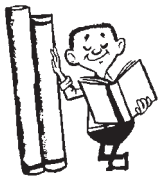
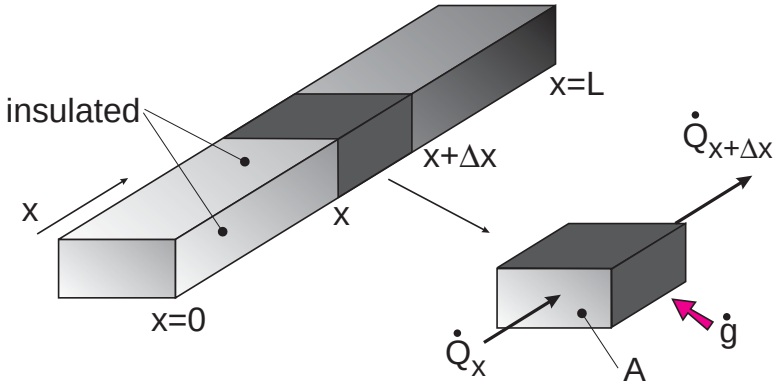
Conduction Heat Transfer



Reading	Problems
8-1 ⟶ 8-9	8-35, 8-63, 8-64, 8-69, 8-75, 8-80, 8-84, 8-89, 8-100, 8-103, 8-112, 8-126, 8-141, 8-150, 8-177
9-1 ⟶ 9-2	9-15, 9-20, 9-21, 9-36, 9-41, 9-42

Fourier Law of Heat Conduction

Consider a long solid bar where heat travels in a one-dimensional manner in the positive $\boldsymbol{x}$ direction as shown below.



$$\frac{\partial E}{\partial t} = \dot{Q}_x - \dot{Q}_{x+\Delta x} - \dot{W}$$

If the volume to the element is given as $\boldsymbol{V = A \cdot \Delta x}$, then the mass of the element is

$$m = \rho \cdot A \cdot \Delta x$$

$$E = m \cdot u = (\rho \cdot A \cdot \Delta x) \cdot u$$

For an incompressible substance the internal energy is $\boldsymbol{du = C\ dT}$ and we can write

$$\frac{\partial E}{\partial t} = \rho C A \Delta x \frac{\partial T}{\partial t}$$

We will assume that any electrical resistance heat addition to the body is denoted at $-\dot{W}$. Therefore

$$-\dot{W} = \dot{g} \cdot V = \dot{g} \cdot A \cdot \Delta x \quad \Leftarrow \quad \dot{g} = \text{volumetric heat addition, } (W/m^3)$$

We can also write the heat flow along the $x-$direction as a product of the temperature difference (the driving force for heat transfer) and a proportionality constant, based on experimental observation.

$$\dot{Q}_x = \frac{kA}{\Delta x}(T_x - T_{x+\Delta x})$$

where k is the thermal conductivity of the material. In the limit as $\Delta x \to 0$

$$\dot{Q}_x = -kA\frac{\partial T}{\partial x}$$

This is *Fourier's law of heat conduction.*

$$\dot{Q}_{x+\Delta x} = \dot{Q} + \frac{\partial \dot{Q}_x}{\partial x}\Delta x$$

when combined with Fourier's equation gives

$$\dot{Q}_{x+\Delta x} = \underbrace{-kA\frac{\partial T}{\partial x}}_{\dot{Q}_x} - \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right)\Delta x$$

Noting that

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} = \frac{\partial E}{\partial t} + \dot{W} = \rho C A \Delta x \frac{\partial T}{\partial t} + A\Delta x \dot{g}$$

By removing the common factor of $A\Delta x$ we can then write the general 1-D conduction equation as

$$\underbrace{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)}_{\substack{\textit{longitudinal} \\ \textit{conduction}}} + \underbrace{\dot{g}}_{\substack{\textit{internal} \\ \textit{heat} \\ \textit{generation}}} = \underbrace{\rho C \frac{\partial T}{\partial t}}_{\substack{\textit{thermal} \\ \textit{inertia}}}$$

Thermal Resistance Networks

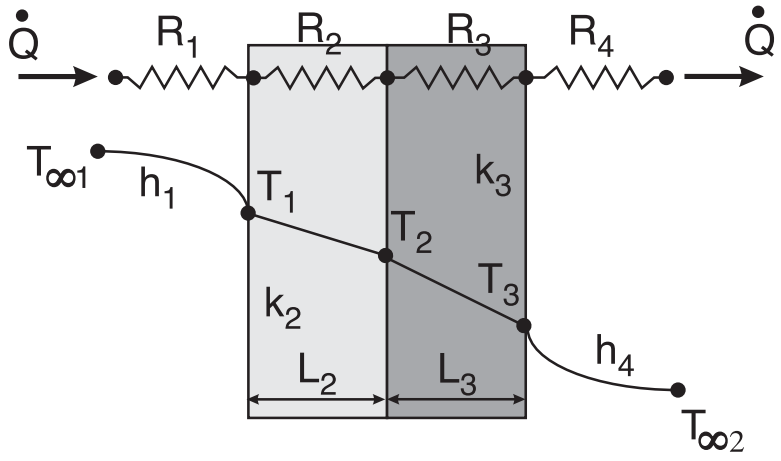
Resistances in Series

The heat transfer across the fluid/solid interface is based on *Newton's law of cooling*

$$\dot{Q} = hA(T_{in} - T_{out}) = \frac{T_{in} - T_{out}}{R_{conv}} \qquad \text{where} \quad R_{conv} = \frac{1}{hA}$$

