JOURNAL OF THERMOPHYSICS AND HEAT TRANSFER
Vol. 13, No. 4, October–December 1999

# Spreading Resistance of Isoflux Rectangles and Strips on Compound Flux Channels

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**The general expression for the spreading resistance of an isoflux, rectangular heat source on a two-layer rectangular flux channel with convective or conductive cooling at one boundary is presented. The general expression depends on several dimensionless geometric and thermal parameters. Expressions are given for some two- and three-dimensional spreading resistances for two-layer and isotropic finite and semi-infinite systems. The effect of heat flux distribution over strip sources on two-dimensional spreading resistances is discussed. Tabulated values are presented for three flux distributions, the true isothermal strip, and a related nonisoflux, nonisothermal problem. For narrow strips, the effect of the flux distribution becomes relatively small. The dimensionless spreading resistance for an isoflux square source on an isotropic square flux tube is discussed, and a correlation equation is reported. The closed-form expression for the dimensionless spreading resistance for an isoflux rectangular source on an isotropic half-space is given.**

## Nomenclature

| | |
|---|---|
| $A$ | = channel conduction area, m$^2$ |
| $A_s$ | = heat source area, m$^2$ |
| $\sqrt{A}$ | = characteristic length of contact area, m |
| $a, b$ | = half-lengths of source area, m |
| $Bi$ | = Biot number, $h\mathcal{L}/k_1$ |
| $c, d$ | = half-lengths of flux channel, m |
| $h$ | = contact conductance or film coefficient, W/m$^2\cdot$K |
| $i$ | = index denoting layer 1 and layer 2 |
| $J_\nu(x)$ | = Bessel function of first kind, order $\nu$ |
| $k, k_1, k_2$ | = thermal conductivities, W/m $\cdot$ K |
| $\mathcal{L}$ | = arbitrary length scale, m |
| $m, n$ | = indices for summations |
| $Q$ | = heat flow rate, W |
| $q$ | = heat flux, W/m$^2$ |
| $R$ | = thermal resistance, K/W |
| $R_s$ | = spreading resistance, K/W |
| $R_{\text{total}}$ | = total resistance, K/W |
| $R_{1\text{D}}$ | = one-dimensional resistance, K/W |
| $\bar{T}_{\text{sink}}$ | = mean sink temperature, K |
| $\bar{T}_{\text{source}}$ | = mean source temperature, K |
| $T_1, T_2$ | = layer temperatures, K |
| $t, t_1, t_2$ | = total and layer thicknesses, m |
| $u$ | = relative local position in strip, $x/c$ |
| $x, y, z$ | = Cartesian coordinates, m |
| $\alpha$ | = conductivity parameter, $(1-\kappa)/(1+\kappa)$ |
| $\beta$ | = eigenvalues, $\sqrt{(\delta^2+\lambda^2)}$ |
| $\Gamma$ | = gamma function |
| $\delta$ | = eigenvalues, $(m\pi/c)$ |
| $\epsilon, \epsilon_1, \epsilon_2$ | = relative contact size; $\epsilon_1 \equiv a/c$ and $\epsilon_2 \equiv b/d$ |
| $\zeta$ | = eigenvalue |
| $\kappa$ | = relative conductivity, $k_2/k_1$ |
| $\lambda$ | = eigenvalues, $(n\pi/d)$ |
| $\mu$ | = heat flux shape parameter, $-\frac{1}{2}, 0, \frac{1}{2}$ |
| $\varrho$ | = aspect ratio of rectangular source area, $a/b \geq 1$ |
| $\tau, \tau_1, \tau_2$ | = relative layer thickness, $t/\mathcal{L}$, $t_1/\mathcal{L}$, and $t_2/\mathcal{L}$, respectively |
| $\phi_m, \phi_n$ | = two-dimensional spreading functions |
| $\phi_{m,n}$ | = three-dimensional spreading function |
| $\psi$ | = dimensionless spreading resistance, $\equiv R_s k_1 \mathcal{L}$ |

Presented as Paper 98-0873 at the 36th Aerospace Sciences Meeting, Reno, NV, 12–15 January 1998; received 29 May 1998; revision received 7 June 1999; accepted for publication 15 June 1999. 

