Modeling Friction Factors in Non-Circular Ducts for Developing Laminar Flow

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Solutions to hydrodynamically developing flow in circular and non-circular ducts are examined. It is shown that the apparent friction factor based upon the square root of the cross-sectional area is a weak function of the shape of the geometry provided an appropriate aspect ratio is defined. A general model which is valid for many duct configurations is developed by combining the developing flow and fully developed flow asymptotes. The new model is simpler than other general models and provides equal or better accuracy. Finally, it is shown that the solution for the elliptic duct geometry may be used to compute accurately, results for 8 singly-connected ducts and 2 doubly-connected ducts, respectively, with an accuracy of ±12 percent.

NOMENCLATURE

A	=	flow area, m^2
a	=	major axis of ellipse or rectangle, m
b	=	minor axis of ellipse or rectangle, m
C	=	empirical constant Eq. (15)
c	=	linear dimension, m
D	=	diameter of circular duct, m
d^*	=	dimensionless diameter ratio, $\equiv D_h/D_{max}$
D_h	=	hydraulic diameter, $\equiv 4A/P$
$\mathbf{E}(\cdot)$	=	complete elliptic integral of the second kind
e	=	eccentricity, m
e^*	=	dimensionless eccentricity, $\equiv e/(r_o - r_i)$
f	=	friction factor $\equiv \overline{\tau}/(\frac{1}{2}\rho\overline{w}^2)$
$g(\epsilon$	=	shape function Eq. (24)
K	=	incremental pressure drop factor
m	=	area mismatch parameter, $\equiv A/A_{D_h}$
N	=	number of sides of a polygon
n	=	correlation parameter Eqs. (21,22)
P	=	perimeter, m
p	=	pressure, N/m^2
p^*	=	dimensionless pressure, $\equiv p/(\frac{1}{2}\rho\overline{w}^2)$
r	=	radius, m
r^*	=	dimensionless radius ratio, $\equiv r_i/r_o$
$Re_{\mathcal{L}}$	=	Reynolds number, $\equiv \overline{w}\mathcal{L}/\nu$
s	=	arc length, m
$\vec{V}$	=	velocity vector, m/s
u, v, w	=	velocity components, m/s
$\overline{w}$	=	average velocity, m/s
x, y, z	=	cartesian coordinates, m
$z^+_{\mathcal{L}}$	=	dimensionless axial position for developing fluid flow, $\equiv z/\mathcal{L}Re_{\mathcal{L}}$

Greek Symbols

ϵ	=	aspect ratio, $\equiv b/a$
μ	=	dynamic viscosity, Ns/m^2
ν	=	kinematic viscosity, m^2/s
ρ	=	fluid density, kg/m^3
ψ	=	shape function
τ	=	wall shear stress, N/m^2

Subscripts

$\sqrt{A}$	=	based upon the square root of flow area
app	=	apparent
c	=	core
D_h	=	based upon the hydraulic diameter
e	=	entrance
fd	=	fully developed
i	=	inner, initial
$\mathcal{L}$	=	based upon the arbitrary length $\mathcal{L}$
o	=	outer
w	=	wall
z	=	local value
∞	=	fully developed limit

Superscripts

$+$	=	denotes dimensionless quantity
$*$	=	denotes dimensionless quantity
$\overline{(\cdot)}$	=	denotes average value of $(\cdot)$

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INTRODUCTION

Predicting the pressure drop under developing flow conditions (see Fig. 1) is quite important in many applications such as heat exchangers where flow passages are typically short in length and may be circular or non-circular. In these applications it is often desirable to have simple models in the early design stages to predict the pressure drop characteristics. A review of the available literature reveals that a few general models exist which are able to predict pressure drop (or friction factor) in non-circular ducts. However, the available models either require tabulated coefficients or consist of several equations requiring intermediate calculations to be performed. The main objective of this paper is to develop a model which will eliminate some of the problems present in the currently available models. The proposed model is compared with results available in the literature for many of the geometries shown in Figs. 2-3.

GOVERNING EQUATIONS AND DIMENSIONLESS GROUPS

Governing Equations

The governing equations and dimensionless groups for hydrodynamically developing flow are presented below. The continuity and momentum equations in vector form are:

$$\nabla \cdot \vec{V} = 0 \tag{1}$$

$$(\vec{V} \cdot \nabla)\,\vec{V} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{V} \tag{2}$$

At the wall of the duct, the fluid is subject to the condition

$$V_w = 0 \tag{3}$$

and at the duct inlet, a uniform inlet velocity

$$V_{in} = w_i \tag{4}$$

is usually prescribed.

Fully Developed Flow

For fully developed flow in a duct of arbitrary cross-section, the Navier-Stokes equations reduce to the momentum equation in the flow direction. The resulting equation is the Poisson equation in one or two dimensions depending upon the cross-sectional geometry. In this case, the source term is the constant pressure gradient along the length of the pipe.

In cartesian coordinates the governing equation for fully developed laminar flow in a constant cross-sectional area duct is

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dz} \tag{5}$$

which represents a balance between the pressure and viscous forces.

Solutions for many of the different duct geometries shown in Figs. 2 and 3 have been obtained using various analytical and numerical methods and are discussed in Shah and London[1] and Shah and Bhatti[2].

Hydrodynamically Developing Flow

In cartesian coordinates the governing equations in the entrance region of the duct are the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{6}$$

and momentum equation

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{dp}{dz} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{7}$$

The pressure gradient term may be written as

$$-\frac{1}{\rho}\frac{dp}{dz} = w_c\frac{dw_c}{dz} \tag{8}$$

where $w_c = w_c(z)$ is the velocity of the accelerating core.

Due to the non-linear terms in Eq. (7) solutions for hydrodynamically developing flows are generally more difficult to obtain than fully developed flows. Developing flows require simultaneous solution to both the continuity, Eq. (6), and the momentum, Eq. (7), equations given above. Despite this difficulty, analytic and approximate analytical solutions for developing flows have been obtained for the circular duct[3], rectangular duct[4], elliptical duct[5], parallel plates[6] and circular annular duct[7]. Solutions for many other geometries have also been obtained numerically and are discussed in Shah and London[1] and Shah and Bhatti[2].

Dimensionless Groups

Numerical and analytical results are often presented in terms of the dimensionless friction factor. The Fanning friction factor is defined as

$$f = \frac{\overline{\tau}_w}{\frac{1}{2}\rho\,\overline{w}^2} \tag{9}$$

which is usually written in terms of the Reynolds number as follows

$$f Re_{\mathcal{L}} = 2\,\frac{\overline{\tau}_w\,\mathcal{L}}{\mu\,\overline{w}} \tag{10}$$

where $\mathcal{L}$ is a characteristic length scale of the duct cross-section, usually chosen to be the hydraulic diameter $D_h = 4A/P$.