The heat flow through a solid material of conductivity, k is

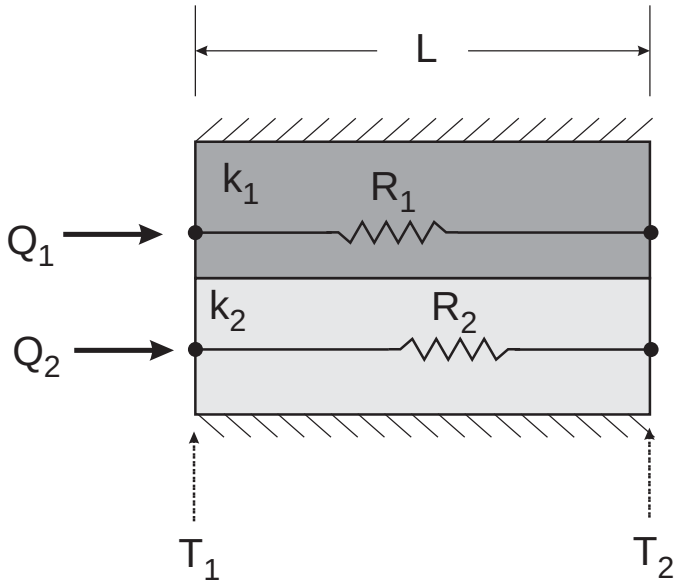
$$\dot{Q} = \frac{kA}{L}(T_{in} - T_{out}) = \frac{T_{in} - T_{out}}{R_{cond}} \qquad \text{where} \quad R_{cond} = \frac{L}{kA}$$



The total heat flow across the system can be written as

$$\dot{\boldsymbol{Q}} = \frac{\boldsymbol{T_{\infty_1}} - \boldsymbol{T_{\infty_2}}}{\boldsymbol{R_{total}}} \qquad \text{where} \quad \boldsymbol{R_{total}} = \sum_{i=1}^{4} \boldsymbol{R_i}$$

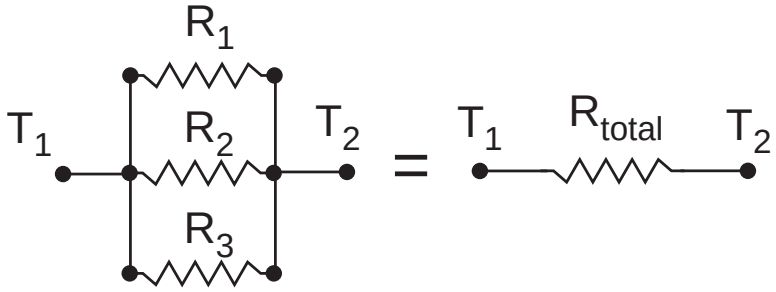
Resistances in Parallel



For systems of parallel flow paths as shown above, we can use the $\boldsymbol{1^{st}}$ law to preserve the total energy

$$\dot{\boldsymbol{Q}} = \dot{\boldsymbol{Q}}_{\boldsymbol{1}} + \dot{\boldsymbol{Q}}_{\boldsymbol{2}}$$

In general, for parallel networks we can use a parallel resistor network as follows:



where

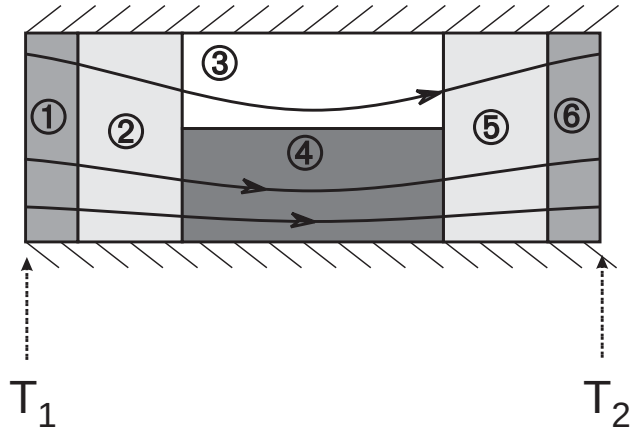
$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$