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## Introduction

THERMAL spreading resistance occurs whenever heat leaves a heat source of finite dimensions and enters into a larger region, as shown in Fig. 1. Figure 1 shows a planar rectangular heat source situated on one end of a compound heat flux channel that consists of two layers having thicknesses $t_1$ and $t_2$ and thermal conductivities $k_1$ and $k_2$, respectively. The heat flux channel is cooled along the bottom surface through a uniform film coefficient or a uniform contact conductance $h$. The heat source area can be rectangular having dimensions $2a$ by $2b$ or it may be a strip of width $2a$, when $2b = 2d$. The dimensions of the heat flux channel are $2c$ by $2d$, as shown in Fig. 1. The lateral boundaries of the heat flux channel are adiabatic.

The heat flow rate through the heat flux channel $Q$ is related to the mean temperature of the heat source $\bar{T}_{\text{source}}$, the mean heat sink temperature $\bar{T}_{\text{sink}}$, and the total system thermal resistance $R_{\text{total}}$ through the relationship

$$Q R_{\text{total}} = \bar{T}_{\text{source}} - \bar{T}_{\text{sink}} \quad (1)$$

The total thermal resistance of the system is defined by the relation

$$R_{\text{total}} = R_s + R_{1\text{D}} \quad (2)$$

where $R_s$ is the thermal spreading resistance of the system and $R_{1\text{D}}$ is the one-dimensional thermal resistance defined as

$$R_{1\text{D}} = t_1/k_1 A + t_2/k_2 A + 1/hA \quad (3)$$

The conduction area in Eq. (3) is $A = 4cd$. For an isoflux source area, the heat flow rate through the system is $Q = qA_s$, where $q$ is the uniform heat flux and $A_s = 4ab$ is the heat source area.

For the general case of a rectangular source area on a rectangular heat flux channel, as shown in Fig. 1, the spreading resistance will depend on several geometric and thermal parameters such as

$$R_s = f(a, b, c, d, t_1, t_2, k_1, k_2, h) \quad (4)$$

This paper has three objectives. One is to obtain a general solution for the system shown in Fig. 1. Second is to report several two- and three-dimensional cases that arise from the general solution. Third is to report several previously unpublished results that examine the

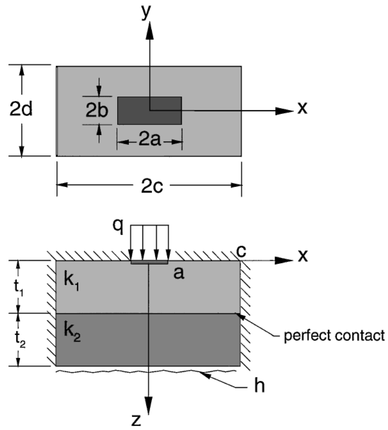


**Fig. 1 Finite compound channel with rectangular heat source.**

effect of heat flux distribution for the two-dimensional strip heat source.

Several investigators[1–7] have examined spreading resistance in rectangular isotropic two- and three-dimensional systems. Recently, Yovanovich[8] reviewed and summarized past work in the area of spreading resistance for circular and rectangular systems. As a result, several solutions of practical interest in engineering systems are now available. The general solution presented in the next section reduces to several cases that were previously unavailable.[8]

## Problem Statement

The temperature distributions $T_1$ and $T_2$ within the two layers must satisfy the Laplace equation

$$\nabla^2 T_i = 0, \qquad i = 1, 2 \tag{5}$$

where for the rectangular heat source/rectangular flux channel system, the three-dimensional Laplacian operator is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Along the common interface $z = t_1$, the two temperatures must satisfy the perfect contact conditions

$$T_1 = T_2, \qquad k_1 \frac{\partial T_1}{\partial z} = k_2 \frac{\partial T_2}{\partial z} \tag{6}$$

Along the lateral boundaries $x = \pm c$ and $y = \pm d$, the two temperatures must satisfy the adiabatic conditions

$$\frac{\partial T_i}{\partial x} = 0, \qquad \frac{\partial T_i}{\partial y} = 0, \qquad i = 1, 2 \tag{7}$$

Along the bottom surface $z = t_1 + t_2$, the Robin boundary condition must be satisfied:

$$\frac{\partial T_2}{\partial z} = -\frac{h}{k_2}(T_2 - \bar{T}_{\mathrm{sink}}) \tag{8}$$

The parameter $h$ can represent a uniform film coefficient or a uniform contact conductance. Over the top surface $z = 0$, the boundary conditions are 1) the isoflux condition

$$\frac{\partial T_1}{\partial z} = -\frac{q}{k_1}, \qquad -a < x < a, \qquad -b < y < b \tag{9}$$

over the heat source area and 2) the adiabatic condition

$$\frac{\partial T_1}{\partial z} = 0 \tag{10}$$

for all points that lie outside the heat source area.

