Fluid Flow and Heat Transfer From Elliptical Cylinders: Analytical Approach

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\[ \xi = \xi_0 \]

\[ U(s) \]

\[ \delta(s) \]

\[ \epsilon = \frac{b}{a} \]

\[ e = \sqrt{1 - \epsilon^2} \]

\[ \mathcal{L} = 4a \frac{E(e)}{\pi} \]
Literature Review

- No analytical work available
- Experimental
- Numerical
Objectives

Develop models for:

- Fluid flow (total drag coefficient)
- Heat transfer (local and average heat transfer coefficients under UWT and UWF)
Assumptions

- Flow normal to cylinder axis
- Steady, laminar and 2-D flow
- Incompressible fluid with constant properties
- Thin hydrodynamic boundary layer
- No energy dissipation in thermal boundary layer
- No slip at cylinder wall
- No mass flow through cylinder wall
- Inviscid flow outside hydrodynamic boundary layer
Governing Equations

Continuity:
\[ \frac{\partial u}{\partial s} + \frac{\partial v}{\partial \eta} = 0 \]

Momentum:
\[ u \frac{\partial u}{\partial s} + v \frac{\partial u}{\partial \eta} = -\frac{1}{\rho} \frac{dP}{ds} + \nu \frac{\partial^2 u}{\partial \eta^2} \]

Energy:
\[ u \frac{\partial T}{\partial s} + v \frac{\partial T}{\partial \eta} = \alpha \frac{\partial^2 T}{\partial \eta^2} \]
Velocity Profile:

\[
\frac{u}{U(s)} = (2\eta - 2\eta^3 + \eta^4) + \frac{\lambda}{6} (\eta - 3\eta^2 + 3\eta^3 - \eta^4)
\]

where \(0 \leq \eta \leq 1\) and \(\lambda = \frac{\delta^2}{\nu} \frac{dU(s)}{ds}\)

Temperature Profiles:

For UWT:

\[
\frac{T - T_\infty}{T_w - T_\infty} = 1 - \frac{3}{2} \eta_T + \frac{1}{2} \eta_T^3
\]

For UWF:

\[
T - T_\infty = \frac{2q \delta_T}{3k_f} \left(1 - \frac{3}{2} \eta_T + \frac{1}{2} \eta_T^3\right)
\]
Fluid Flow Model

Specify Approach Velocity and Velocity Profile

θ-Momentum Equation

Pressure Drag Coefficient ($C_{Dp}$)

Momentum Integral Equation

Hydrodynamic Boundary Layer Thickness ($\delta$)

Newton’s Law of Viscosity

Friction Drag Coefficient ($C_{Df}$)

Total Drag Coefficient

$C_D = C_{Df} + C_{Dp}$
Heat Transfer Model

Specify Temperature Profile Inside Thermal Boundary Layer

Velocity Profile and Hydrodynamic Boundary Layer Thickness

Energy Integral Equation

Thermal Boundary Layer Thickness $\delta_T$

Newton’s Law of Cooling

Local Heat Transfer Coefficient

Average Heat Transfer Coefficient
Analytical Results (Fluid Flow)

Total Drag Coefficient:

\[
C_D = \frac{1.353 + 4.43\epsilon^{1.35}}{\sqrt{Re_L}} + \left(1.1526 + \frac{1.26}{Re_L}\right)\epsilon^{0.95}
\]

when \(\epsilon \to 1\)

\[
C_D = \frac{5.786}{\sqrt{Re_D}} + 1.152 + \frac{1.260}{Re_D}
\]

when \(\epsilon \to 0\)

\[
C_D = \frac{1.353}{\sqrt{Re_L}}
\]
Heat Transfer Parameter:

\[
\frac{Nu_L}{Re_L^{1/2} Pr^{1/3}} = \begin{cases} 
0.75 - 0.16 \exp \left( \frac{-0.018}{\epsilon^{3.1}} \right) & \text{UWT} \\
0.91 - 0.31 \exp \left( \frac{-0.09}{\epsilon^{1.79}} \right) & \text{UWF}
\end{cases}
\]

when \( \epsilon \rightarrow 1 \)

\[
\frac{Nu_D}{Re_D^{1/2} Pr^{1/3}} = \begin{cases} 
0.5930 & \text{UWT} \\
0.6321 & \text{UWF}
\end{cases}
\]

when \( \epsilon \rightarrow 0 \)

\[
\frac{Nu_L}{Re_L^{1/2} Pr^{1/3}} = \begin{cases} 
0.750 & \text{UWT} \\
0.912 & \text{UWF}
\end{cases}
\]
Local Shear Stress

\[ C_{r} \sqrt{R_e} \]

\[ \theta \text{ (Radians)} \]

\( \varepsilon = 1 \) (Circular Cylinder)
\( \varepsilon = \frac{1}{2} \) (Elliptical Cylinder)
\( \varepsilon = \frac{1}{3} \) (Elliptical Cylinder)
\( \varepsilon = \frac{1}{4} \) (Elliptical Cylinder)
Total Drag Coefficient with Reynolds Number

![Graph showing the relationship between drag coefficient (C_D) and Reynolds number (Re_L).](image)

- **Analytical (Present Model)**
- **Experimental (Wieselsberger)**
- **Analytical (Present Model, \( \varepsilon = 0.5 \))**
- **Analytical (Present Model, \( \varepsilon = 0.25 \))**
- **Theoretical (Van Dyke)**

Legend:
- **Circular Cylinder**
- **Elliptical Cylinder**
- **Finite Plate**
Average Nusselt Number with Reynolds Number

\[ \text{Pr} = 0.71 \]
Average Nusselt Number with Reynolds Number

\[ \epsilon = 1:2 \]
Average Nusselt Number with Reynolds Number

\begin{equation}
\varepsilon = 1:4
\end{equation}
Average Nusselt Number with Axis Ratio

\[ UWF \quad Pr = 0.71 \]

\[
\begin{align*}
\text{Present Model} & \quad \text{Ota et al. (1984)} \\
\text{Present Model} & \quad \text{Ota et al. (1984)} \\
\end{align*}
\]

\[
\begin{align*}
\text{Re}_L & = 10000 \\
\text{Re}_L & = 40000 \\
\end{align*}
\]

\[
\begin{align*}
\text{UWF} \\
\text{Pr} & = 0.71 \\
\end{align*}
\]
Approximate analytical method gives:

- One model for drag coefficient
- Two models for heat transfer coefficient

These models can be used for:

- Laminar range ($40 \leq \text{Re}_D \leq 10^5$)
- Large Prandtl numbers ($\text{Pr} \geq 0.71$)
- Any axis ratio ($0 \leq \epsilon \leq 1$)
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