and

$$\dot{Q} = \frac{T_1 - T_2}{R_{total}}$$

Flow in a Composite Wall

For a composite system where $k_3 << k_4$ the heat flow lines would look like

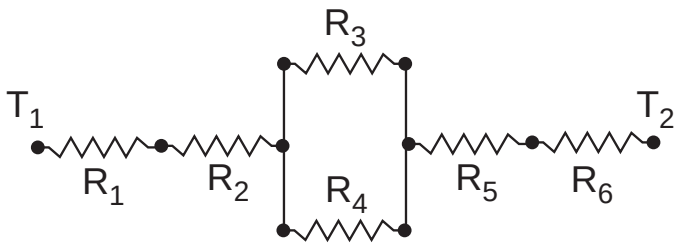


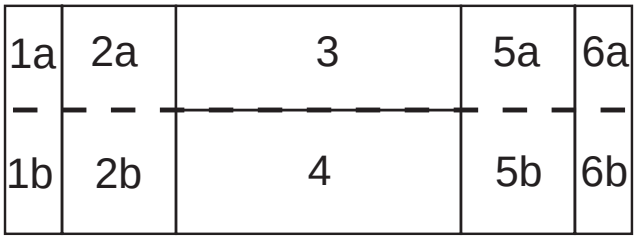
This problem cannot be easily solved using only a series network or a parallel network *but* we can use these two approaches as bounds on the problem and hopefully find an average of the two limits that will provide a reasonable compromise.

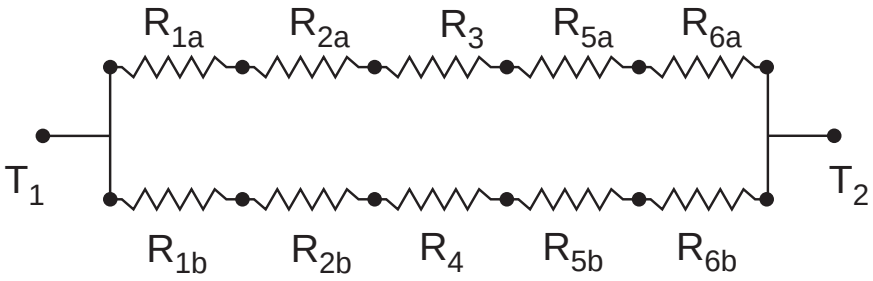
a) Upper bound on heat transfer: If we assume that heat flows in a 1-D manner within each section we can use a combined series/parallel resistor network as follows:

Since there is no turning in the heat flow path, the thermal resistance will be minimized and we will have an upper bound on the heat transfer.

b) Lower bound on heat transfer: The lower bound on heat transfer (upper bound on resistance) is achieved by forcing heat flow through an upper and a lower path as shown below.

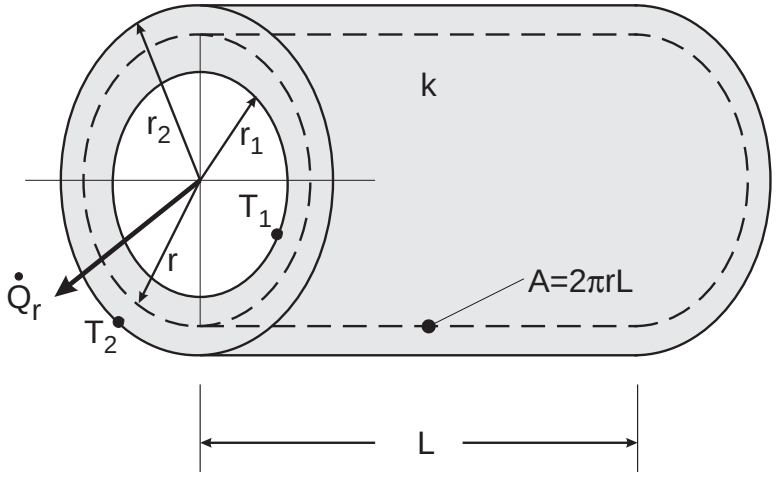






- equivalent to inserting a thin insulator along the dotted line
  - forcing heat flow through the upper and lower paths
  - prevents heat flow from finding a path of least resistance
  - leads to a lower bound on heat transfer
- since the solid resistance in each element is $\boldsymbol{R = \frac{L}{kA}}$ the area used in the upper and lower sections will be adjusted to reflect the cross sectional flow area of that section

Cylindrical Systems



Steady, 1D heat flow from $\boldsymbol{T_1}$ to $\boldsymbol{T_2}$ in a cylindrical systems occurs in a radial direction where the lines of constant temperature (isotherms) are concentric circles, as shown by the dotted line in the figure above and $\boldsymbol{T = T(r)}$.

Performing a $\boldsymbol{1^{st}}$ law energy balance on a *control mass* from the annular ring of the cylindrical cylinder

$$\frac{\boldsymbol{dE_{C.M.}}}{\boldsymbol{dt}}^{\boldsymbol{0}} = \boldsymbol{\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen}}^{\boldsymbol{0}}$$

Therefore

$$\boldsymbol{\dot{Q}_r - \left[\dot{Q}_r + \left(\frac{d\dot{Q}_r}{dr}\right) dr\right] = 0 \qquad \Rightarrow \qquad \frac{d\dot{Q}_r}{dr} = 0}$$

