary conditions results in a system of four equations which may be solved for the four unkown constants. The solution to this system of equations is easily obtained using the computer algebra system (CAS) *Maple*[18].

Having solved for the constants, all that remains is to define the transformed boundary temperatures $\overline{T}_{1,\infty}(s)$ and $\overline{T}_{2,\infty}(s)$ and obtain the inverse Laplace transform. Obtaining the inverse transform is often quite difficult or impossible for a system such as the one above. Therefore, it would be beneficial to obtain the inverse transform through some other means such as numerical methods. Several methods exist for numerically inverting the transformed solution[19−21]. They are discussed in the next section.

Numerical Inversion of the Laplace Transform

Several methods for numerically inverting the Laplace transform are available and are discussed in Chu et al.[19], Cheng et al.[20], and Honig and Hurdes[21]. These methods provide a simple means of obtaining the inverse Laplace transform to problems with no known inverse or for problems whose inverse requires considerable mathematical effort.

In Chu et al.[19], a method based upon using a Fourier series approximation of the complex inversion integral is developed. This method is then applied to obtain the solution to two dimensional transient heat conduction in composite fins[15]. In Mujahid and Zedan[16,17], this method is also applied in obtaining the solution to one-dimensional transient conduction in a composite wall under both sudden and periodic temperature changes at a boundary. Several other methods have been developed and are discussed in Cheng et al.[20] and Honig and Hurdes[21]. Each of these methods works well for particular types of functions or over a particular time interval. One method from Cheng et al.[20] and the method presented in Chu et al.[19] are discussed in this section.

Two numerical inversion algorithms which have been examined are given below. In each case the inverse transform is represented by $\phi(t)$ and its Laplace transform represented by $\overline{\phi}(s)$.

Fourier Series Method[19]

$$\phi(t) \approx \frac{e^{ct}}{\tau}\left[\frac{1}{2}\overline{\phi}(c) + \Re\left(\sum_{k=1}^{N}\overline{\phi}\left(c + \frac{jk\pi}{t}\right)(-1)^k\right)\right] \tag{4}$$

Stehfest Method[20]

$$\phi(t) \approx \frac{\ln 2}{t}\sum_{n=1}^{N} c_n \overline{\phi}\left(\frac{n\ln 2}{t}\right) \tag{5}$$

where

$$c_n = (-1)^{n+\frac{N}{2}} \sum_{k=floor(\frac{n+1}{2})}^{min(n,\frac{N}{2})} \frac{k^{\frac{N}{2}}(2k)!}{(\frac{N}{2}-k)!k!(k-1)!(n-k)!(2k-n)!}$$

The inversion given in Eq.(4) is referred to as the Fourier Series method, while that given by Eq.(5) is referred to as the Stehfest method. Both methods have been examined in detail by Muzychka[22]. The advantages and disadvantages of each method are summarized in Table 2.

Each method works well for a wide variety of functions. The primary advantage of the Stehfest method is its computational speed and accuracy. However, its primary disadvantage is that it does not invert functions which are periodic or discontinuous. The Fourier series method, although not as efficient computationally, is more robust. It is capable of inverting periodic and discontinuous functions, such as the triangular wave function, square wave function, and combinations of ramp and step inputs. Approximately 50 terms are required when using the Fourier series method and 12 or 14 terms when using the Stehfest method, to produce extremely accurate results. In addition, the Fourier series method requires an additional parameter ct which is generally taken to be $ct = 3$ with little effect on error. The effect of this parameter and the number of terms in the inversion series are discussed in Muzychka[22].