The separation of variables method was employed to find the solutions for $T_1$ and $T_2$, by assuming solutions of the form $T_i(x, y, z) = X_i(x) \times Y_i(y) \times Z_i(z)$. The computer algebra system MAPLE V (Ref. 9) was used to accomplish all of the required algebraic manipulations to obtain the two temperature distributions. The spreading resistance was obtained by means of the definition proposed by Mikic and Rohsenow[1]:

$$R_s = (\bar{T}_{\mathrm{source}} - \bar{T}_{\mathrm{contact\ plane}})/Q \tag{11}$$

The mean temperature of the heat source area is obtained from

$$\bar{T}_{\mathrm{source}} = \frac{1}{4ab}\int_{-a}^{a}\int_{-b}^{b} T_1(x, y, 0)\,\mathrm{d}x\,\mathrm{d}y \tag{12}$$

and the mean temperature of the contact plane $z = 0$ is obtained from

$$\bar{T}_{\mathrm{contact\ plane}} = \frac{1}{4cd}\int_{-c}^{c}\int_{-d}^{d} T_1(x, y, 0)\,\mathrm{d}x\,\mathrm{d}y \tag{13}$$

## General Spreading Resistance Expression

The methodology just described was used to obtain the solution for the general problem defined earlier. The spreading resistance is obtained from the following general expression that shows the explicit and implicit relationships with the geometric and thermal parameters of the system:

$$\begin{aligned} R_s = {} & \frac{1}{2a^2cdk_1}\sum_{m=1}^{\infty}\frac{\sin^2(a\delta_m)}{\delta_m^3}\cdot\phi(\delta_m) \\ & + \frac{1}{2b^2cdk_1}\sum_{n=1}^{\infty}\frac{\sin^2(b\lambda_n)}{\lambda_n^3}\cdot\phi(\lambda_n) \\ & + \frac{1}{a^2b^2cdk_1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin^2(a\delta_m)\sin^2(b\lambda_n)}{\delta_m^2\lambda_n^2\beta_{m,n}}\cdot\phi(\beta_{m,n}) \end{aligned} \tag{14}$$

The general expression for the spreading resistance consists of three terms. The single summations account for two-dimensional spreading in the $x$ and $y$ directions, respectively, and the double summation term accounts for three-dimensional spreading from the rectangular heat source. Figure 2 shows the superposition of the two strip solutions and the rectangular solution that yield the general expression.

The eigenvalues are $\delta_m = m\pi/c$, $\lambda_n = n\pi/d$, and $\beta_{m,n} = \sqrt{(\delta_m^2 + \lambda_n^2)}$. The eigenvalues $\delta$ and $\lambda$, corresponding to the two strip solutions, depend on the flux channel dimensions and the indices $m$ and $n$, respectively. The eigenvalues $\beta$ for the rectangular solution are functions of the other two eigenvalues.

The contributions of the layer thicknesses $t_1$ and $t_2$, the layer conductivities $k_1$ and $k_2$, and the uniform conductance $h$ to the spreading resistance are determined by means of the general expression

$$\phi(\zeta) = \frac{(\alpha e^{4\zeta t_1} + e^{2\zeta t_1}) + \varphi\left(e^{2\zeta(2t_1 + t_2)} + \alpha e^{2\zeta(t_1 + t_2)}\right)}{(\alpha e^{4\zeta t_1} - e^{2\zeta t_1}) + \varphi\left(e^{2\zeta(2t_1 + t_2)} - \alpha e^{2\zeta(t_1 + t_2)}\right)} \tag{15}$$