Integrating this result we obtain

$$\boldsymbol{\dot{Q}_r = constant}$$

Using Fourier's law of heat conduction

$$\boldsymbol{\dot{Q}_r = -k\ A\ \frac{dT}{dr} = constant}$$

where the cross sectional area for heat flow is given as the product of the circumference of the circular ring and the length of the cylinder

$$\boldsymbol{A = 2\pi r\mathcal{L}}$$

Separating the above equation gives

$$\boldsymbol{dT = -\frac{\dot{Q}}{2\pi k\mathcal{L}}\,\frac{dr}{r}}$$

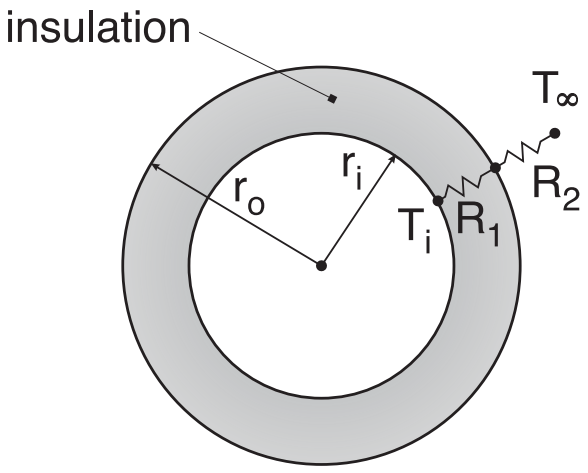
Integrating from $\boldsymbol{T = T_1}$ @ $\boldsymbol{r = r_1}$ to $\boldsymbol{T = T_2}$ @ $\boldsymbol{r = r_2}$

$$\boldsymbol{T_2 - T_1 = -\frac{\dot{Q}_r}{2\pi k\mathcal{L}}(\ln r_2 - \ln r_1) = -\frac{\dot{Q}_r}{2\pi k\mathcal{L}}\ln\frac{r_2}{r_1}}$$

Therefore we can write

$$\boldsymbol{\dot{Q}_r = \frac{T_1 - T_2}{\left(\frac{\ln(r_2/r_1)}{2\pi k\mathcal{L}}\right)}} \qquad \text{where} \quad \boldsymbol{R = \left(\frac{\ln(r_2/r_1)}{2\pi k\mathcal{L}}\right)}$$

Critical Thickness of Insulation



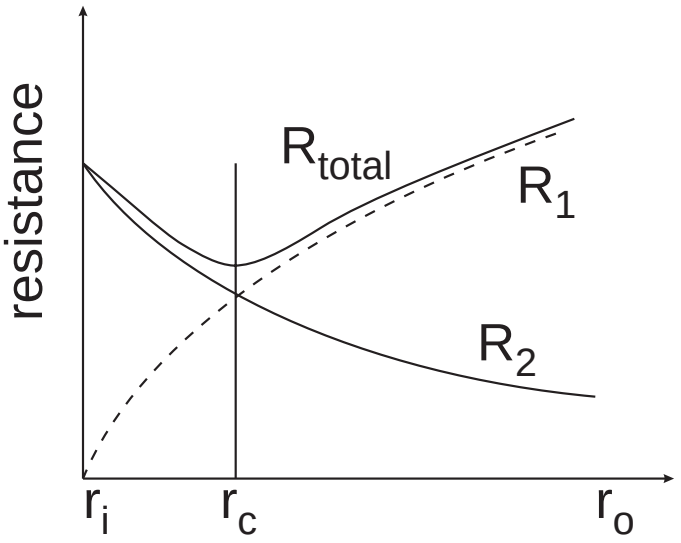
Could there be a situation in which adding insulation <u>increases</u> the overall heat transfer?

To find the critical radius, r_c, where adding more insulation begins to decrease heat transfer, set

$$\frac{dR_{total}}{dr_o} = 0$$

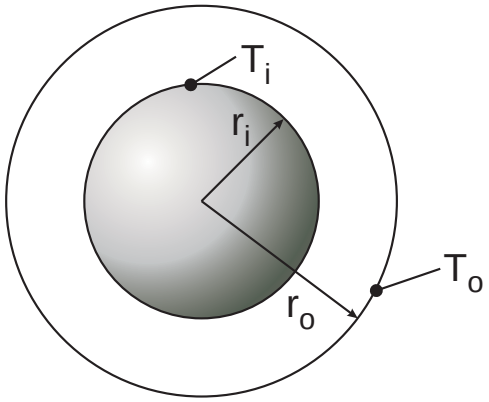
$$\frac{dR_{total}}{dr_o} = \frac{1}{2\pi k r_o \mathcal{L}} - \frac{1}{h 2\pi r_o^2 \mathcal{L}} = 0$$

$$r_c = \frac{k}{h}$$



There is always a value of r_c, but there is a minimum in heat transfer only if $r_c > r_i$

Spherical Systems



For steady, 1D heat flow in spherical geometries we can use a $\mathbf{1}^{st}$ law energy balance similar to the analysis for cylindrical systems to obtain

$$\dot{\boldsymbol{Q}} = \left(-k\frac{\boldsymbol{dT}}{\boldsymbol{dr}}\right)\underbrace{4\pi \boldsymbol{r}^2}_{\boldsymbol{A}} = \textbf{constant}$$