A general form of the friction factor Reynolds group in terms of the solution for the velocity distribution is

$$f Re_{\mathcal{L}} = 2\mathcal{L}\,\frac{\dfrac{1}{P}\displaystyle\oint -\frac{\partial w}{\partial n}\bigg|_w ds}{\dfrac{1}{A}\displaystyle\iint_A w\,dA} \tag{11}$$

where $\left.\dfrac{\partial w}{\partial n}\right|_s$ represents the velocity gradient at the duct wall with respect to an inward directed normal.

If the flow is developing, an apparent friction factor[1] which accounts for the wall shear and increase in momentum in the inviscid core can be defined as

$$f_{app} Re_{\mathcal{L}}(z^+) = f Re_{\mathcal{L}} + \frac{K(z^+)}{4z^+} = \frac{\Delta p^*}{4z^+} \tag{12}$$

where the incremental pressure drop $K(z^+)$ is defined as the difference between the total pressure drop in the duct and the pressure drop if the flow were fully developed at every point along the duct, or

$$K(z^+) = \Delta p^* - \left(\frac{dp^*}{dz^+}\right)_{fd} z^+ \qquad (13)$$

where $\Delta p^* = \dfrac{p_i - p_z}{\frac{1}{2}\rho \overline{w}^2}$ and $z^+ = \dfrac{z/\mathcal{L}}{Re_{\mathcal{L}}}$.

PRESENT MODELS

The apparent friction factor in a circular or non-circular duct may be computed from the following expressions[1,2]

$$f_{app}Re = \begin{cases} \dfrac{3.44}{\sqrt{z^+}} & z^+ < 0.001 \quad \text{(a)} \\ fRe_{fd} + \dfrac{K_\infty}{4z^+} & z^+ > 0.05 \quad \text{(b)} \end{cases} \qquad (14)$$

where K_∞ is the value of the incremental pressure drop when the flow becomes fully developed. The solutions for $f_{app}Re$ given above are only valid for very short ducts or very long ducts. To establish the complete $f_{app}Re$ relationship, knowledge of the incremental pressure drop $K(z^+)$ is necessary.

Several models for predicting pressure drop for hydrodynamically developing flows in ducts of constant cross-section are available in the literature. These models are based upon the early work of Bender[8]. Bender[8] combined the result of Shapiro et al.[9] Eq. (14a), with the result for the "long" duct Eq. (14b), to provide a model which is valid over the entire length of a circular duct. Shah[10] later extended the model of Bender[8] to predict results for the equilateral triangle, the circular annulus, and the rectangular duct geometries. Shah[10] achieved this by generalizing the form of the model of Bender[8], and tabulating coefficients for each special case. The model proposed by Shah[10] is given below.

$$f_{app}Re_{D_h} = \frac{3.44}{\sqrt{z^+}} + \frac{fRe_{fd} + K_\infty/4z^+ - 3.44/\sqrt{z^+}}{1 + C/(z^+)^2} \qquad (15)$$

Recently, Yilmaz[11] has proposed a more general model of Shah[10]. Rather than tabulating coefficients, Yilmaz[11] developed expressions for the fully developed friction factor fRe_{fd}, incremental pressure drop K_∞, and the constant C which appear in the Shah[10] extension of the Bender[8] model. The model of Yilmaz[11] takes the following form:

$$f_{app}Re_{D_h} = \frac{3.44}{\sqrt{z^+}} + \frac{16\psi + K/4z^+ - 3.44/\sqrt{z^+}}{1 + 0.98 \times 10^{-4} K^{3.14}/(z^+)^2} \qquad (16)$$

where

$$\psi = 1 + \frac{(\psi_\infty - 1)}{1 + 0.33 d^{*2.25}/(m-1)} \qquad (17)$$

and

$$\psi_\infty = \frac{3}{8} d^{*2}(3 - d^*) \qquad (18)$$

are shape factors relating the non-circular duct to a circular duct.

The incremental pressure drop for the arbitrary geometry is

$$K = \frac{1.33}{1 + (1.33/K_\infty - 1)/[1 + 0.74 d^{*2}/(m-1)]} \qquad (19)$$

where

$$K_\infty = \frac{12}{5}(3 - d^*)^2 \left[\frac{9}{7}\left(\frac{3 - d^*}{7 - 3d^*}\right) - \frac{1}{5 - 2d^*}\right] \qquad (20)$$

with $m = A/A_{D_h}$, $d^* = D_h/D_{max}$, where A_{D_h} is the area based upon the hydraulic diameter, $A_{D_h} = \pi D_h^2/4$, and D_{max} is the diameter of the maximum inscribed circle.

This model is more general than that of Shah[10] but quite complex. Despite its complexity, the model of Yilmaz[11] is accurate over the entire range of the entrance and fully developed regions. The primary drawback of the simple model proposed by Shah[10] is the requirement of tabulated coefficients and parameters for each geometry, i.e. fRe_{fd}, K_∞, and C, thus limiting interpolation for geometries such as the rectangular duct whose solution varies with aspect ratio. In the case of the model developed by Yilmaz[11], interpolation is no longer a problem, however this is achieved at the cost of simplicity. Although the model of Yilmaz[11] is only a function of two geometry specific parameters, m and d^*, there is some difficulty in determining D_{max} for many geometries. The proposed model is only a function of one geometric parameter, the aspect ratio of the duct, and allows for accurate prediction of the apparent friction factor for many geometries given in Shah and London[1] and Shah and Bhatti[2].

MODELING

The proposed model for hydrodynamically developing flows takes the form of

$$y(z^+) = [y^n_{z^+\to 0} + y^n_{z^+\to\infty}]^{1/n} \qquad (21)$$

where $y_{z^+\to 0}$ and $y_{z^+\to\infty}$ are asymptotic solutions for small and large values of the independent variable z^+ and n is the fitting parameter. The method of superposition of asymptotic solutions is discussed in detail by Churchill and Usagi[12].

For the particular case of predicting the apparent friction factor in a duct, this would involve combining the developing flow, Eq. (14a), and fully developed flow, fRe_{fd}, asymptotes in the following form:

$$f_{app}Re_{D_h} = \left\{ \left(\frac{3.44}{\sqrt{z^+}} \right)^n + fRe_{fd}^n \right\}^{1/n} \quad (22)$$

The authors have determined that the optimal value for the parameter n is $n \approx 2$. The above model accurately predicts the data for all of the geometries examined in this study. The proposed model is considerably simpler than that of Shah[10] and Yilmaz[11] and does not contain the incremental pressure drop factor K_∞. The proposed model does require knowledge of the fully developed friction factor Reynolds number product fRe_{fd}. It would be beneficial to develop a means to predict the fully developed results for many geometries as done by Yilmaz[11], to further simplify the model.