where

$$\varphi = \frac{\zeta + Bi/\kappa\mathcal{L}}{\zeta - Bi/\kappa\mathcal{L}}$$

$$\alpha = (1 - \kappa)/(1 + \kappa)$$

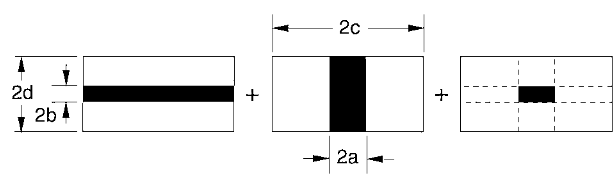


**Fig. 2 Superposition of solutions.**

**Table 1 Summary of solutions for isoflux source**

| Figure | Configuration | Limiting values |
|---|---|---|
| | *Rectangular heat source: three-dimensional solutions* | |
| 1 | Finite compound rectangular flux channel | $a, b, c, d, t_1, t_2, k_1, k_2, h$ |
| 3 | Semi-infinite compound rectangular flux channel | $t_2 \to \infty$ |
| 4 | Finite isotropic rectangular flux channel | $k_1 = k_2$ |
| 5 | Semi-infinite isotropic rectangular flux channel | $t_1 \to \infty$ |
| 6 | Isotropic half-space | $c \to \infty, d \to \infty, t_1 \to \infty$ |
| 7 | Compound half-space | $c \to \infty, d \to \infty, t_2 \to \infty$ |
| | *Strip heat source: two-dimensional solutions* | |
| 8 | Finite compound rectangular flux channel | $a, c, b = d, t_1, t_2, k_1, k_2, h$ |
| 9 | Semi-infinite compound rectangular flux channel | $t_2 \to \infty$ |
| 10 | Finite isotropic rectangular flux channel | $k_1 = k_2$ |
| 11 | Semi-infinite isotropic rectangular flux channel | $t_1 \to \infty$ |

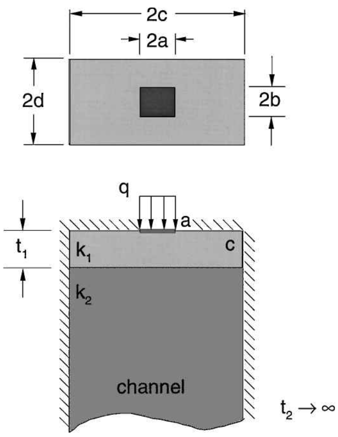

Fig. 3 Semi-infinite coated channel with rectangular heat source.

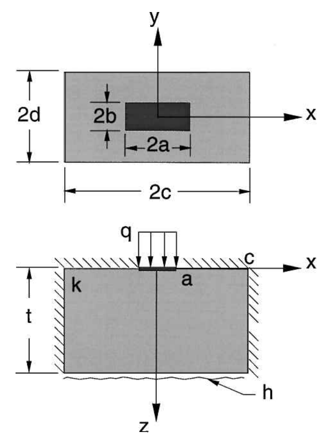

Fig. 4 Finite isotropic channel with rectangular heat source.

with $\kappa = k_2/k_1$ and $Bi = h\mathcal{L}/k_1$, where $\mathcal{L}$ is an arbitrary length scale employed to define the dimensionless spreading resistance:

$$\psi = R_s k_1 \mathcal{L} \tag{16}$$

that is based on the thermal conductivity of the layer adjacent to the heat source. Various system lengths may be used, and the appropriate choice depends on the system of interest. In all summations $\phi(\zeta)$ is evaluated in each series using $\zeta = \delta_m$, $\lambda_n$, and $\beta_{m,n}$.

## Spreading Resistance for Three-Dimensional Systems

The dimensionless spreading resistance $\psi$ depends on six independent dimensionless parameters such as 1) the relative sizes of the rectangular source area ($\epsilon_1 = a/c$ and $\epsilon_2 = b/d$), 2) the layer conductivity ratio ($\kappa = k_2/k_1$), 3) the relative layer thicknesses ($\tau_1 = t_1/\mathcal{L}$ and $\tau_2 = t_2/\mathcal{L}$), and 4) the Biot number ($Bi = h\mathcal{L}/k_1$). Thus, correlation of the general solution or graphical representation of the resistance is not possible. However, the general solution reduces to many special cases, such as those shown in Figs. 3–11. This section examines all of the three-dimensional solutions that may be obtained from the general solution given by Eqs. (14) and (15). All of the special three-dimensional cases are summarized in Table 1.

#### Semi-Infinite Compound Rectangular Flux Channel

The general expression for $\phi(\zeta)$ reduces to a simpler expression when $t_2 \to \infty$ (Fig. 3). The solution for this particular case arises from Eq. (14) with

$$\phi(\zeta) = \frac{(e^{2\zeta t_1} - 1)\kappa + (e^{2\zeta t_1} + 1)}{(e^{2\zeta t_1} + 1)\kappa + (e^{2\zeta t_1} - 1)} \tag{17}$$

where the influence of the contact conductance has vanished.