Then integrating from the inner to the outer radius

$$\begin{aligned}\int_{T_i}^{T_o} \boldsymbol{dT} &= -\int_{r_i}^{r_o} \frac{\dot{\boldsymbol{Q}}}{4\pi \boldsymbol{r}^2 \boldsymbol{k}}\, \boldsymbol{dr} \\ \boldsymbol{T_o - T_i} &= \frac{\dot{\boldsymbol{Q}}}{4\pi \boldsymbol{k}}\left[\frac{\boldsymbol{1}}{\boldsymbol{r_o}} - \frac{\boldsymbol{1}}{\boldsymbol{r_i}}\right]\end{aligned}$$

We can write the heat transfer as

$$\dot{\boldsymbol{Q}} = \frac{4\pi \boldsymbol{k r_i r_o}}{(\boldsymbol{r_0 - r_i})}\,(\boldsymbol{T_i - T_o}) = \frac{(\boldsymbol{T_i - T_o})}{\boldsymbol{R}} \qquad \text{where} \quad \boldsymbol{R} = \frac{\boldsymbol{r_o - r_i}}{4\pi \boldsymbol{k r_i r_o}}$$

Heat Generation in a Solid

Heat can be generated within a solid as a result of:

- resistance heating in wires
- chemical reactions
- nuclear reactions

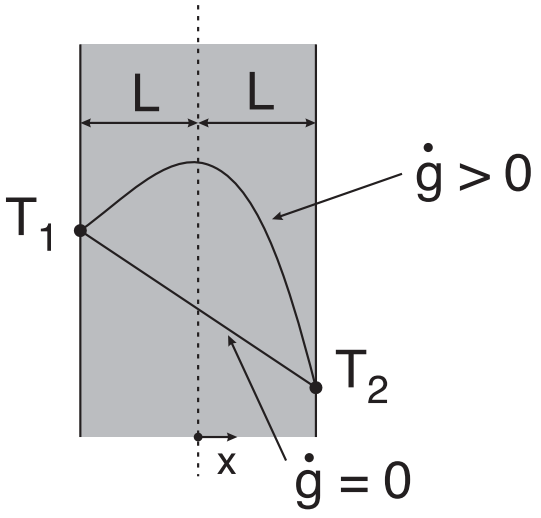
A volumetric heat generation terms will be defined as follows:

$$\dot{g} = \frac{\dot{E}_g}{V} \quad (W/m^3)$$

for heat generation in wires, we will define $\dot{g}$ as

$$\dot{g} = \frac{I^2 R_e}{\pi r_o^2 \mathcal{L}}$$

Slab System



Assume steady, 1D conduction with constant $\dot{g}$ in the slab and constant k.

We can write Fourier's law of conduction for a steady system as

$$k\frac{d^2T}{dx^2} + \dot{g} = 0$$

with boundary conditions

$$T = T_1 \quad @ \quad x = -L$$

$$T = T_2 \quad @ \quad x = +L$$

Integrating Fourier's equation twice gives

$$T = -\frac{\dot{g}x^2}{2k} + C_1 x + C_2$$

From the boundary conditions we can find

$$C_2 = \frac{(T_1 - T_2)}{2} + \frac{\dot{g}L^2}{2k}$$

$$C_1 = -\left(\frac{T_1 - T_2}{2L}\right)$$

Therefore

$$T = \frac{T_1 + T_2}{2} - \left(\frac{T_1 - T_2}{2}\right)\frac{x}{L} + \frac{\dot{q}L^2}{2k}\left(1 - \left(\frac{x}{L}\right)^2\right)$$

Case 1: if $T_1 = T_2 = T_3$

$$T = T_s + \frac{\dot{q}L^2}{2k}\left(1 - \left(\frac{x}{L}\right)^2\right)$$

This is a concave-down parabola. Why must the shape be of this form?

Case 2: If there is also heat loss from the outside edges to T_∞

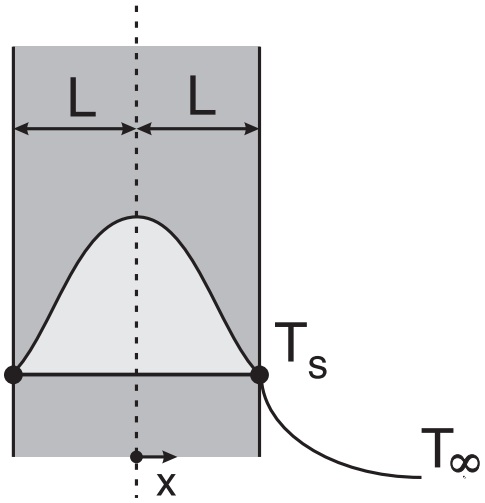
$$\dot{q} = -\left[k\frac{\partial T}{\partial x}\right]_{x=L^-} = -k\left\{\frac{\dot{g}L^2}{2k}\left(-\frac{2x}{L^2}\right)\right\}_{x=L^-} = +\dot{g}L$$