Entrance Region

In the entrance region where the boundary layer thickness is small, the results are similar for all ducts regardless of geometry. An analytical result for the apparent friction factor in the entrance region of the circular duct was derived by Shapiro et al.[9] using several methods. The leading term in the solution for any characteristic length $\mathcal{L}$ is given by

$$f_{app}Re_{\mathcal{L}} = \frac{3.44}{\sqrt{z_{\mathcal{L}}^+}} \quad (23)$$

which is valid for $z^+ = z/(Re_{\mathcal{L}}\mathcal{L}) \leq 0.001$. This solution is independent of the duct shape and is analogous to the short time solution in conduction heat transfer. In Yovanovich et al.[13], the authors developed a simple model for computing transient heat flow from arbitrarily shaped convex isothermal bodies into infinite domains, by combining the short time and steady state asymptotes in a similar manner.

Fully Developed Region

In the fully developed region of the duct, many solutions have been obtained. However, the solutions are not independent of duct geometry as is the case in the entrance region. It is desirable to have a solution in the fully developed region which is a weak function of the shape of the duct. This may be achieved by selecting a more appropriate characteristic length which will collapse most of the results for non-circular ducts onto a single curve.

In the heat transfer and fluid flow literature, the conventional selection is the hydraulic diameter, $D_h = 4A/P$. This characteristic length arises from a simple control volume force balance performed on an arbitrary slug of fluid in a duct of arbitrary shape.[14] For a circular duct $D_h = D$, where D is the diameter of the duct. The hydraulic diameter concept is much more accurate in turbulent flow theory[7], however, since this paper is concerned with laminar flow problems, an alternative to the hydraulic diameter which will collapse the results of many non-circular ducts onto a single curve.

One notable drawback of the hydraulic diameter concept is the fact that the area computed from the hydraulic diameter is not the same as the true area of the duct in question. This "mismatch" in areas is often assumed to be the cause of the mismatch between the results for the circular duct geometry and non-circular geometries.[14-16] In the model developed by Yilmaz[11] a mismatch parameter which is defined as the ratio of the true duct cross-sectional area to the area computed from the hydraulic diameter is proposed in the development of the model.

The appropriate characteristic length should minimize the differences between solutions for different geometries when the results are non-dimensionalized. Three obvious choices for a characteristic length are the perimeter $\mathcal{L} = P$, the hydraulic diameter $\mathcal{L} = 4A/P$, and the square root of the flow area $\mathcal{L} = \sqrt{A}$. In a recent paper[17] the authors showed by means of dimensional analysis that the square root of the cross-sectional flow area is a more appropriate characteristic length for presenting friction factors of non-circular ducts than the hydraulic diameter. It was shown that most numerical and analytical results for the fully developed friction factor-Reynolds number product are predicted to within ±10 percent by the closed form solution for the elliptic duct when the characteristic length is $\mathcal{L} = \sqrt{A}$. The square root of the cross-sectional area as a characteristic length is essentially the same as defining an equivalent diameter which preserves the cross-sectional flow area.

Table 1
Definitions of Aspect Ratio

Geometry	Aspect Ratio
Regular Polygons	$\epsilon = 1$
Singly-Connected‡	$\epsilon = \frac{b}{a}$
Trapezoid	$\epsilon = \frac{2b}{a+c}$
Annular Sector	$\epsilon = \frac{1-r^*}{(1+r^*)\Phi}$
Circular Annulus	$\epsilon = \frac{(1-r^*)}{\pi(1+r^*)}$
Eccentric Annulus	$\epsilon = \frac{(1+e^*)(1-r^*)}{\pi(1+r^*)}$

‡ All except annular sector and trapezoid.

The results of Yovanovich and Muzychka[17] for many geometries are presented in Figs. 4-5. The numeric results are accurately predicted by

$$fRe_{\sqrt{A}} = 8\sqrt{\pi} \left(\frac{\pi}{4} \frac{(1+\epsilon^2)}{\sqrt{\epsilon}\,\mathbf{E}(\sqrt{1-\epsilon^2})} \right) . \quad (24)$$

where $\mathbf{E}(\cdot)$ is the complete elliptic integral of the second kind, and ϵ is an appropriate aspect ratio of the duct.

To eliminate the problem of evaluating the elliptic integral, an approximate expression was developed for the shape function $g(\epsilon)$ defined as

$$g(\epsilon) = \left(\frac{\pi}{4} \frac{1+\epsilon^2}{\sqrt{\epsilon}E(\sqrt{1-\epsilon^2})}\right) \quad (25)$$

such that

$$fRe_{\sqrt{A}} = 8\sqrt{\pi}g(\epsilon) \quad (26)$$

The shape function $g(\epsilon)$ may be accurately computed from the following expression:

$$g(\epsilon) \approx \left[1.086957^{1-\epsilon}(\sqrt{\epsilon} - \epsilon^{3/2}) + \epsilon\right]^{-1} \quad (27)$$

Equation (27) is valid over the range $0.05 \leq \epsilon \leq 1$ with an RMS error of 0.70 percent and a maximum error less than ±2 percent. Definitions of the appropriate aspect ratio for all of the geometries are summarized in Table 1. The aspect ratio ϵ is defined as the ratio of the maximum width and height of each geometry with the constraint that $0 < \epsilon < 1$. If the duct is doubly connected such as the annulus or eccentric annulus, the aspect ratio is taken to be the ratio of the maximum duct spacing and the average duct perimeter.

Full Model

A model which is valid over the entire duct length for many different geometries may be obtained by combining the results given above in the form of Eq (21). Using the Churchill-Usagi[12] asymptotic correlation method, the new model is obtained as

$$f_{app}Re_{\sqrt{A}} = \left[\left\{\frac{3.44}{\sqrt{z^+}}\right\}^2 + \left\{8\sqrt{\pi}\left(\frac{\pi}{4}\frac{(1+\epsilon^2)}{\sqrt{\epsilon}\,\mathbf{E}(\sqrt{1-\epsilon^2})}\right)\right\}^2\right]^{1/2} \quad (28)$$

where the characteristic length for $f_{app}Re$, fRe, and z^+ is now $\mathcal{L} = \sqrt{A}$ rather than the hydraulic diameter.

The correlation parameter n may be chosen such that the RMS differences between the predicted results and the numerical or analytical results is minimized. Table 2 compares the percent difference and the RMS difference of the proposed model with the models of Shah[10] and Yilmaz[11] for a number of geometries. Also presented in Table 2 is the optimal value of the parameter n which minimizes the RMS difference for each geometry. It is apparent that choosing a single value of $n = 2$ does not introduce large errors and simplifies the model considerably.

One notable feature of the new model is that it does not contain the incremental pressure drop term K_∞ which appears in the models of Bender[8], Shah[10], and Yilmaz[11]. Since the solution of Shapiro et al.[9] for the entrance region accounts for both the wall shear and the increase in momentum due to the accelerating core, there is no need to introduce the term K_∞. Thus the proposed model is now only a function of duct length z^+ and aspect ratio ϵ, whereas the models of Shah[10] and Yilmaz[11] are functions of many more parameters.