Table 2: Summary of Numerical Inversion Algorithms

Characteristic	Fourier Series Method	Stehfest Method
Periodic Functions	Yes	No
Non-Periodic Functions	Yes	Yes
Discontinuous Functions	Yes	No
Extra Parameters	$2 \leq ct \leq 6$	None
Number of Terms	$20 \leq N \leq 50$	$10 \leq N \leq 14 (Even)$
Divergence	$t \to 0$	$t \to 0$
		N odd or $N \geq 20$
Error	Term Dependent ($\leq 2\%$)	Term Dependant ($\leq 1\%$)

Steady Thermal Stress Model

In the previous sections the formulation and solution of the one dimensional transient temperature field in layered materials with arbitrary boundary conditions was presented. Classic thermoelastic analysis allows the solution of the stress field and temperature field to be computed independently if the mechanical deformation of the system does not alter the temperature field[11,12]. This section discusses the prediction of thermal stresses in a bimaterial system which is subjected to a steady thermal gradient.

Thermal stresses in bimaterial strips and plates were first examined by Timoshenko[10]. Timoshenko[10] used a mechanics of materials approach to predict the axial stresses and deflections of bimetallic strips and plates under uniform thermal loading and various restraint conditions. The main advantage of this model is its simplicity and accuracy in predicting axial stresses and forces. However, it does not predict the edge stresses which are a result of self equilibriating forces which arise due to the thermal loading. Timoshenko[10] recognized their presence, but stated that they were of a "local" type concentrated near the edges. These local stresses are primarily responsible for edge cracking and delamination in bimaterials. Extension of Timoshenko's model for a through thickness thermal gradient is discussed later.

Various models to predict the interlaminar stresses in a bimaterial beam or strip have been developed by a number of researchers. All of these models typically fall into one of two classes: either a mechanics of materials approach or a theory of elasticity approach. The simpler of these two types of models is the mechanics of materials approach.

Simple models proposed by Chen and Nelson[3], Grimado[4], Mirman[5−7], and Suhir[8,9] are based on the mechanics of materials approach. A number of these models were originally developed for analysis of laminated beams and layered microelectronic components. The models proposed by Chen and Nelson[3] and Grimado[4] may be obtained as special cases of the model generalized by Mirman[5−7]. The models of Chen and Nelson[3] and Grimado[4] consider two materials bonded together with an adhesive, which is treated as a third material. The model presented by Mirman[5−7], which is also referred to as the Built Up Bar (BUB) theory, also considers materials which are joined without an adhesive. Finally, Suhir[8,9] has also developed a number of approximate solutions for bimaterial and multilayered structures.

The models which will be examined more closely are the model of Timoshenko[10] for computing axial stresses and the model of Mirman[5−7] for computing interfacial stresses. The results of Timoshenko[10] may be obtained from the more general model presented by Mirman[5–7]. These models have been chosen for their simplicity and for their accurate determination of the interlaminar and axial stresses. Comparisons to numerical data obtained through finite element analysis[23–25] demonstrate the accurate approximation of applying a mechanics of materials approach. The development of these models is discussed below.

The details of the derivation of the basic equations may be found in Mirman[5–7] and will not be presented here. Application of the mechanics of materials approach yields two differential equations, one for the axial force $P(x)$ and one for the bending moment $M(x)$. They are given below:

$$\frac{1}{\Psi}\frac{d^2P}{dx^2} - AP(x) - BM(x) = \epsilon_2^T - \epsilon_1^T + \frac{M_2^T H_2}{2(EI)_2} + \frac{M_1^T H_1}{2(EI)_1} \tag{6}$$

and

$$\frac{1}{\Phi}\frac{d^4M}{dx^4} + CM(x) + BP(x) = \frac{M_2^T H_2}{(EI)_2} - \frac{M_1^T H_1}{(EI)_1} \tag{7}$$

where

$$A = \left\{\frac{1}{E_1A_1} + \frac{1}{E_2A_2} + \frac{H_1^2}{4E_1I_1} + \frac{H_2^2}{4E_2I_2}\right\} \tag{8}$$

$$B = \left\{\frac{H_1}{2E_1I_1} - \frac{H_2}{2E_2I_2}\right\} \tag{9}$$

$$C = \left\{\frac{1}{E_1I_1} + \frac{1}{E_2I_2}\right\} \tag{10}$$

are constants which depend upon the material properties and dimensions of the bimaterial system. The interfacial compliances are computed from the following equations