but Newton's law of cooling tells us

$$\dot{q} = h(T_s - T_\infty)$$

Therefore

$$T_s = T_\infty + \frac{\dot{g}L}{h}$$



Cylindrical System

The general conduction equation for a cylindrical system is given as

$$\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right)+\dot{g}=0$$

Integrating once gives

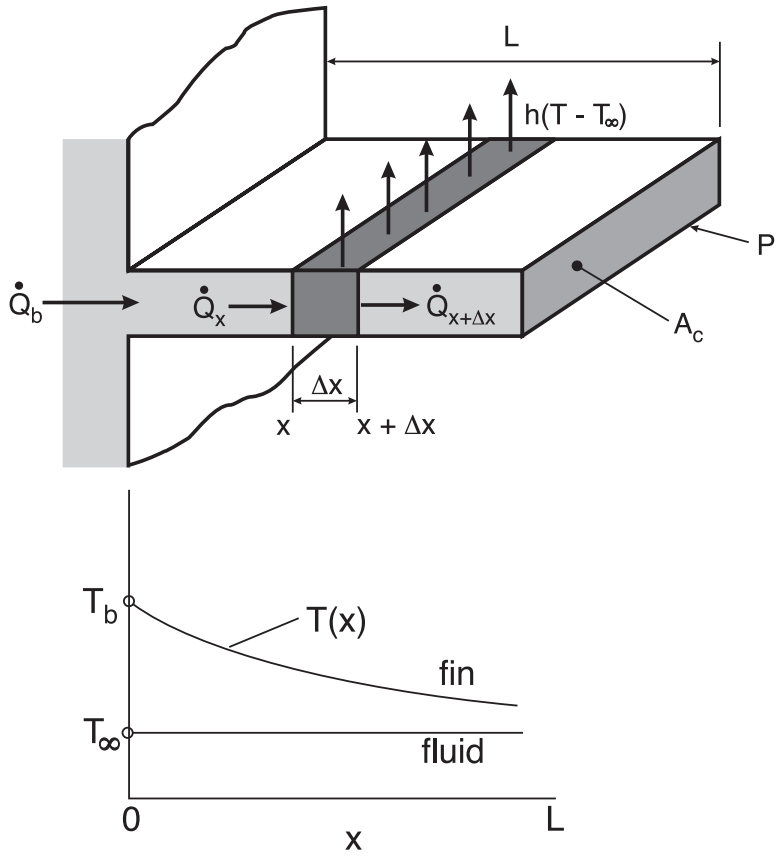
$$-2\pi r\mathcal{L}k\frac{dT}{dr}=(\pi r^2\mathcal{L})\dot{q}$$

While integrating a second time and noting the boundary condition $T=T_s$ @$r=r_0$, gives

$$T=T_s+\frac{\dot{g}r_0^2}{4k}\left(1-\left(\frac{r}{r_0}\right)^2\right)$$

Heat Transfer from Finned Surfaces

- heat transfer is improved by $\boldsymbol{h} \uparrow$ and $\boldsymbol{A} \uparrow$
- it is difficult to obtain a significant increase in $\boldsymbol{h}$
- an increase in $\boldsymbol{A}$ is easily achieved through extended surfaces or fins



We will assume a constant cross sectional area and that the temperature inside the fin is *only* a function of $\boldsymbol{x}$.

Just as in the development for Fourier's law of conduction, we can establish a $\boldsymbol{1^{st}}$ law balance over the thin slice of the fin between $\boldsymbol{x}$ and $\boldsymbol{x + \Delta x}$ such that

$$\dot{\boldsymbol{Q}}_{\boldsymbol{x}} - \dot{\boldsymbol{Q}}_{\boldsymbol{x+\Delta x}} - \boldsymbol{P \Delta x h (T - T_\infty)} = \boldsymbol{0}$$

From Fourier's law we know

$$\dot{\boldsymbol{Q}}_{\boldsymbol{x}} - \dot{\boldsymbol{Q}}_{\boldsymbol{x+\Delta x}} = \boldsymbol{k A_c \frac{d^2 T}{dx^2} \Delta x}$$

Therefore the conduction equation for a fin with constant cross section is

$$\underbrace{kA_c\frac{d^2T}{\partial x^2}}_{\substack{longitudinal\\conduction}} - \underbrace{hP(T-T_\infty)}_{\substack{lateral\\convection}} = 0$$

The temperature difference between the fin and the surroundings (temperature excess) is usually expressed as

$$\theta = T(x) - T_\infty$$

which allows the 1-D fin equation to be written as

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

where the fin parameter m is

$$m = \left(\frac{hP}{kA_c}\right)^{1/2}$$

and the boundary conditions are

$$\theta = \theta_b \quad @ \quad x = 0$$

$$\theta \to 0 \quad \text{as} \quad x \to \infty$$

The solution to the differential equation for θ is

$$\theta(x) = C_1 \sinh(mx) + C_2 \cosh(mx)$$

substituting the boundary conditions to find the constants of integration

$$\theta = \theta_b \frac{\cosh[m(L-x)]}{\cosh(mL)}$$

The heat transfer flowing through the base of the fin can be determined as

$$\begin{aligned} Q_b &= A_c\left(-k\frac{dT}{dx}\right)_{@x=0} \\ &= \theta_b(kA_chP)^{1/2}\ \tanh(mL) \end{aligned}$$