RESULTS

Comparisons between numerical data and the new model for the geometries in Table 2 are presented in Figs. 6-11 for a range of z^+. With the exception of the eccentric annular duct at large values of r^* and e^*, the proposed model predicts all of the developing flow data available in the literature to within ±12 percent. The proposed model provides equal or better accuracy than the model of Yilmaz[11] and is also much simpler. A comparison of the model with the data for the parallel plate channel is also provided. For this geometry $\sqrt{A} \rightarrow \infty$, however, this geometry may be accurately modeled as a rectangular duct with $\epsilon = 0.01$ or a circular annular duct with $r^* = 0.05$. Good agreement is obtained with the current model when the parallel plate channel is modeled as a finite area duct with small aspect ratio.

SUMMARY

A simple model was developed for predicting the apparent friction factor Reynolds number product in noncircular ducts for developing laminar flow. The new model only requires two parameters, the aspect ratio of the duct and the dimensionless duct length. Whereas the model of Shah[10] requires tabulated values of three parameters, and the model of Yilmaz[11] consists of several equations. The new model predicts most of the developing flow data within ± 12 percent for 8 singly connected ducts and 2 doubly connected ducts. The new model may also be used to predict results for ducts which no solutions or tabulated data exist. It was also shown that the square root of the cross-sectional flow area was a more effective characteristic length scale than the hydraulic diameter for collapsing the numerical results of geometries having similar shape and aspect ratio.

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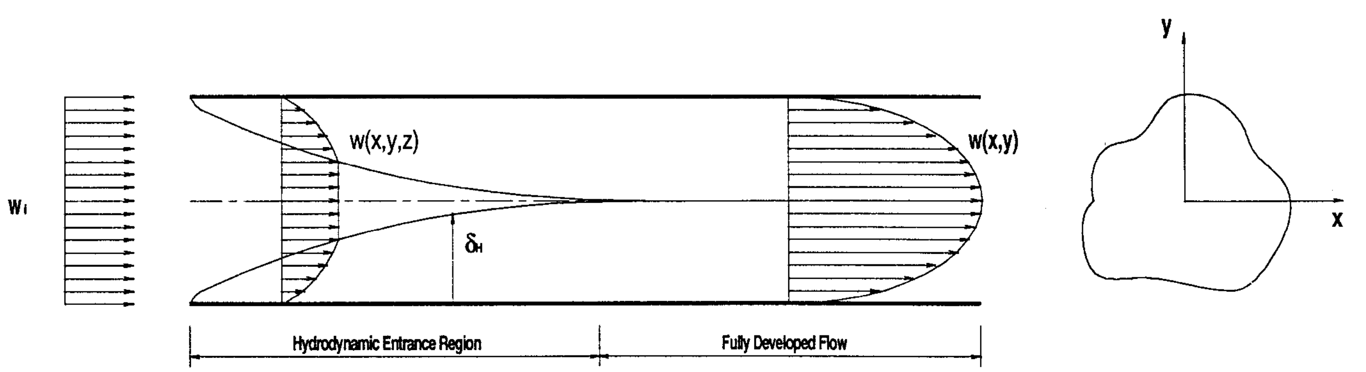

Fig. 1 Hydrodynamic entrance problem.

Table 2
Comparison of RMS and Percent Differences* in Developing Flow Models

Ref.	Geometry	Shah (1978) min/max	Yilmaz (1990) min/max	Proposed Model min/max	Proposed Model RMS	Proposed Model $n^{\dagger}$	($n = 2$) RMS
24	Circle	± 1.9	-0.3/2.7	-2.73/1.05	1.20	2.01	1.71
24	Circular Annulus $r_i/r_o = 0.05$	± 2.0	-17.0/1.2	-2.61/1.18	1.27	2.18	2.51
24	Circular Annulus $r_i/r_o = 0.10$	± 1.9	-17.4/1.9	-1.97/1.09	0.77	2.14	1.73
24	Circular Annulus $r_i/r_o = 0.50$	± 2.2	-10.2/0.9	-1.13/6.96	1.87	2.07	1.99
24	Circular Annulus $r_i/r_o = 0.75$	± 2.1	-5.4/1.3	-1.44/7.17	2.12	2.04	2.16
25	Square $b/a = 1$	± 2.3	-2.4/1.6	-1.48/2.27	1.41	1.95	1.61
25	Rectangle $b/a = 0.5$	± 1.9	-2.1/6.7	-1.48/2.27	1.14	1.98	1.16
25	Rectangle $b/a = 0.2$	± 1.7	-1.5/5.0	-1.11/1.89	1.10	2.15	2.04
26	Parallel Plates $b/a \to 0$	± 2.4	-1.6/1.8	-1.22/0.86	0.60	2.32	3.33
27	Isosceles Triangle $2\phi = 30°$	-	-1.1/0.9	-1.30/4.41	1.75	1.71	4.91
28	Isosceles Triangle $2\phi = 60°$	± 2.4	-0.6/1.1	-0.63/5.97	2.35	1.70	5.16
28	Isosceles Triangle $2\phi = 90°$	-	1.6/5.2	-7.28/0.85	2.04	2.03	2.08
29	Eccentric Annulus $e^* = 0.5$, $r^* = 0.5$	-	-5.5/3.0	-2.06/1.97	1.71	1.50	8.72
29	Eccentric Annulus $e^* = 0.5$, $r^* = 0.1$	-	-9.1/16	-2.22/2.29	1.66	1.66	5.08
29	Eccentric Annulus $e^* = 0.7$, $r^* = 0.3$	-	-10.3/3.1	-10.89/8.39	7.56	1.86	7.72
29	Eccentric Annulus $e^* = 0.9$, $r^* = 0.1$	-	-11.4/0.2	-13.96/5.87	5.94	2.38	7.30
29	Eccentric Annulus $e^* = 0.9$, $r^* = 0.5$	-	-9.9/-3.5	-35.13/9.44	13.34	3.61	18.14
5	Ellipse $b/a = 1$	-	-	-2.97/3.75	2.53	1.96	2.56
5	Ellipse $b/a = 0.5$	-	-	-2.98/5.77	3.77	1.97	3.85
5	Ellipse $b/a = 0.2$	-	-	-5.75/7.59	5.62	1.69	7.01
30	Circular Sector $2\phi = 11.25°$	-	-	-9.42/3.03	3.67	2.01	3.68
30	Circular Sector $2\phi = 22.5°$	-	-	-1.70/5.10	1.57	1.73	4.22
30	Circular Sector $2\phi = 45°$	-	-	-1.14/12.1	3.73	1.62	6.69
30	Circular Sector $2\phi = 90°$	-	-	-1.60/16.72	4.63	1.75	5.69
31	Pentagon	-	-	-3.46/12.75	5.79	1.76	6.41
31	Trapezoid $\phi = 72$, $b/a = 1.123$	-	-	-4.96/11.35	6.05	1.55	8.68