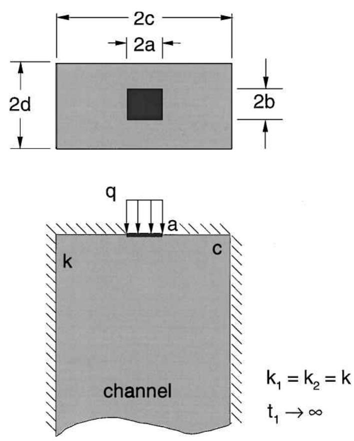

Fig. 5 Semi-infinite isotropic channel with rectangular heat source.

#### Finite Isotropic Rectangular Flux Channel

The general expression for $\phi(\zeta)$ reduces to a simpler expression when $\kappa = 1$ (Fig. 4). The solution for this particular case arises from Eq. (14) with

$$\phi(\zeta) = \frac{(e^{2\zeta t} + 1)\zeta - (1 - e^{2\zeta t})Bi/\mathcal{L}}{(e^{2\zeta t} - 1)\zeta + (1 + e^{2\zeta t})Bi/\mathcal{L}} \tag{18}$$

where the influence of $\kappa$ has vanished.

#### Semi-Infinite Isotropic Rectangular Flux Channel

When the relative thickness $\tau$ is sufficiently large, $\phi \to 1$, for the three basic solutions of Eq. (14), then $\psi = \psi(\epsilon_1, \epsilon_2)$ is independent of $\tau$ and $Bi$. This corresponds to the case of a rectangular heat source on a semi-infinite rectangular flux channel (Fig. 5). This solution was first reported by Mikic and Rohsenow.[1]

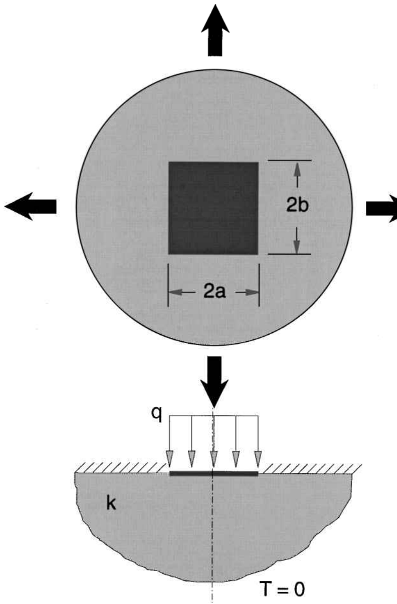


Fig. 6 Isotropic half-space with rectangular heat source.

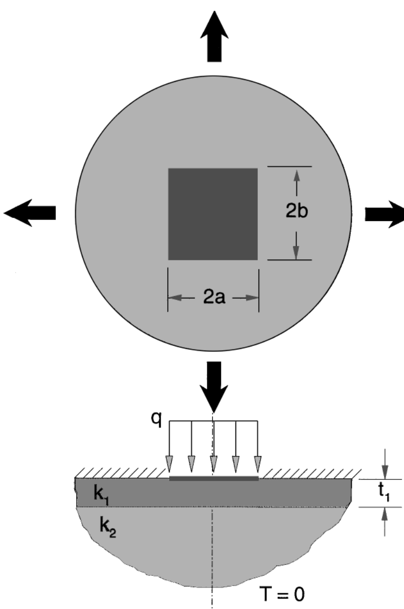


Fig. 7 Compound half-space with rectangular heat source.

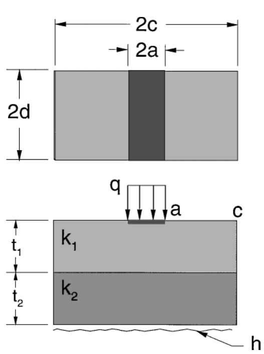


Fig. 8 Finite compound channel with strip heat source.

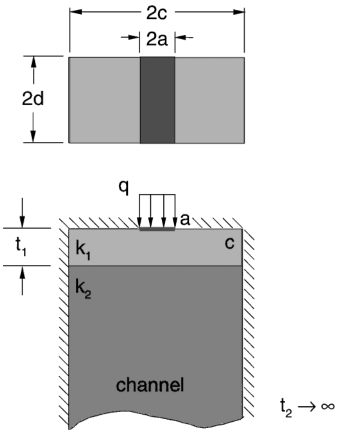


Fig. 9 Semi-infinite compound channel with strip heat source.