Fin Efficiency and Effectiveness

The dimensionless parameter that compares the actual heat transfer from the fin to the ideal heat transfer from the fin is the *fin efficiency*

$$\eta = \frac{\text{actual heat transfer rate}}{\text{maximum heat transfer rate when the entire fin is at } T_b} = \frac{\dot{Q}_b}{hPL_c\theta_b}$$

If the fin has a constant cross section then

$$\eta = \frac{\tanh(mL)}{mL}$$

An alternative figure of merit is the *fin effectiveness* given as

$$\epsilon_{fin} = \frac{\text{total fin heat transfer}}{\text{the heat transfer that would have occurred through the base area in the absence of the fin}} = \frac{\dot{Q}_b}{hA_c\theta_b}$$

Transient Heat Conduction

Various applications of transient heat conduction include:

- surface hardness control
- stress relieving
- processes requiring chemical reaction (i.e. curing of rubber)
- wire drawing and coating
- welding

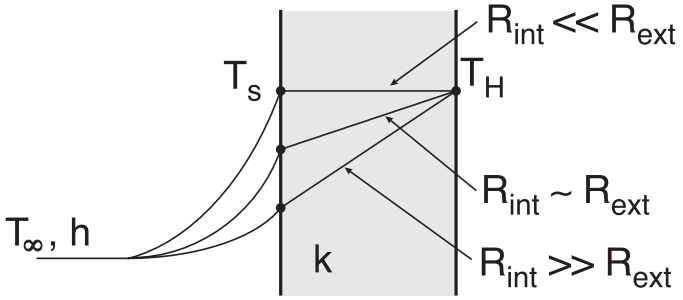
Performing a $\mathbf{1}^{st}$ law energy balance on a plane wall gives

$$\dot{E}_{in} - \dot{E}_{out} \quad \Rightarrow \dot{Q}_{cond} = \frac{T_H - T_s}{L/(k \cdot A)} = \dot{Q}_{conv} = \frac{T_s - T_\infty}{1/(h \cdot A)}$$

where

$$\frac{T_H - T_s}{T_s - T_\infty} = \frac{L/(k \cdot A)}{1/(h \cdot A)} = \frac{\text{internal resistance to H.T.}}{\text{external resistance to H.T.}}$$

$$= \frac{hL}{k} = Bi \equiv \text{Biot number}$$



$\boldsymbol{R_{int} << R_{ext}}$**:** the Biot number is small and we can conclude

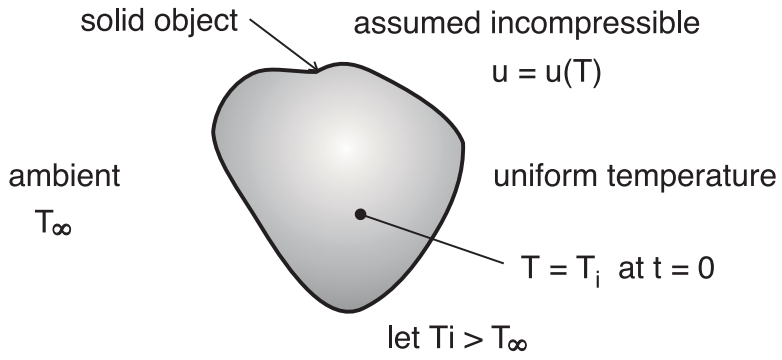
$$T_H - T_s << T_s - T_\infty \qquad \text{and in the limit} \quad T_H \approx T_s$$

$\boldsymbol{R_{ext} << R_{int}}$:

$\boldsymbol{R_{int} << R_{ext}}$: the Biot n umber is large and we can conclude

$$\boldsymbol{T_s - T_\infty << T_H - T_s} \qquad \text{and in the limit} \quad \boldsymbol{T_s \approx T_\infty}$$

Transient Conduction Analysis

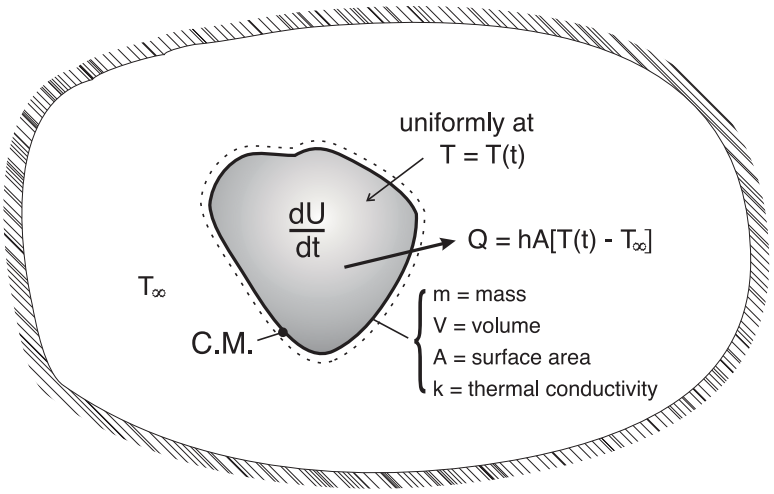


For the 3-D body of volume $\boldsymbol{V}$ and surface area $\boldsymbol{A}$, we can use a lumped system analysis if

$$\boldsymbol{Bi = \frac{hV}{kA} < 0.1} \qquad \Leftarrow \text{results in an error of less that } \mathbf{5\%}$$