$$\Psi = \left\{\frac{H_a}{G_a} + K\left(\frac{H_1}{G_1} + \frac{H_2}{G_2}\right)\right\}^{-1} \tag{11}$$

and

$$\Phi = \left\{\frac{H_a}{E_a} + K\left(\frac{H_1}{E_1} + \frac{H_2}{E_2}\right)\right\}^{-1} \tag{12}$$

where H_a is the thickness of the adhesive layer if present. K is a constant which is usually set to $K = 0$ if the adhesive layer is relatively thick. If no adhesive is present, the value of K may be chosen to be $K = 1$ if the adherands are of equal thickness for both the shear Ψ and transverse Φ compliances, and $K = 1/3$ if the adherands are of unequal thickness for transverse compliance only. The choice of values for this constant is discussed in Mirman[6] and in Muzychka[22] who examined the effect of this parameter by comparing the stress distributions predicted by solution of Eqs. (6,7) with data obtained from finite element analysis presented in Eischen et al.[23].

The thermal loading terms in Eqs. (6,7) are defined as follows:

$$\epsilon_i^T = \beta_i(T_{i,avg} - T_{i,o}) = \beta_i \left\{ \frac{1}{H_i} \int_{-H_i/2}^{H_i/2} T_i(y_i) dy_i - T_{i,o} \right\} \tag{13}$$

and

$$M_i^T = \int_{-H_i/2}^{H_i/2} \sigma(y_i) y_i dy_i = \int_{-H_i/2}^{H_i/2} E_i \beta_i T_i(y_i) y_i dy_i \tag{14}$$

The system of equations given by Eqs. (6,7) are subjected to the following boundary conditions:

$$\frac{dP(0)}{dx} = 0, \quad \frac{dM(0)}{dx} = 0, \quad \frac{d^3M(0)}{dx^3} = 0 \tag{15}$$

$$P(L) = 0, \quad M(L) = 0, \quad \frac{dM(L)}{dx} = 0 \tag{16}$$

Once the general solutions for the axial force $P(x)$ and bending moment $M(x)$ are obtained, the interfacial stresses per unit width may be obtained from the following relationships:

$$\tau(x) = -\frac{d}{dx} P(x) \tag{17}$$

and

$$\sigma(x) = \frac{d^2}{dx^2} M(x) \tag{18}$$

The axial stresses may be computed from the expressions derived by Timoshenko[10] given below

$$\sigma_{xx,1} = -\frac{P}{A_1} + \frac{E_1 y_1}{\rho} \tag{19}$$

and

$$\sigma_{xx,2} = \frac{P}{A_2} + \frac{E_2 y_2}{\rho} \tag{20}$$

where

$$\rho = \frac{E_1 I_1 + E_2 I_2}{(H_1 + H_2) P} \tag{21}$$

The axial stress distributions computed from Eqs.(19,20) are exactly the same as those computed from the equations developed by Timoshenko[10] when the axial force P is evaluated at a position far removed from the edges, i.e. $P = P(0)$.

Finally, if the state of stress is plane strain rather than plane stress, then each layer is treated as a bar of unit width ($W = 1$) having a coefficient of thermal expansion which is replaced by an effective value defined by $\beta_i = \beta_i(1 + \nu_i)$, and an elastic modulus replaced with $E_i = E_i/(1 - \nu_i^2)$ (see Mirman[6,7]).