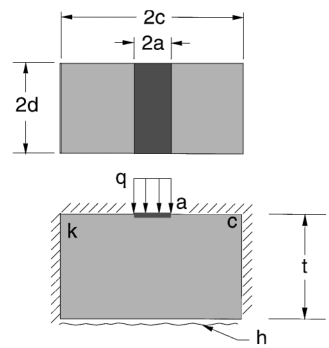


Fig. 10 Finite isotropic channel with strip heat source.

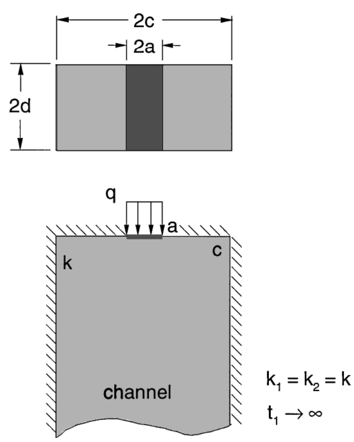


Fig. 11 Semi-infinite isotropic channel with strip heat source.

The general solution may also be used to obtain the solution for an isoflux square area on the end of a square semi-infinite flux tube.[5] For the special case of a square heat source on a semi-infinite square, isotropic flux tube, the general solution reduces to a simpler expression that depends on one parameter only. The solution[1] was recast into the form[5]

$$k\sqrt{A_s}R_s = \frac{2}{\pi^3\epsilon}\left[\sum_{m=1}^{\infty}\frac{\sin^2(m\pi\epsilon)}{m^3} + \frac{1}{\pi^2\epsilon^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin^2(m\pi\epsilon)\sin^2(n\pi\epsilon)}{m^2n^2\sqrt{m^2+n^2}}\right] \tag{19}$$

where the characteristic length was selected as $\mathcal{L} = \sqrt{A_s}$. The relative size of the heat source was defined as $\epsilon = \sqrt{(A_s/A_c)}$, where

$A_c$ is the flux tube area. A correlation equation was reported for Eq. (19):

$$k\sqrt{A_s}R_s = 0.47320 - 0.62075\epsilon + 0.1198\epsilon^3 \tag{20}$$

in the range $0 \le \epsilon \le 0.5$, with a maximum relative error of approximately 0.3%. The constant on the right-hand side of the correlation equation is the value of the dimensionless spreading resistance of an isoflux square source on an isotropic half-space when the square root of the source area is chosen as the characteristic length.

### Isoflux Rectangular Heat Sources on a Half-Space

The spreading resistance for an isoflux rectangular source of dimensions $2a \times 2b$ on an isotropic half-space (Fig. 6) whose thermal conductivity is $k$ has a closed-form solution[6]:

$$k\sqrt{A_s}R_s = \frac{\sqrt{\varrho}}{\pi}\left\{\sinh^{-1}\left(\frac{1}{\varrho}\right) + \frac{1}{\varrho}\sinh^{-1}\varrho + \frac{\varrho}{3}\left[1 + \frac{1}{\varrho^3} - \left(1 + \frac{1}{\varrho^2}\right)^{\frac{3}{2}}\right]\right\} \tag{21}$$

where $\varrho = a/b \ge 1$ is the aspect ratio of the rectangle. When the scale length is $\mathcal{L} = \sqrt{A_s}$, the dimensionless spreading resistance becomes a weak function of $\varrho$. For a square heat source, the numerical value of the dimensionless spreading resistance is $k\sqrt{A_s}\,R_s = 0.4732$, which is very close to the numerical value for the isoflux circular source on an isotropic half-space and other singly connected heat source geometries, such as an equilateral triangle and a semicircular heat source.

The solution for the rectangular heat source on a compound half space (Fig. 7) can be obtained from the general solution for the finite compound flux channel, provided that $t_2 \to \infty$, $c \to \infty$, and $d \to \infty$. No closed-form solution such as that given by Eq. (21) exists. The size of the computational domain may be determined by comparing the series solution with the closed-form solution [Eq. (21)] using the dimensions of the source and the conductivity of the more conductive material to determine the approximate outer dimensions, which have little influence on the isotropic result.

## Spreading Resistance for Two-Dimensional Systems

Several two-dimensional solutions may be obtained from the general solution presented earlier. Four special cases that are summarized in Table 1 are discussed next.