* %diff= $(Analytical - Predicted)/(Analytical) \times 100$

† Optimal value

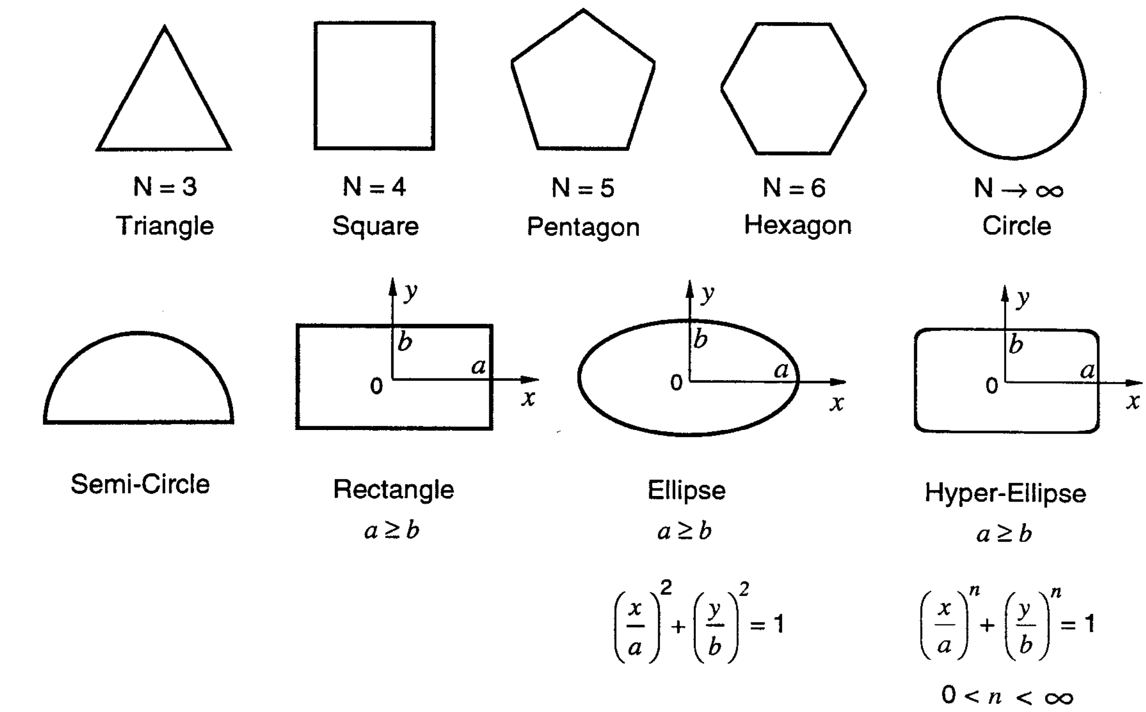

Fig. 2 Common singly-connected geometries.

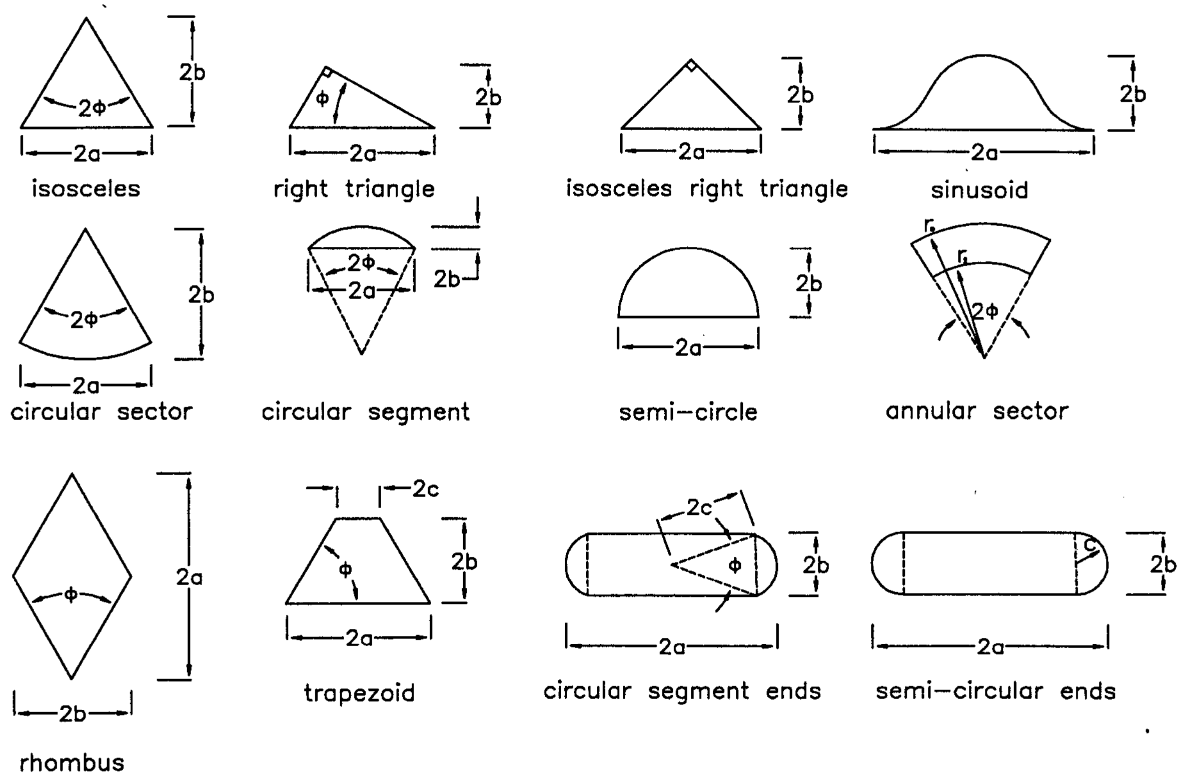

Fig. 3 - Other singly connected geometries.