Solution Procedure

The solution to the system of equations given by Eqs.(6,7) subject to the boundary conditions given by Eqs.(15,16) is easily obtained. First, Eq.(6) is substituted into Eq.(7) resulting in a new sixth order differential equation for the axial force $P(x)$ which is given below

$$\frac{d^6}{dx^6}P(x) - \Psi A\frac{d^4}{dx^4}P(x) + \Phi C\frac{d^2}{dx^2}P(x) \tag{22}$$
$$+\Phi\Psi(B^2 - CA)P(x) = -\Phi\Psi(B\Delta_3 + C(\Delta_1 - \Delta_2))$$

where

$$\Delta_1 = \epsilon_1^T - \epsilon_2^T \tag{23}$$
$$\Delta_2 = \frac{M_1^T H_1}{2E_1 I_1} + \frac{M_2^T H_2}{2E_2 I_2} \tag{24}$$
$$\Delta_3 = \frac{M_1^T}{E_1 I_1} - \frac{M_2^T}{E_2 I_2} \tag{25}$$

are the thermal loading terms. The auxiliary equation for Eq. (22) is given by

$$m^6 - \Phi A m^4 + \Phi C m^2 + \Phi\Psi(B^2 - CA) = 0 \tag{26}$$

Equation (26) has six roots, two real and four complex. The solution for these roots is easily obtained in all (CAS) such as *Maple*[18]. The form of these roots are:

$$r, -r, p + qj, p - qj, -p + qj, -p - qj \tag{27}$$

The general solution for $P(x)$ in terms of the real roots r and the real and complex parts, p and q respectively, of the complex roots is

$$P(x) = D_1\cosh(rx) + D_2\sinh(rx) + (D_3\cosh(px) + D_4\sinh(px))\cos(qx) \tag{28}$$
$$+(D_5\cosh(px) + D_6\sinh(px))\sin(qx) + \frac{C(\Delta_2 - \Delta_1) - B\Delta_3}{B^2 - CA}$$

Finally, the solution for the bending moment $M(x)$ is obtained by substituting Eq.(28) into Eq.(6). Application of the boundary conditions Eqs.(15,16) results in six equations in six unknowns which may be solved for using a (CAS) such as *Maple*[18]. This completes the solution for the interfacial and axial stresses in a bimaterial.

Transient Thermal Stress Model

In the previous section the solution to the thermal stress equations were presented in terms of $T_i(y_i)$, the temperature distribution in each layer. However, the solution in terms of $T_i(y_i, t)$ is desired for the arbitrarily prescribed boundary conditions $T_{1,\infty}(t)$ and $T_{2,\infty}(t)$. Using the results presented in the first section, the temperature distribution in the transform domain is substituted into the solution for the interfacial stresses and numerically inverted using Eqs.(4,5). The thermal loading terms are now defined as follows:

$$\epsilon_i^T = \beta_i \left\{ \frac{1}{l_i} \int_{-l_i/2}^{l_i/2} \overline{T}_i(y_i, s) dy_i - T_o \right\} \tag{29}$$

and

$$M_i^T = \int_{-l_i/2}^{l_i/2} E_i \beta_i \overline{T}_i(y_i, s) y_i dy_i \tag{30}$$

The entire procedure for determining transient thermal stresses in bimaterial systems subject to arbitrary time dependant boundary conditions is outlined in Table 3.

Table 3: General Procedure to Compute Transient Thermal Stresses

Step	Description
1	Solve system defined by Eq.(3) and Table 1 for B_1, B_2, C_1, and C_2
2	Use Eq.(28) to solve Eqs.(6,15,16) for $M(x)$ and $D_1 - D_6$
3	Define $\overline{T}_{1,\infty}(s)$ and $\overline{T}_{2,\infty}(s)$
4	Define the thermal and mechanical properties $k_i, h_i, \alpha_i, \beta_i, E_i, G_i, \nu_i$ and the dimensions H_i, L, W
5	Define the state of stress i.e. Plane Stress or Plane Strain
6	Compute the roots to Eq.(26) and find r, p, q
7	Compute $\Delta_1, \Delta_2, \Delta_3$ using Eqs.(29,30) and the results from step 1
8	Compute $\overline{\sigma}(x, s), \overline{\sigma}_{xx}(y, s), \overline{\tau}(x, s)$
9	Invert using Eqs.(4,5) to get $\sigma(x, t), \sigma_{xx}(y, t), \tau(x, t)$ for all t
10	Compute $\sigma(x, t), \sigma_{xx}(y, t), \tau(x, t)$