### Finite Compound Rectangular Flux Channel

When the rectangular heat source on the system shown in Fig. 1 has dimensions such that $2b = 2d$, the system shown in Fig. 8 results. The solution may be written as

$$R_s = \frac{1}{a^2 c k_1}\sum_{m=1}^{\infty}\frac{\sin^2(a\delta_m)}{\delta_m^3}\cdot\phi(\delta_m) \tag{22}$$

where $\phi$ is given by Eq. (15).

### Semi-Infinite Compound Rectangular Flux Channel

The solution for $R_s$ when $t_2 \to \infty$ (Fig. 9) is obtained from Eq. (22) with $\phi$ defined by Eq. (17).

### Finite Isotropic Rectangular Flux Channel

The solution for $R_s$ when $\kappa = 1$ (Fig. 10) is obtained from Eq. (22) with $\phi$ defined by Eq. (18). For this system the appropriate scale length may be chosen to be $\mathcal{L} = c$, the half-width of the flux channel. The general solution may then be written in an alternative form:

$$kR_s = \frac{1}{\pi^3\epsilon^2}\sum_{n=1}^{\infty}\frac{\sin^2(n\pi\epsilon)}{n^3}\left[\frac{n\pi + Bi\tanh(n\pi\tau)}{n\pi\tanh(n\pi\tau) + Bi}\right] \tag{23}$$

with $\epsilon = a/c$, $\tau = t/c$, and $Bi = hc/k$.

### Semi-Infinite Isotropic Rectangular Flux Channel

When the relative thickness exceeds the critical value $\tau > 2.65/\pi$, the earlier result reduces to the result for the case shown in Fig. 11, an isoflux strip on an isotropic, semi-infinite flux channel for which the spreading resistance is obtained from the expression[1]

$$kR_s = \frac{1}{\pi^3\epsilon^2}\sum_{n=1}^{\infty}\frac{\sin^2(n\pi\epsilon)}{n^3} \tag{24}$$

that depends on the relative strip size only.

### Effect of Heat Source Flux Distribution

The effect of the heat flux distribution on a semi-infinite isotropic strip source was examined by Yovanovich.[7] Flux distributions are of the form $f(u) = (1 - u^2)^{\mu}$, where $u = x/a$ is the arbitrary relative position in the strip source and the flux shape parameter is $\mu$. Yovanovich[7] reported the general result

$$kR_s = \frac{1}{\pi^2}\Gamma\left(\mu + \frac{3}{2}\right)\frac{1}{\epsilon}\sum_{n=1}^{\infty}\frac{\sin(n\pi\epsilon)}{n^2}\left[\frac{2}{n\pi\epsilon}\right]^{\mu+\frac{1}{2}}J_{\mu+\frac{1}{2}}(n\pi\epsilon) \tag{25}$$

where $J_{\mu+1/2}$ is the Bessel function of the first kind of order $\mu + \frac{1}{2}$. By means of the general expression, Yovanovich[7] obtained results for three flux distributions: 1) equivalent isothermal flux distribution, when $\mu = -\frac{1}{2}$, 2) isoflux strip, when $\mu = 0$, and 3) parabolic flux distribution, when $\mu = \frac{1}{2}$. The general expression with $\mu = -\frac{1}{2}$ for the equivalent isothermal flux distribution reduces to the previously reported result[1]

$$kR_s = \frac{1}{\pi^2\epsilon}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(n\pi\epsilon)J_0(n\pi\epsilon) \tag{26}$$

This expression can be compared against the true isothermal closed-form expression[2,3]

$$kR_s = (1/\pi)\ln(\{\sin[(\pi/2)\epsilon]\}^{-1}) \tag{27}$$

For $\epsilon < 0.2$, the earlier result approaches the asymptote $kR_s = \pi^{-1}\ln(2/\pi\epsilon)$.

The parabolic flux distribution result[7] was obtained by setting $\mu = \frac{1}{2}$:

$$kR_s = \frac{2}{\pi^3\epsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin(n\pi\epsilon)J_1(n\pi\epsilon) \tag{28}$$

Numerical values of $\psi = kR_s$ are given in Table 2 for the flux distributions defined by the flux distribution parameter $\mu = -\frac{1}{2}$, 0, $\frac{1}{2}$