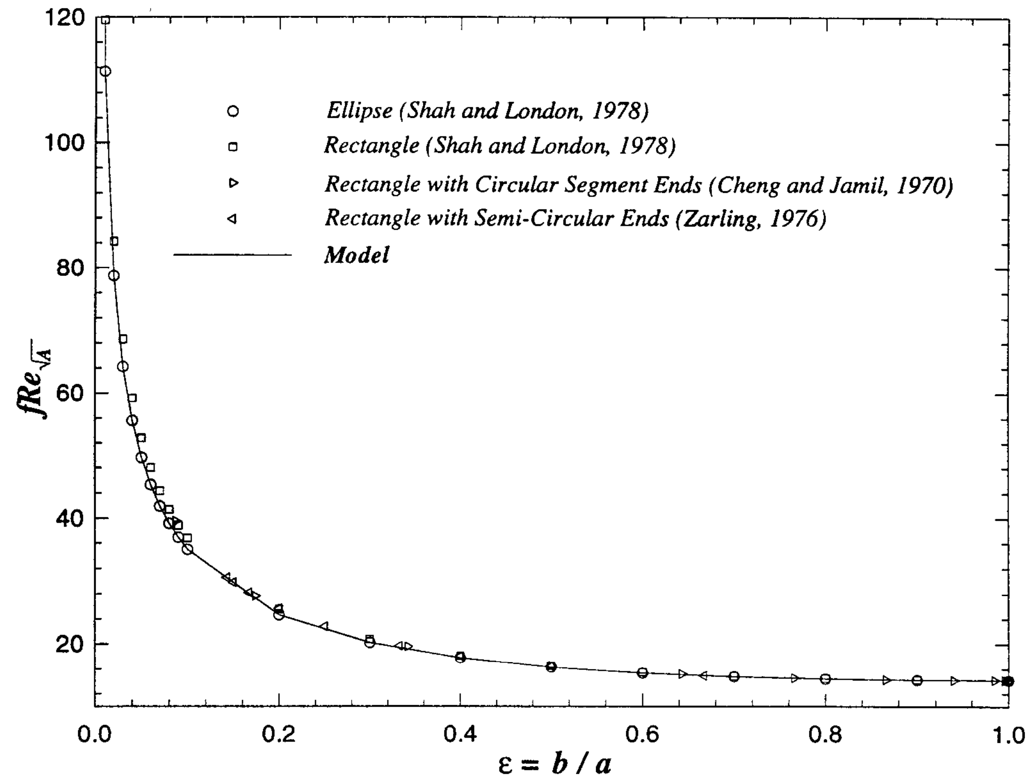

Fig. 4 - $fRe_{\sqrt{A}}$ for Various Flat Ducts

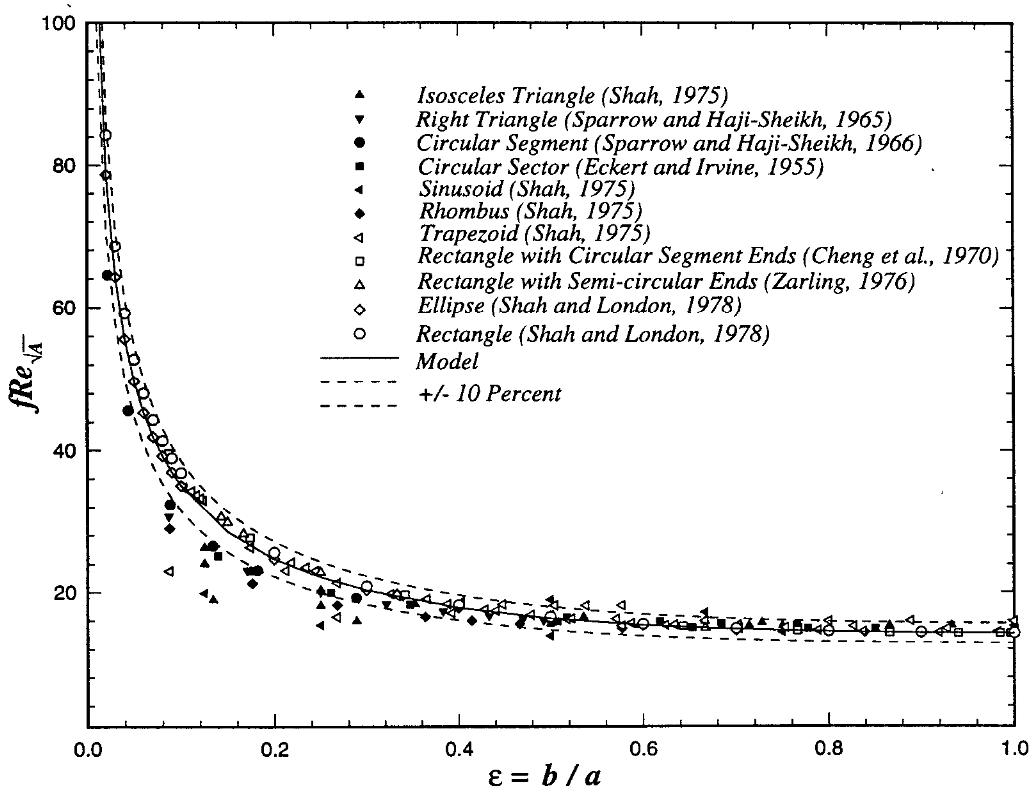

Fig. 5 - $fRe_{\sqrt{A}}$ for Many Common Ducts

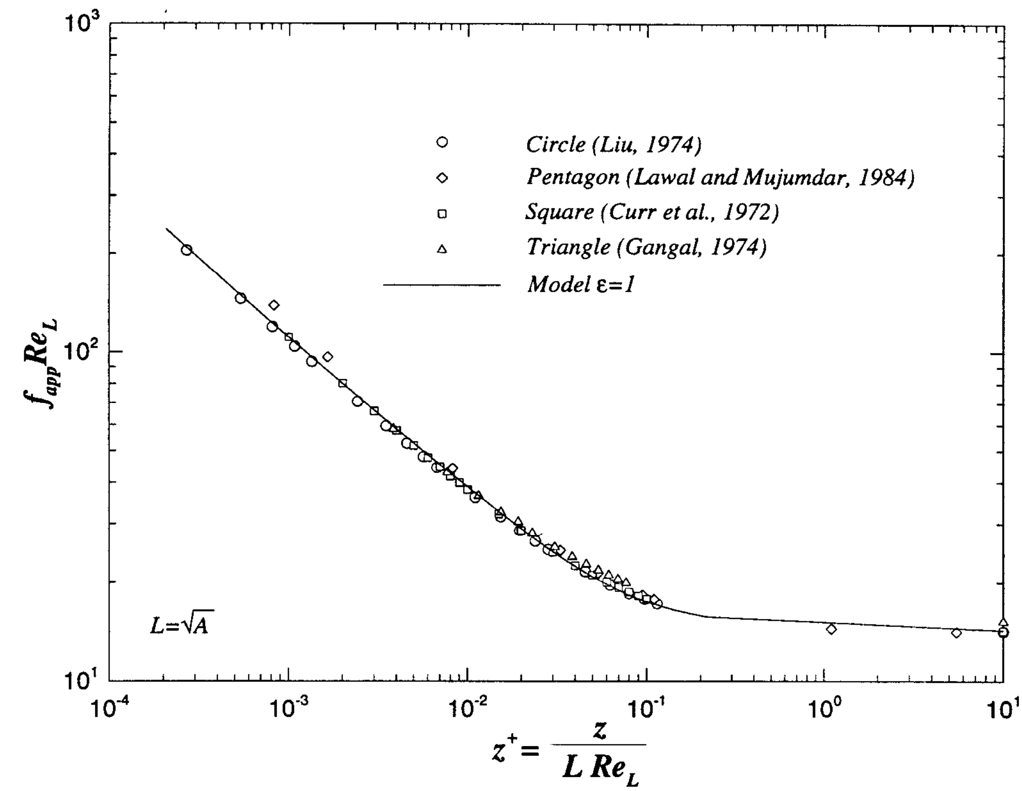

Fig. 6 - $f_{app}Re_{\sqrt{A}}$ for Polygonal Ducts