Following the procedure outlined in Table 3, computation of the transient thermal stresses in a bimaterial system is given below for the following thermal boundary conditions:

$$\begin{cases} T_{1,\infty}(t) = T_o + \dfrac{(T_{max} - T_o)}{t_o} H(t_t - t) + (T_{max} - T_o) H(t - t_t) \\ T_{2,\infty}(t) = T_o \end{cases} \tag{31}$$

where T_o is the initial temperature, T_{max} is the maximum temperature, t_t the time at which the maximum temperature is reached, and $H(\cdot)$ is the Heaviside step function. The $T_{1,\infty}(t)$ boundary condition represents a ramp startup condition followed by a uniform temperature. The Laplace transform of these boundary conditions is

$$
\begin{cases} \overline{T}_{1,\infty}(s) = \dfrac{T_o}{s} + \dfrac{(T_{max} - T_o)(1 - \exp^{-t_t s})}{t_t s^2} \\ \overline{T}_{2,\infty}(s) = \dfrac{T_o}{s} \end{cases} \tag{32}
$$

For this particular example the thermal and mechanical properties of the bimaterial system are given in Table 4. In addition to the properties presented in Table 4, $T_{max} = 75\,C$, $T_o = 25\,C$, $t_t = 1800\,s$, and the heat transfer coefficients $h_i = \infty$, i.e. perfect contact. The state of stress is taken as plain strain assuming $\nu_i = 0.3$. The complete solution was programmed using the symbolic programming language of *Maple*[18]. Subroutines or procedures were written to solve each system of equations symbolically, compute the expressions for each stress, compute the roots to Eq.(26), and numerically invert the Laplace transform solution into the time domain. In order to provide a computationally efficient procedure, the numerical inversion was computed symbolically and the general expression evaluated at each point in time, rather than numerically invert each point in time. This process is similar to expanding a function in a series representation.

Table 4: Mechanical and Thermal Properties

Property	Aluminum	PMMA
$k\ (W/mK)$	230	0.185
$\alpha\ (m^2/s)$	$8.45e{-}5$	$1.10e{-}7$
$E\ (GPa)$	70	3
$\beta\ ((m/m)/K)$	$24e{-}6$	$63e{-}6$
$H\ (mm)\,[in]$	2 [1/16], 4 [1/8], 7 [1/4]	13 [1/2]
$2L, 2W\ (mm)$	150	150

The results for the maximum interfacial stresses $\sigma(L,t)$ and $\tau(L,t)$ of this example are presented in Figs. 2 and 3. It can be seen that the maximum shear stress follows the prescribed thermal boundary condition with the maximum stress occurring much later than the maximum surface temperature. However, the transverse normal or peeling stress does not behave the same way. In this particular example the maximum peel stress occurs in the early stages of heating before the maximum surface temperature is reached. Also, the steady state value of the peel stress is much less than that experienced in the early stages of heating, demonstrating the need for a transient analysis.

Summary and Conclusions

A general procedure to analytically compute transient thermal stresses in bimaterials for arbitrary thermal boundary conditions was presented. The model was developed using the computer algebra system *Maple*[18]. This (CAS) provided the symbolic solution for two complicated systems of equations which describe the temperature and stress distributions in the bimaterial.

The solution to the temperature problem was obtained using the Laplace transform method and results obtained in the time domain by means of a numerical inversion scheme. This method of solution enables the prediction of the interfacial and axial thermal stresses for complex time dependant boundary conditions such as triangular waves, square waves, and/or combinations of ramp and step inputs.

The thermal stress model was based upon a mechanics of materials approach which was generalized by Mirman[7,8]. This model is an extension of a simpler model originally proposed by Timoshenko[10] for computing the axial stress distribution in a bimaterial subject to a uniform temperature change.