**Table 2 Numerical values of $\psi$ for $\mu = -\frac{1}{2}$, 0, and $\frac{1}{2}$**

| Parameter | $\epsilon$ | 0.02 | 0.04 | 0.06 | 0.08 | 0.10 | 0.20 | 0.40 | 0.60 | 0.80 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mu = -\frac{1}{2}$ | Eq. (26) | 1.1011 | 0.8808 | 0.7518 | 0.6609 | 0.5902 | 0.3729 | 0.1658 | 0.0607 | 0.0067 |
| $\mu = 0$ | Eq. (24) | 1.1377 | 0.9172 | 0.7883 | 0.6970 | 0.6263 | 0.4083 | 0.1984 | 0.0882 | 0.0255 |
| $\mu = \frac{1}{2}$ | Eq. (28) | 1.1545 | 0.9340 | 0.8051 | 0.7138 | 0.6430 | 0.4247 | 0.2134 | 0.1007 | 0.0338 |
| $T$ = const | Eq. (27) | 1.1015 | 0.8811 | 0.7523 | 0.6611 | 0.5905 | 0.3738 | 0.1691 | 0.0675 | 0.0160 |

**Table 3 Typical numerical values of Eq. (29) and the average of Eqs. (24) and (26)**

| $\epsilon$ | 0.02 | 0.20 | 0.40 | 0.60 | 0.80 |
|---|---|---|---|---|---|
| Eq. (29) | 1.122 | 0.3936 | 0.1860 | 0.0794 | 0.0214 |
| [Eq. (24) + Eq. (26)]/2 | 1.120 | 0.3911 | 0.1838 | 0.0779 | 0.0208 |
| % Difference | 0.24 | 0.65 | 1.21 | 1.95 | 3.04 |

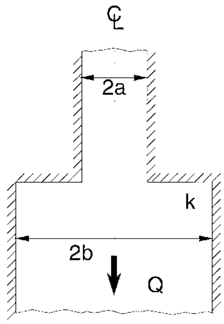


**Fig. 12 Infinite channel with abrupt change in channel width.**

and the true isothermal result for a range of the relative strip size parameter $\epsilon$.

For completeness, the analytical, closed-form result for the flux channel shown in Fig. 12 is reported. In this case the flux channel is isotropic, the cross section changes abruptly from a width of $2a$ to a width of $2b$. The boundary condition over the interface between the upper and lower parts is not known. For the general case, $\epsilon = a/b < 1$, the boundary condition is neither isothermal nor isoflux. The true condition is an unknown variable temperature distribution and an unknown variable flux distribution. When $\epsilon = 1$, the temperature and flux distributions are known; however, the spreading resistance is not present. The spreading resistance can be obtained by means of the closed-form result[4]

$$kR_s = \frac{1}{2\pi}\left[\left(\epsilon + \frac{1}{\epsilon}\right)\ln\left[\frac{1+\epsilon}{1-\epsilon}\right] + 2\ln\left[\frac{1-\epsilon^2}{4\epsilon}\right]\right] \tag{29}$$

We observe that the numerical values for the equivalent isothermal flux distribution [Eq. (26)] and the true isothermal [Eq. (27)] approach each other as $\epsilon \to 0$; however, there are large differences in the numerical values for $\epsilon > 0.6$. The numerical values for the parabolic distribution are greater than the isoflux values, which are greater than the values for the isothermal strip. For very narrow strips, $\epsilon < 0.02$, the maximum difference between the highest values corresponding to $\mu = \frac{1}{2}$ and the lowest values corresponding to $\mu = -\frac{1}{2}$ differ by less than 5%. This implies that the spreading resistance for very narrow strips depends weakly on the heat flux distribution.

In Table 3 the numerical values obtained from Eq. (29) are compared against the mean values of Eqs. (24) and (26) for a range of the relative strip sizes. The differences are less than 1% for $\epsilon \leq 0.20$, and the differences become negligible for $\epsilon \to 0$.

## Conclusion

A general expression for the spreading resistance of an isoflux rectangular source on the surface of a finite compound rectangular flux channel is presented. The series solution consists of three summations that correspond to two strip solutions and a rectangle solution. In general, the dimensionless spreading resistance depends on several dimensionless geometric and thermal parameters.

Results are presented for isotropic finite and semi-infinite rectangular flux channels for the strip source. Results are also presented for the isoflux rectangular and square source areas on an isotropic half-space. A correlation equation is reported for the three-dimensional spreading resistance for an isoflux square source on an isotropic semi-infinite square flux tube.

Expressions that show the effect of heat flux distribution over the strip source area are presented. Tabulated values of the dimensionless spreading resistance for various flux distributions are also given.

## Acknowledgment

The first author acknowledges the continued financial support of the Natural Sciences and Engineering Research Council of Canada.

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