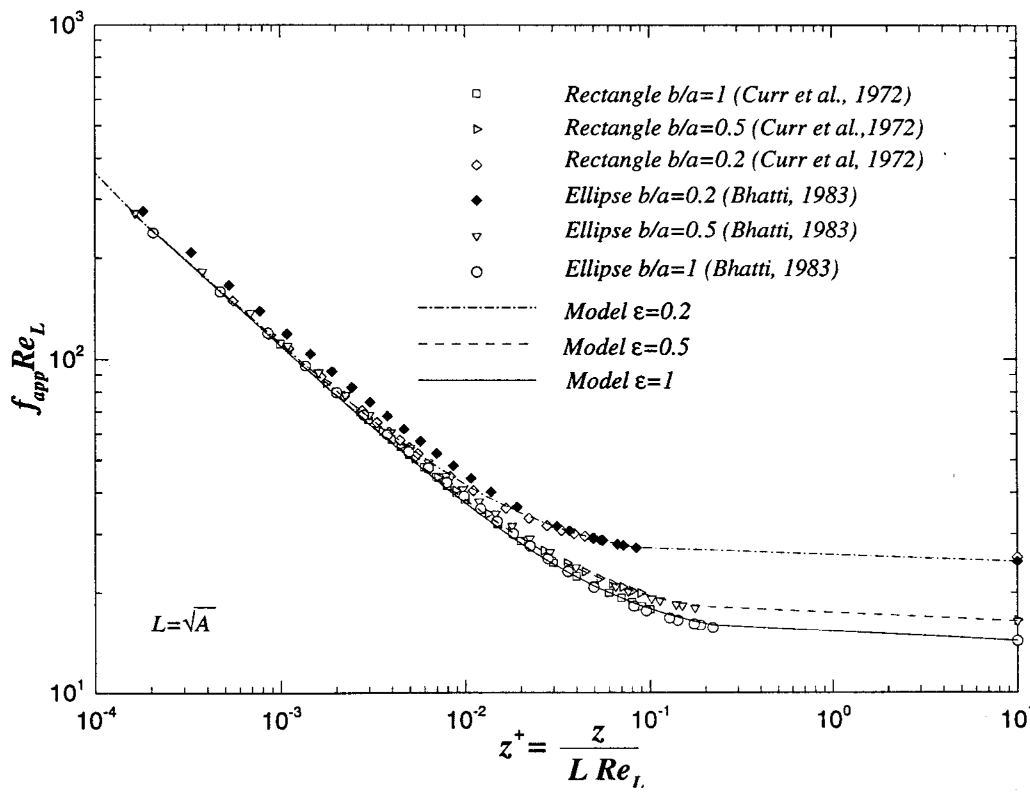

Fig. 7 - $f_{app}Re_{\sqrt{A}}$ for Flat Ducts

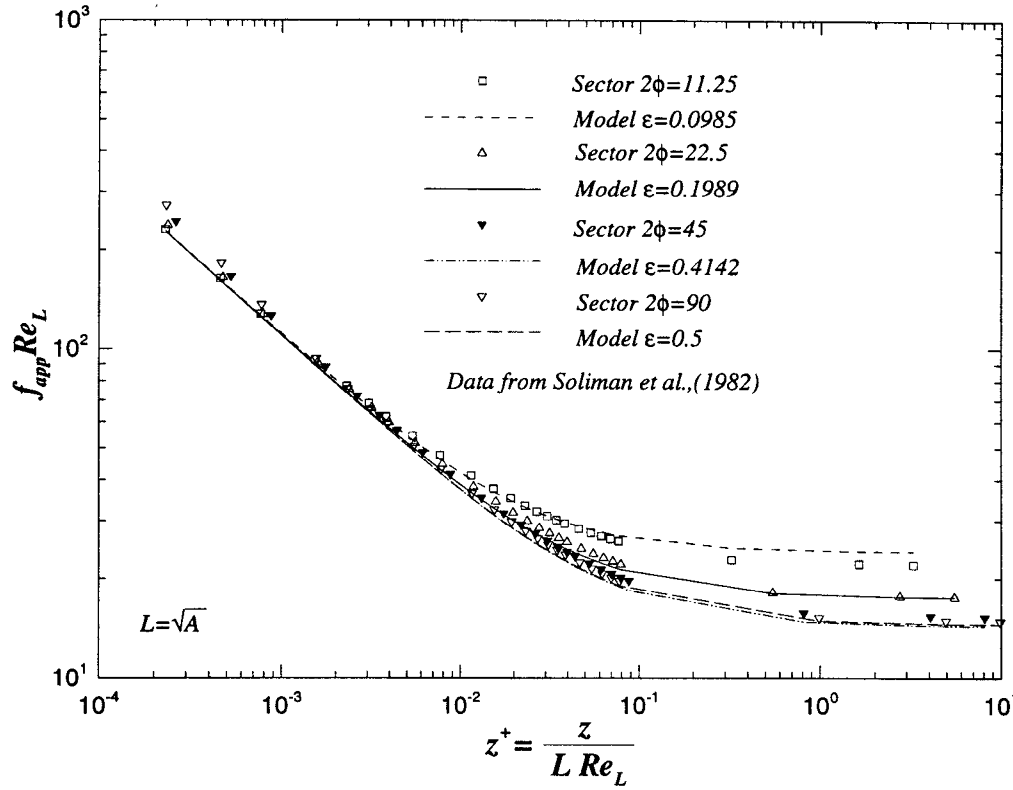

Fig. 8 - $f_{app}Re_{\sqrt{A}}$ for Circular Sector Ducts

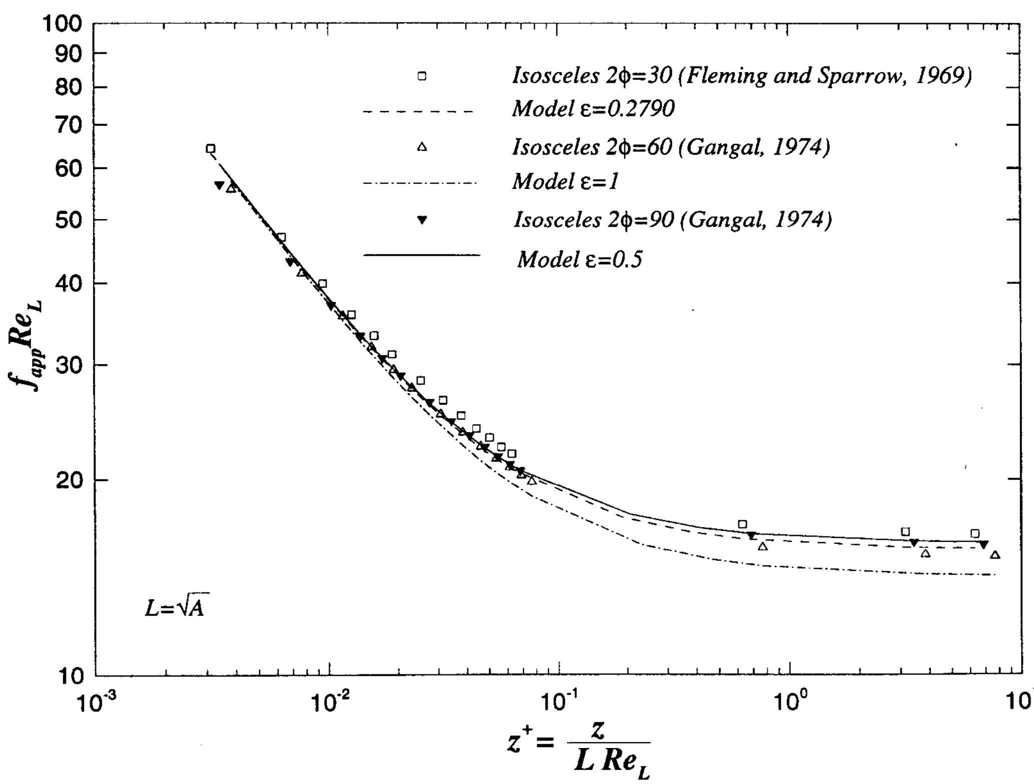