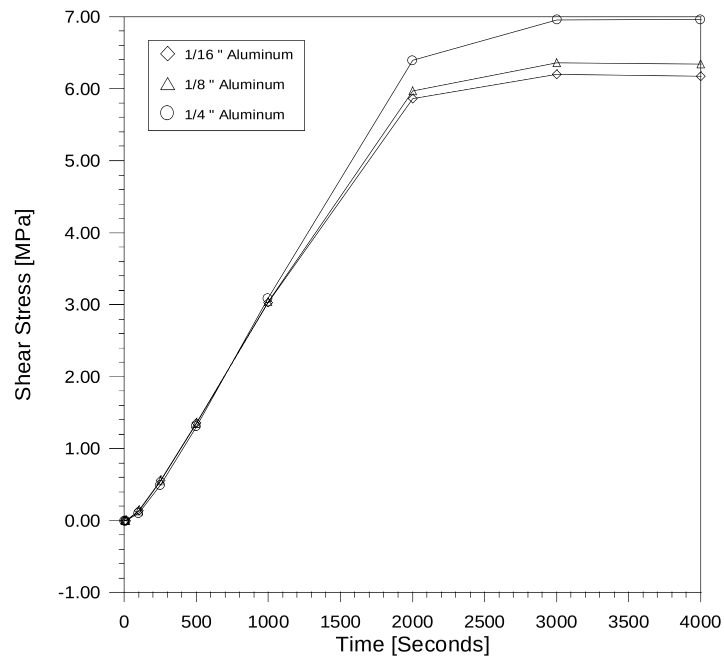

Fig.2: Plot of the maximum transient shear stress $\tau(L,t)$

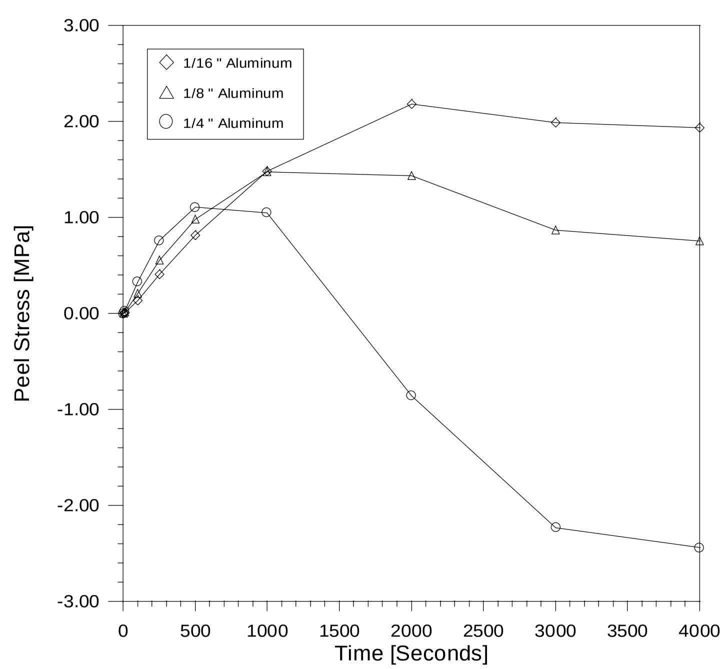

Fig.3: Plot of the maximum transient peel stress $\sigma(L,t)$

Finally, illustration of the method for a transient boundary condition composed of a ramp and step input showed that prediction of the maximum peel stress under steady conditions would show smaller or negative stresses, where as, the transient solution shows that a very large positive maximum stress occurs during the initial heating stage.

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References

[1] Suhir, E., 1988, "Thermal Stress Failures in Microelectronic Components - Review and Extension", in *Advances in Thermal Modeling of Electronic Components and Systems*, eds. A. Bar-Cohen and A. D. Kraus, Hemisphere Publishing.

[2] Tummala, R. R. and Rymaszewski, E. J., 1989, *Microelectronics Packaging Handbook*, Van Nostrand Reinhold.

[3] Chen, W. T. and Nelson, C. W., 1979, "Thermal Stresses in Bonded Joints", *IBM Journal of Research and Development*, Vol. 23, pp. 179-188.