The characteristic length for the 3-D object is given as $\boldsymbol{\mathcal{L} = V/A}$**.** Other characteristic lengths for conventional bodies include:

Slab	$\boldsymbol{\frac{V}{A_s} = \frac{WH2L}{2WH} = L}$
Rod	$\boldsymbol{\frac{V}{A_s} = \frac{\pi r_o^2 L}{2\pi r_0 L} = \frac{r_0}{2}}$
Sphere	$\boldsymbol{\frac{V}{A_s} = \frac{4/3\pi r_o^3}{4\pi r_0^2} = \frac{r_0}{3}}$



At $\boldsymbol{t > 0,\ \ T = T(x, y, z, t)}$, however, when $\boldsymbol{Bi < 0.1}$ then we can assume $\boldsymbol{T \approx T(t)}$.

Performing a $\boldsymbol{1^{st}}$ law energy balance on the control volume shown below

$$\frac{\boldsymbol{dE_{C.M.}}}{\boldsymbol{dt}} = \boldsymbol{\dot{E}_{in} - \dot{E}_{out} + \cancelto{0}{\dot{E}_g}}$$

If we assume $\boldsymbol{PE}$ and $\boldsymbol{KE}$ to be negligible then

$$\frac{\boldsymbol{dU}}{\boldsymbol{dt}} = \boldsymbol{-\dot{Q}} \qquad \Leftarrow \frac{\boldsymbol{dU}}{\boldsymbol{dt}} < \boldsymbol{0} \text{ implies } \boldsymbol{U} \text{ is decreasing}$$

For an incompressible substance specific heat is constant and we can write

$$\underbrace{\boldsymbol{mC}}_{\equiv \boldsymbol{C_{th}}} \frac{\boldsymbol{dT}}{\boldsymbol{dt}} = - \underbrace{\boldsymbol{Ah}}_{\boldsymbol{1/R_{th}}} \boldsymbol{(T - T_\infty)}$$

where

$\boldsymbol{C_{th}} =$ lumped capacitance

It should be clearly noted that we have neglected the spatial dependence of the temperature within the object. This type of an approach is only valid for $\boldsymbol{Bi = \frac{hV}{kA} < 0.1}$

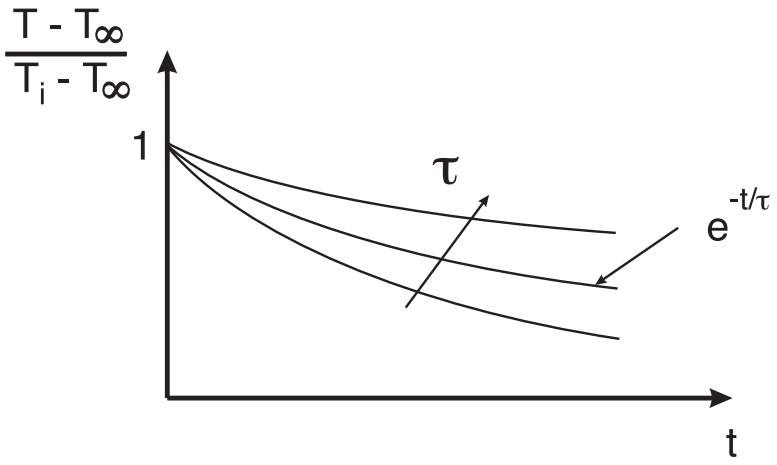
$$\boldsymbol{C_{th}} \frac{\boldsymbol{dT}}{\boldsymbol{dt}} = -\frac{\boldsymbol{1}}{\boldsymbol{Rth}} \boldsymbol{(T - T_\infty)}$$

We can integrate and apply the initial condition, $\boldsymbol{T = T_i}$ $\boldsymbol{@t = 0}$ to obtain

$$\frac{\boldsymbol{T(t) - T_\infty}}{\boldsymbol{T_i - T_\infty}} = \boldsymbol{e^{-t/(R_{th} \cdot C_{th})} = e^{-t/\tau}}$$

where

$$\boldsymbol{\tau = R_{th} \cdot C_{th}} = \text{thermal time constant} = \frac{\boldsymbol{mC}}{\boldsymbol{Ah}}$$



The total heat transfer rate can be determined by integrating $\boldsymbol{\dot{Q}}$ with respect to time.

$$\begin{aligned}
\boldsymbol{\dot{Q}_{total}} &= \boldsymbol{\int_0^{t^*} \dot{Q}\, dt = \int_0^{t^*} hA(T - T_\infty)\, dt} \\
&= \boldsymbol{ha(T_i - T_\infty) \int_0^{t^*} e^{-t/\tau}\, dt} \\
&= \boldsymbol{hA(T_i - T_\infty(\tau)[1 - e^{-t^*/\tau}]}
\end{aligned}$$

Therefore

$$\boldsymbol{\dot{Q}_{total} = \dot{m}C(T_i - T_\infty)[1 - e^{-t^*/\tau}]}$$