Fig. 9 - $f_{app}Re_{\sqrt{A}}$ for Isosceles Triangle Ducts

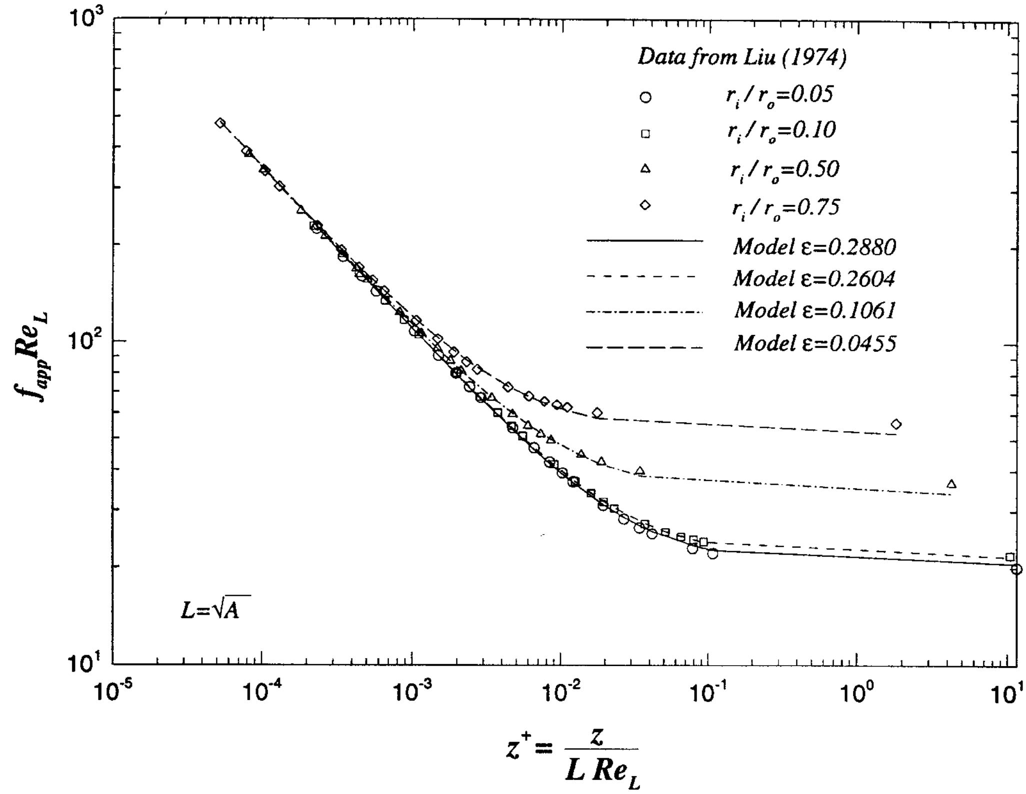

Fig. 10 - $f_{app}Re_{\sqrt{A}}$ for Circular Annular Ducts

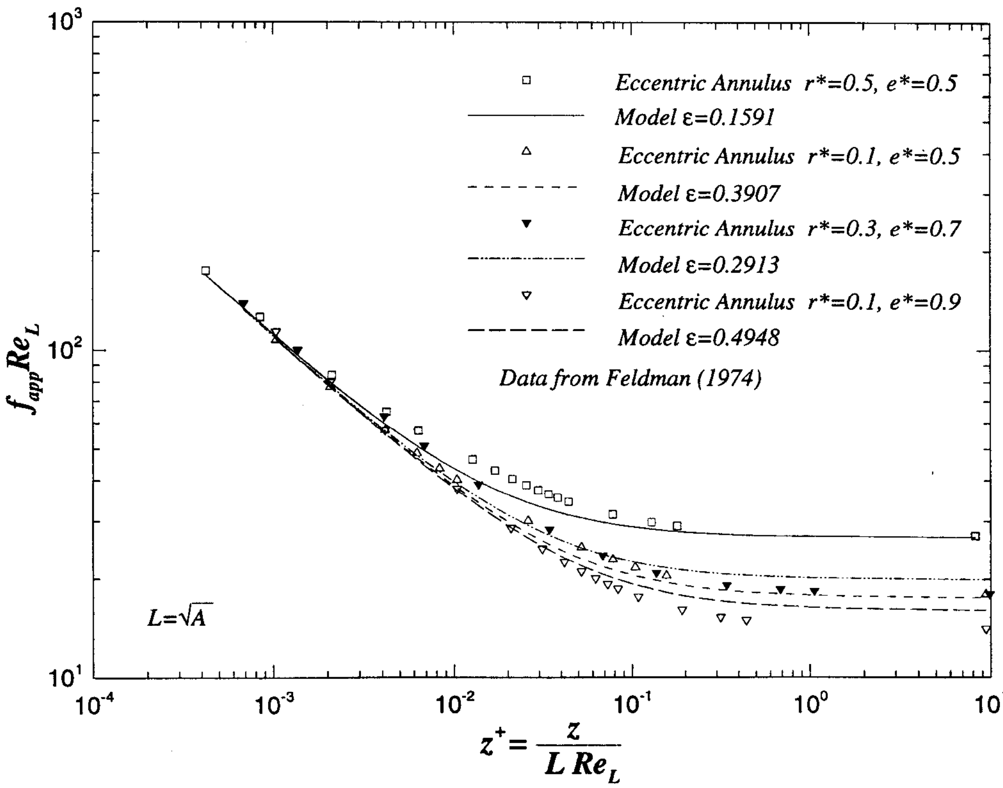

Fig. 11 - $f_{app}Re_{\sqrt{A}}$ for Eccentric Annular Ducts