[4] Grimado, P. B., 1978, "Interlaminar Thermoelastic Stresses in Layered Beams", *Journal of Thermal Stresses*, Vol. 1, pp. 75-86.

[5] Mirman, B. A. and Knecht, S., 1990, "Creep Strains in an Elongated Bond Layer", *IEEE Transactions on Components, Hybrids, and Manufacturing Technology*, Vol. 13, pp. 914-928.

[6] Mirman, B. A., 1992, "Microelectronics and the Built Up Bar Theory", *Journal of Electronic Packaging*, Vol. 114, pp. 384-388.

[7] Mirman, B. A., 1992, "Interlaminar Stresses in Layered Beams", *Journal of Electronic Packaging*, Vol. 114, pp. 389-396.

[8] Suhir, E. 1989, "Thermally Induced Interfacial Stresses in Elongated Bimaterial Plates", *Applied Mechanics Reviews*, Vol. 42, pp. S253-S262.

[9] Suhir, E., 1989, "Interfacial Stresses in Bimetal Thermostats", *Journal of Applied Mechanics*, Vol. 56, pp. 595-600.

[10] Timoshenko, S., 1925, "Analysis of Bimetal Thermostats", *Journal of the Optical Society of America*, Vol. 11, pp. 233-255.

[11] Boley, B.A. and Weiner, J.H., 1960, *Theory of Thermal Stresses*, John Wiley and Sons Inc.

[12] Timoshenko, S. and Goodier, J.N., 1970, *Theory of Elasticity*, McGraw-Hill.

[13] Gebhart, B. G., 1993, *Heat Conduction and Mass Diffusion*, McGraw-Hill.

[14] Ozisik, M.N, 1980, *Heat Conduction*, John Wiley and Sons Inc.

[15] Chu, H., Weng, C. and Chen, C., 1983, "Transient Response of a Composite Fin", *Journal of Heat Transfer*, Vol. 105, pp. 307-311.

[16] Mujahid, A. M. and Zedan, M. F., 1990, "Transient Heat Conduction Response of a Composite Plane Wall", *Warme und Stoffubertragung*, Vol. 26, pp. 33-39.

[17] Mujahid, A. M. and Zedan, M. F., 1993, "Laplace Transform Solution for Heat Transfer in Composite Walls with Periodic Boundary Conditions", *Journal of Heat Transfer*, Vol. 115, pp. 263-265.

[18] *Maple V3*, 1981-1996, Waterloo Maple Software, Waterloo, Ontario, Canada.

[19] Chu, H., Weng, C. and Chen, C., 1982, "Application of Fourier Series Technique to Transient Heat Transfer Problem", *Chemical Engineering Communications*, Vol. 16, pp. 215-225.

[20] Cheng, A. H-D., Sidauruk, P. and Abousleiman, Y., 1992, "Approximate Inversion of the Laplace Transform", *The Mathematica Journal*, Vol. 4, pp. 76-81.

[21] Honig, G. and Hurdes, U., 1984, "A Method for the Numerical Inversion of Laplace Transforms", *Journal of Computational and Applied Mathematics*, Vol. 10, pp. 113-132.

[22] Muzychka, Y.S., 1995, *Analytic Models to Compute Transient Thermal Stresses in Bimaterial Structures*, M.A.Sc. Thesis, University of Waterloo.

[23] Eischen, J. W., Chung, C. and Kim, J. H., 1990, "Realistic Modelling of Edge Effect Stresses in Bimaterial Elements", *Journal of Electronic Packaging*, Vol. 112, pp. 16-23.

[24] Glaser, J. C., 1990, "Thermal Stresses in Compliantly Joined Materials", *Journal of Electronic Packaging*, Vol. 112, pp. 24-29.

[25] Poinke, C. D. and Wempner, G., 1991, "The Various Approximations of the Bimetallic Thermostatic Strip", *Journal of Applied Mechanics*, Vol. 58, pp. 1015-1020.